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A not so fair game

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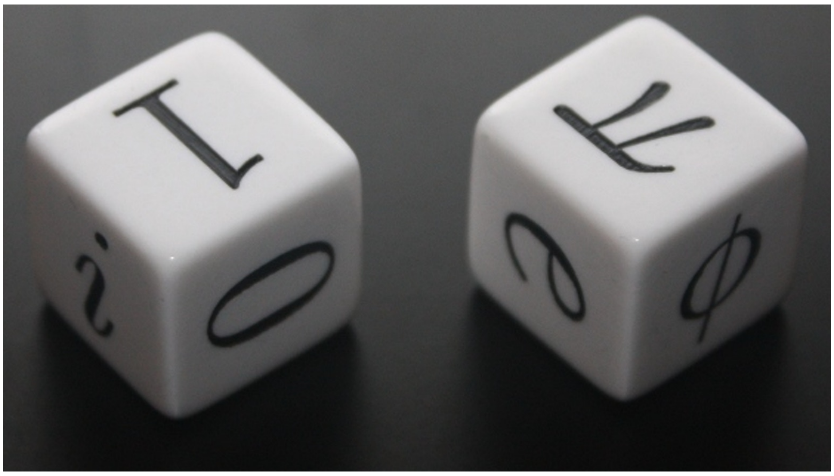
Subject.

Our research aims to create a mathematic model for a funny gamble, to explain and teach us how to calculate the chances of success, based on statistical – probability calculation. Also, it can show the chances of winning, but it does not guarantee it.



Our problem involves a craps game between Pacala and Tandala, two famous fictional characters in the Romanian folklore, literature and humour.

Pacala created three different dice, modifying the number of dots on each face. Therefore, the dice introduced are as follows:

Die 1:	5	7	8	9	10	18
Die 2:	2	3	4	15	16	17
Die 3:	1	6	11	12	13	14

For each die, all of the faces have equal chance of appearance. Each player picks a die and keeps it until the end of the competition. One game consists of each player throwing their chosen die, and the one who accumulates the most points wins the game. This type of game can be repeated multiple times, in identical and independent conditions.

a) Apparently polite, Pacala invites Tandala to choose his die first. We are asked to prove that, no matter what die Tandala picks, Pacala has the possibility of choosing a better die from the ones left. In other words, in each game, Pacala has better odds of winning.

b) In each round, the player who scores higher receives one € from the other player. We are asked to find Pacala's average gain after 60 games.

c) The next question we have to answer to is the following: "What is the probability that, after 10 rounds, Tandala has at least 5 €?"

d) In the beginning of the game, both players had 10 €. In each game, the winner receives 1 € from the other player.We are asked to find the average number of games required for one of the players to become broke.

e) Tandala understands Pacala's trick and decides to bring in three new dice, numbered from one to six, with the possibility of repetition, so that he may have a higher chance of winning.We must help him create his dice.

Results.

a) *Apparently polite, Pacala invites Tandala to choose his die first. We are asked to prove that, no matter what die Tandala picks, Pacala has the possibility of choosing a better die from the ones left. In other words, in each game, Pacala has better odds of winning.*

Solution.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(d_2 > d_1 | T = d_1) = \frac{\text{favourable cases}}{\text{possible cases}} = \frac{15}{36}$$

$$P(d_3 > d_1 | T = d_1) = \frac{21}{36} > \frac{1}{2}$$

We denote by: T-Tandala; d-die [1].
Favourable cases for $d_2 > d_1$: (15,5), (16,5), (17,5), (15,7), (16,7), (17,7), (15,8), (16,8), (17,8), (15,9), (16,9), (17,9), (15,10), (16,10), (17,10).

$$\#\{\text{favourable cases}\} = 15.$$

If Tandala picks die 1, Pacala has to pick die 3.

$$P(d_1 > d_2 | T = d_2) = \frac{21}{36} > \frac{1}{2}$$
$$P(d_3 > d_2 | T = d_2) = \frac{15}{36} < \frac{1}{2}$$

If Tandala picks die 2, Pacala has to pick die 1.

$$P(d_2 > d_3 | T = d_3 = \frac{21}{36} > \frac{1}{2}$$
$$P(d_1 > d_3 | T = d_3) = \frac{15}{36} < \frac{1}{2}$$

If Tandala picks die 3, Pacala has to pick die 2.

Conclusion: if Tandala picks die 1, Pacala has to pick die 3.Using the same reasoning, we obtain the solutions that, if Tandala picks die 2, Pacala picks die 1, and if Tandala picks die 3, Pacala picks die 2.

Player	T	P
Dice	d_1	d_3
Dice	d_2	d_1
Dice	d_3	d_2

We can see that, no matter what die Tandala picks, Pacala can still choose a better one and win the game.

b) *In each round, the player who scores higher receives 1 € from the other player. We are asked to find Pacala's average gain after 60 games.*

Solution.
It is important to mention that, if these were ideal dice, each player's chances of winning would be:

C	1	2	3	4	5	6
p	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

C =The player's gain [2]
p =The player's chances of winning

$$E(C) = \sum_{k=1}^{6} k \times \frac{1}{6} = \frac{7}{2}$$

We can make an association with the ideal coin in a heads-or-tails-game, $\{H, T\} = \Omega$

C	H	T
p	$\frac{1}{2}$	$\frac{1}{2}$

In one round of the craps game, $\Omega = \{-1, 1\}$. A rank k game, with k from 1 to 60, would look like this:

C_k	P	T
p	$\frac{21}{36}$	$\frac{15}{36}$

and considering Pacala's gain...:

C_k	1	-1
p	$\frac{21}{36}$	$\frac{15}{36}$

$$E(C_k) = 1 \times \frac{21}{36} + (-1) \times \frac{15}{36} = \frac{6}{36} = \frac{1}{6}$$

After 60 games $C = \sum_{k=1}^{60} C_k$,

$$E(C) = E\left(\sum_{k=1}^{60} C_k\right) = \sum_{k=1}^{60} E(C_k) = \sum_{k=1}^{60} \frac{1}{6} = 10 \text{ €}.$$

This is a proper example of a Bernoulli Distribution (it can also be used to represent a coin toss, where 1 and 0 would mean "head" and "tail", respectively. In particular, unfair coins would have $p \neq 0.5$). The Bernoulli Distribution is a special case of the Binomial Distribution, where a single experiment is conducted.

c) *The next question we have to answer to is the following:"What is the probability that, after 10 rounds, Tandala has at least 5 €?"*

Solution.
We will be using the letter T in order to express Tandala's gain after 10 games.

$A \cup \overline{A} = \Omega$, $A \cap \overline{A} = \oslash$, where A and $\overline{A}$ are two complementary events.

Let $A = \{T \geq 5\}$. This represents the event in which Tandala wins at least 5 €.

$$\begin{aligned}\overline{A} &= \{T \leq 4\} = \{T < 5\} \\ &= \{(T=0), (T=1), (T=2), (T=3), (T=4)\}\end{aligned}$$

$P(A) + P(\overline{A}) = P(\Omega) \to P(A) = 1 - P(\overline{A})$

We will be using the Markov-Polya Urn model: [3]

$$P(T) = \sum_{k=0}^{4} C_{10}^{k} \left(\frac{15}{36}\right)^{k} \left(\frac{21}{36}\right)^{10-k}$$

$$\begin{aligned}P(T=0) &= \left(\frac{21}{36}\right)^{10} \\ P(T=1) &= 10\left(\frac{21}{36}\right)^{9} \left(\frac{15}{36}\right)^{1} \\ P(T=2) &= C_{10}^{2}\left(\frac{15}{36}\right)^{2}\left(\frac{21}{36}\right)^{8} \\ P(T=3) &= C_{10}^{3}\left(\frac{15}{36}\right)^{3}\left(\frac{21}{36}\right)^{7} \\ P(T=4) &= C_{10}^{4}\left(\frac{15}{36}\right)^{4}\left(\frac{21}{36}\right)^{6} \\ P(T \leq 4) &= \sum_{k=0}^{4} P(T=k) \\ P(T \geq 5) &= 1 - P(T \leq 4).\end{aligned}$$

d) *In the beginning of the game, both players had 10 €. In each game, the winner receives 1 € from the other player. We are asked to find the average number of games required for one of the players to become broke.*

Solution.

We will be using the notation t_n to express the average expected number of games required for Tandala to become broke, if his initial amount of money is n, with $n \in \mathbf{N}$. [4]

We will use the results of the first game in order to find the solution.

1) If Tandala wins the first game, then his amount of money becomes $(n+1)$€ and the expected number of games required to get him broke turns into t_{n+1}. The probability of Tandala winning the first game is $q = \frac{15}{36}$.

2) If Tandala loses the first game, then he has $(n-1)$€ left and the expected number of games until he becomes broke turns into t_{n-1}. The probability of Tandala losing the first game is $p = \frac{21}{36}$.

The equation becomes:

(1) $$t_n = q \cdot t_{n+1} + p \cdot t_{n-1} + 1,\ n \geq 1$$

The expected number of games until Tandala is left with no money at all is equal to 1 (the first game) + the expected number of games following the first game.

For $n = 0,\ t_0 = 0$ (The average number of games required starting from $n = 0$ is 0.)
For $n = 20,\ t_{20} = 0$ (If Tandala has all the money, the game stops.)

The first equation can also be written as follows:

(2) $$q \cdot t_{n+1} - t_n + p \cdot t_{n-1} = -1,\ n \geq 1$$

In order to solve this equation, we will firstly consider the homogenous equation: [5]

(3) $$q \cdot t_{n+1} - t_n + p \cdot t_{n-1} = 0,\ n \geq 1$$

The characteristic equation attached is:

$$q \cdot \alpha^2 - \alpha + p = 0 \Leftrightarrow (1-p) \cdot \alpha^2 - \alpha + p = 0 \Leftrightarrow q \cdot \alpha^2 - \alpha + 1 - q = 0,$$

whose solutions are $\alpha_{1,2} = \dfrac{1 \pm (2q-1)}{2q}$.

$$\begin{aligned} \alpha_1 &= 1 \\ \alpha_2 &= \frac{2-2q}{2q} = \frac{p}{q} \end{aligned}$$

The general solution of the recurrence (3) is: [6]

$$(t_n)^o = A \cdot 1^n + B \cdot \left(\frac{p}{q}\right)^n,\ n \geq 1,\ (A, B \in \mathbf{R})$$

In order to solve the recurrence(2), we are looking for a particular solution which looks like this:

(4) $$(t_n)^p = C \cdot n,\ n \geq 1,\ (C \in \mathbf{R}).$$

After introducing (4) in the equation (2), we get:

$$\begin{aligned} q \cdot C \cdot (n+1) - C \cdot n + p \cdot C \cdot (n-1) &= -1 \\ C \cdot [q \cdot (n+1) - n + p \cdot n - p] &= -1 \\ C \cdot [q \cdot n + p \cdot n + q - n - p] &= -1 \\ C \cdot [(q+p) \cdot n + q - p - n] &= -1 \\ C \cdot (n + q - p - n) &= -1 \end{aligned}$$

$$C = \frac{1}{p-q}$$

Therefore, a particular solution of (2) is $(t_n)^p = \frac{n}{p-q}\cdot$

The general solution of the recurrence (2) is:

$$t_n = (t_n)^o + (t_n)^p = A + B\cdot\left(\frac{p}{q}\right)^n + \frac{n}{p-q}, \quad n \geq 0 \tag{5}$$

Using the conditions, $t_0 = 0 \Leftrightarrow A + B = 0,\ A = -B$

$$\begin{aligned}
t_{20} = 0 \Rightarrow \quad & A + B\cdot\left(\frac{p}{q}\right)^{20} + \frac{20}{p-q} = 0 \\
& A - A\cdot\left(\frac{p}{q}\right)^{20} = \frac{20}{q-p} \\
& A\cdot\left[1-\left(\frac{p}{q}\right)^{20}\right] = \frac{20}{q-p} \\
& A = \frac{20}{q-p}\cdot\frac{1}{1-\left(\frac{p}{q}\right)^{20}} \\
& B = \frac{20}{p-q}\cdot\frac{1}{1-\left(\frac{p}{q}\right)^{20}}
\end{aligned}$$

After inserting A and B in (5), we get:

$$\begin{aligned}
t_n &= \frac{20}{q-p}.\frac{1}{1-\left(\frac{p}{q}\right)^{20}} + \frac{20}{p-q}\cdot\frac{1}{1-\left(\frac{p}{q}\right)^{20}}\cdot\left(\frac{p}{q}\right)^n + \frac{n}{p-q},\ n \geq 0 \\
&= \frac{20}{q-p}\cdot\frac{1}{1-\left(\frac{p}{q}\right)^{20}}\left[1-\left(\frac{p}{q}\right)^n\right] + \frac{n}{p-q},\ n \geq 0 \\
&= \frac{20}{q-p}\cdot\frac{1-\left(\frac{p}{q}\right)^n}{1-\left(\frac{p}{q}\right)^{20}} + \frac{n}{p-q},\ n \geq 0
\end{aligned}$$

For n=10 (Tandala's initial amount of money):

$$t_{10} = \frac{20}{\frac{15}{36}-\frac{21}{36}}\cdot\frac{1-\left(\frac{7}{5}\right)^{10}}{1-\left(\frac{7}{5}\right)^{20}} + \frac{10}{\frac{21}{36}-\frac{15}{36}} \simeq 56.$$ [7]

The number of games required for Tandala to become broke starting from n=10 is 56.

e) *Tandala understands Pacala's trick and decides to bring in three new dice, numbered from one to six, with the possibility of repetition, so that he may have a higher chance of winning. We must help him create his dice.*

Solution.
One possible example of three dice that satisfy the conditions is: [8]

Die 1:	1	2	3	3	4	5
Die 2:	1	2	2	3	6	6
Die 3:	1	3	3	3	3	4

Player	*P*	*T*
Dice	d_3	d_1
Dice	d_1	d_2
Dice	d_2	d_3

Editing notes.

[1] d_i denotes either the choice of die i (in $T = d_i$), or the outcome when it is thrown.
[2] The game here is different, since only one die is thrown and the gain C is equal to its outcome. Below, E denotes the *expectation*, that is the mean value.
[3] We have a Binomial distribution – the distribution of the number of successes in a series of n independent Bernoulli trials with same probability p (with $n = 10$ and $p = 15/36$ here). The probability of k successes ($0 \leq k \leq n$) is equal to $C_n^k p^k (1-p)^{n-k}$, where $C_n^k = n!/\big(k!(n-k)!\big)$ is the binomial coefficient.
[4] In fact, the value of t_n which is computed below corresponds to the expected number of games until *either one of the players* (not Tandala only) becomes broke, as stated in the question.
Of course, since Tandala has less than 1 chance over 2 to win each game, the probability that he eventually becomes broke is higher than Pacala's one, but it would have been interesting to compute this probability.
[5] Here, the authors apply the classical method for soving linear recurrence equations. If you do not know this method, go directly to the general solution (5) and check that it satisfies the equation.
[6] The result here, as well as the value found for C below, are valid since $p \neq q$ in the present exemple. When $p = q$ the form of the general solution of (3) and of a particular solution of (2) are different.
[7] The expression of t_{10} can be simplified into $60 - 120/\big(1 + (\frac{7}{5})^{10}\big)$ and $1 + (\frac{7}{5})^{10} \simeq 30$.
Again, this is the expected number of games until one of the players, Tandala *or* Pacala, becomes broke.
[8] We should know the rule in case when the outcomes of both dice are equal – likely, the players throw them again. Besides, it would have been interesting to know how the authors found these values (e.g. are there other solutions?), and to know the probabilities for each pair of dies, even if the computations are left to the reader.