The Paving's World

2024 – 2025 School Year

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1. Introduction

1.1. Overview

The purpose of this problem was to both stimulate our analytical thinking and to use our interest in solving and exploring mathematics. This problem has taught us to approach a thing from as many perspectives as possible but also encouraged us to use our logical thinking and creativity.

1.2. The problem

You have to entirely cover the plane copying infinitely a certain parallelogram, using multiple transformations, such as translation, rotation, symmetry or slipping.

How many possibilities do you have?

Find as many as you can and justify them.

1.3. Summary results

Using the 4 different types of transformation, we were able to discover 17 ways in which we can cover the plane. These were as many unique ways as we could find, any other experiment becoming the same process as another one we had discussed before.

2. Conceptualization

2.1. Observations

We take into consideration the fact that squares, rectangles and rhombuses are also parallelograms, only to create a bigger variety of designs, but the total number of models / arrangements remains the same (17).

Even though we will only use the four transformations given: translation, rotation, symmetry and slipping, we will begin by giving the definition for each of the above (plus homothety / enlargement) and some of their properties in order to become familiar with these concepts and how we are going to use them.

2.2. Defining the main concepts

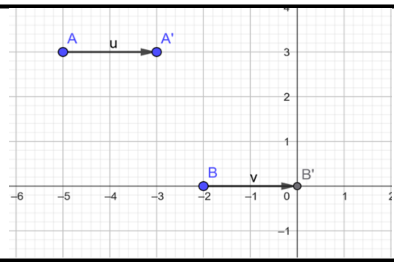
A. Translation

A translation is a geometric transformation that moves every point of a figure, shape or space by the same distance in a given direction. In a two-dimensional space , a translation can also be interpreted as the addition of a constant vector to every point, or as shifting the origin of the coordinate system.

$$T : \pi \to \pi \; (translation\ of\ vector\ \overline{u})$$
$$T(A) = A' \; such\ that\ AA' = \overline{u}$$

The proprieties of translation are:

- Isometric (having equal dimensions);
- Keeps the collinearity (for 3 or more points);
- Keeps the parallelism of lines.



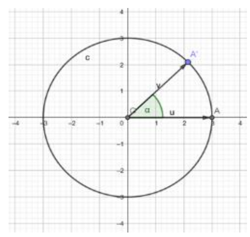
B. Rotation

A rotation is a transformation in which the object is rotated about a fixed point with a respective angle. The direction of rotation can be clockwise or counterclockwise. The fixed point in which the rotation takes place is called the center of rotation. We will take the center of rotation the center of the base pattern and the direction counterclockwise.

$$R(O,\alpha) : \pi \to \pi \; (rotation\ of\ center\ O\ and\ angle\ \alpha)$$

The proprieties of rotation are:

- Isometric;
- Keeps the length of the segments;
- Same orientation.



C. Symmetry

A symmetry is a mapping from a Euclidean space to itself that is an isometry with a hyperplane as a set of fixed points; this set is called the axis (in dimension 2) or plane (in dimension 3) of reflection. We will only analyze the symmetries in a two-dimensional space.

Point and line symmetry

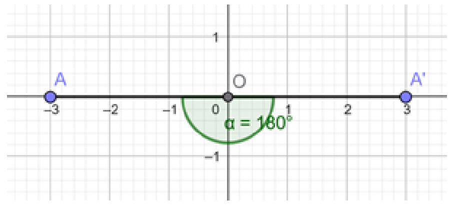
$$s_d : \pi \to \pi$$

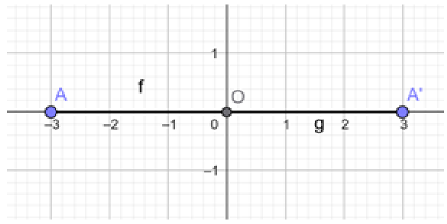
$$s_d(A) = A' \text{ such that } A' = Sim_d A$$

$s_d(A) = A'$ *such that d is the mediator of thesegment* $[AA']$

The proprieties of symmetry are:

- $A' = Sim_o A \Leftrightarrow A = Sim_o A'$;
- $(s_o \circ s_o)(X) = s_o(s_o(X)) = X \Leftrightarrow s_o^2 = 1_P$;
- $A \in d \Rightarrow s_d(A) = A' \in d$;
- $X' = s_d(X) \Rightarrow X = s_d(X')$;
- Axial symmetry is involatile;
- Axial symmetry is isometric.





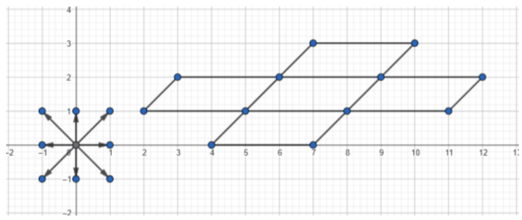
D. Slipping

We say that slipping is a particular case of translation, that corresponds to the vector $\vec{u} \in \{\pm \vec{i}; \pm \vec{j}\}$ where $\vec{i}$ and $\vec{j}$ are the versors of the system of axes, not necessarily perpendicular.

$$T_1 : \pi \to \pi,\ T_1(M) = M'$$
$$T_u : \pi \to \pi,\ T_u(M) = M'$$

The proprieties of slipping are:

- Isometric (having equal dimensions);
- Keeps the collinearity (for 3 or more points);
- Keeps the parallelism of lines.



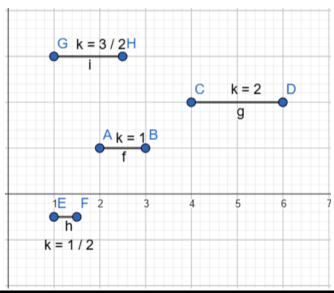
E. Enlargement

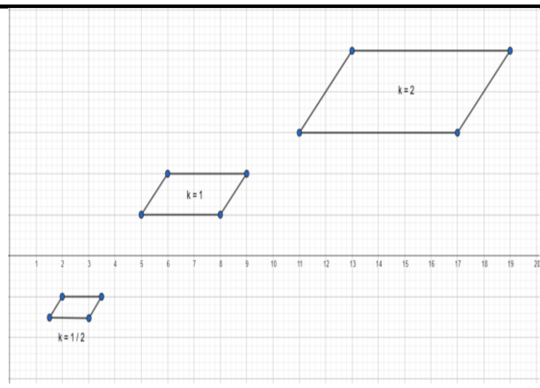
Enlargement is a geometric transformation defined by a center O and a ratio k, which scales the dimension of figures proportionally while preserving their shape. For any point A in the plane, its transformed point A' satisfies the conditions: $OA' = k \times OA$; $\overline{OA'}$ *is colliniar with* $\overline{OA}$.

Enlargement applied to a line

The proprieties:

- A line becomes a parallel line; the line remains fixed;
- Points on the line move proportionally relative to the center according to k



Enlargement applied to a polygon (in our case a parallelogram) The properties: • Side lengths: All side lengths scale proportionally: $L' = k \times L$ • Angles: Interior and exterior angles remain unchanged; • Diagonals: Diagonals are scaled by k ; • Area: The polygon's area changes according to: $S' = k^2 \times S$.	

3. Methodology

For each model that we have discovered we started by generating a base, using only one type of parallelogram (square, rhombus, rectangle or regular), then we created a design through a creative and limitless way (in order to get rid of the monotony and make a little connection with the artistic segment of our problem), and finishing by applying the same design for each parallelogram, in order to obtain a bigger repeating pattern.

4. Results and discussions

4.1. Asymmetrical parallelogram

We start by entirely covering a plane with parallelograms, using only slipping (Figure 1). Then we create the design using a printing stamp (a torus), and it should roll on a 'phantom' plane, which would allow itself to be crossed by it (Figure 2). After that, applying the design to every parallelogram, we obtain the final pattern (Figure 3).

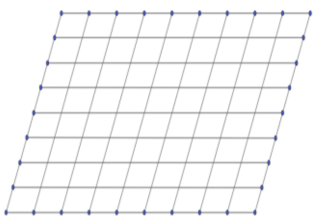
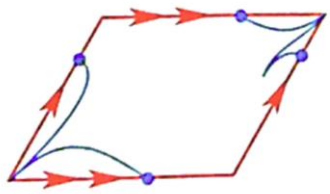
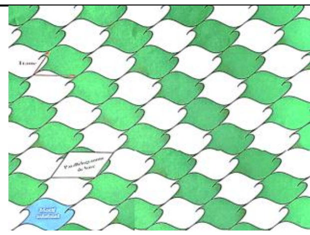
		

Figure 1 – Base	Figure 2 – Design	Figure 3 – Result

4.2. Symmetrical parallelogram

We start by covering a plane with rectangles, using only slipping (Figure 4). Then, for our design, we consider a closed envelope with a given design and place it in a base parallelogram (Figure 5). Repeating it infinitely, we obtain the final pattern (Figure 6).

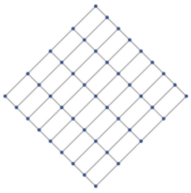
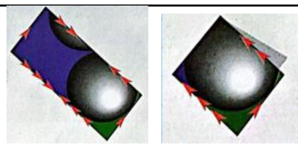
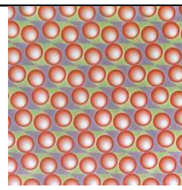
		

Figure 4 – Base	Figure 5 – Design	Figure 6 – Result

4.3. Hexagons with rotational symmetry

We start by covering the plane with parallelograms, but using both translation, to repeat the pattern across the plane, and rotation, to generate rotational symmetry within the parallelogram (Figure 7). As for the design, it is an equilateral triangular envelope rolling on the plane (Figure 8). Appling the stamp on all the parallelograms, we obtain the final result (Figure 9).

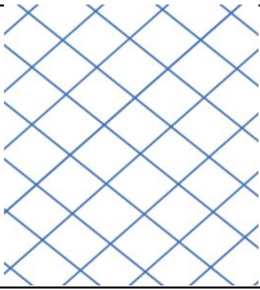
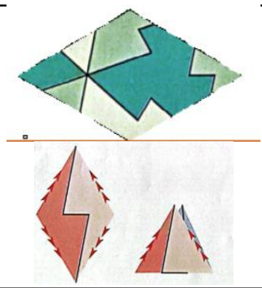
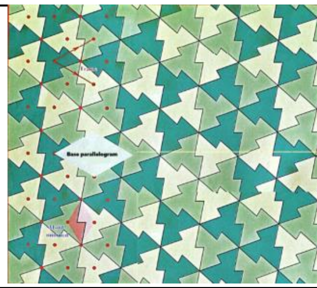
		

Figure 7 – Base	Figure 8 – Design	Figure 9 – Result

4.4. Square with rotational symmetry

We start by covering a plane with squares, but using both translation, to repeat the pattern across the plane, and rotation, to generate rotational symmetry within the parallelogram (Figure 10). The design is formed using one smaller motif which by being rotated forms a bigger square (Figure11). Appling the bigger square on all the base's squares helps us obtain the final image (Figure 12).

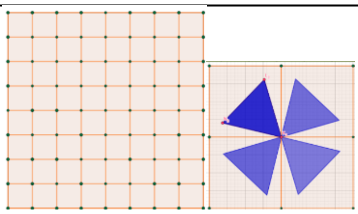
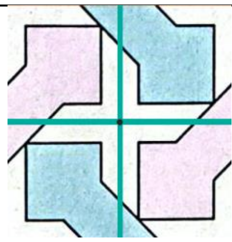
	

Figure 10 – Base	Figure 11 – Design	Figure 12 – Result

4.5. Hexagon with 6-rotation symmetry

We start by covering a plane with parallelograms, using both translation, to repeat the pattern across the plane, and rotation, to generate rotational symmetry within the parallelogram (Figure 13). The design is formed by using the parallelogram's two sets of opposing corners to form a repeating hexagonal pattern (Figure 14). Applying the parallelograms on the base grid, we achieve the result (Figure 15).

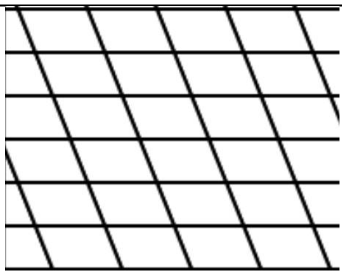
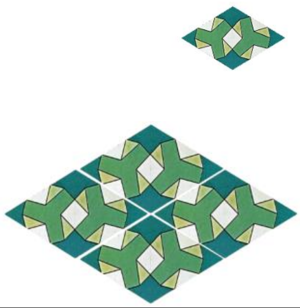
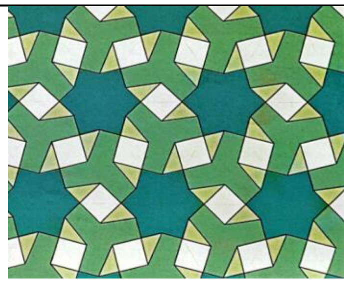
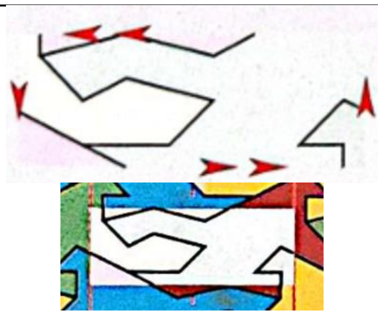
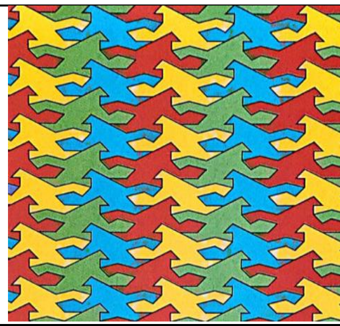
		

Figure 13 – Base	Figure 14 – Design	Figure 15 – Result

4.6. Sliding Rectangle

We start by covering a plane with rectangles, using only slipping (Figure 16). The design is formed by using carefully drawn lines that break the monotony of the rectangle and unite forming a more interesting picture (Figure 17). Applying the rectangles on the base grid, we achieve the result (Figure 18).

		
Figure 16 – Base	Figure 17 – Design	Figure 18 – Result

4.7. Rectangular Biglissant

We start with a plane covered in rectangles (Figure 19). The design consists of several parallelograms, of different colors, arranged in a zigzag pattern (Figure 20). When they are put together using slipping and rotation, we obtain an interesting pattern (Figure 21).

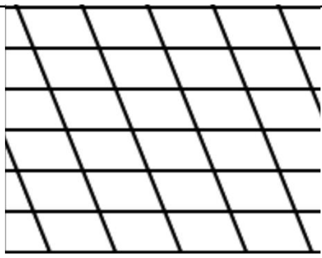
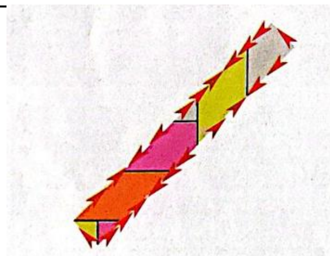
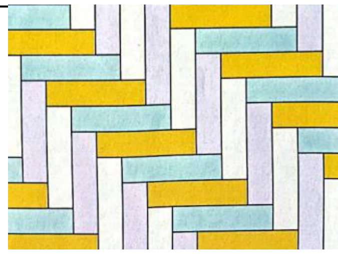
		

Figure 19 - Base	Figure 20 - Design	Figure 21 - Result

4.8. Monosymmetric rectangular

We begin with a simple surface covered in rectangles (Figure 22). The design is created by reflecting a drawn trapezoid in one of the base rectangles using a symmetry transformation (Figure 23). This transformation is then repeated across the surface to obtain the result (Figure 24).

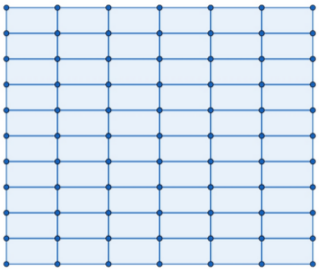


Figure 22 - Base

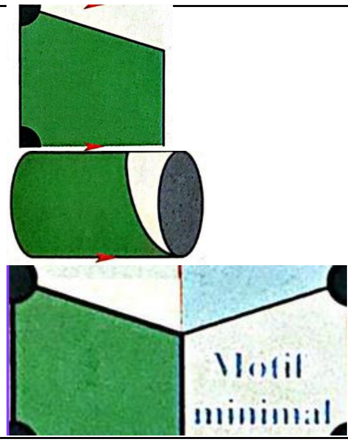


Figure 23 - Design

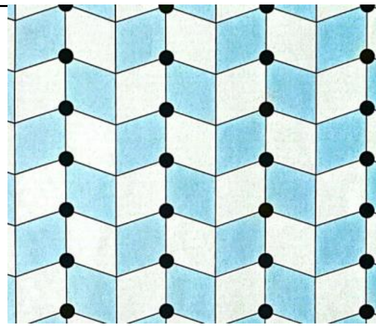


Figure 24 - Result

4.9. Rhombic monosymmetry

We start by covering the plane with parallelograms placed on a rectangular base and divided into two symmetrical parts (Figure 25). As for the design, it uses both horizontal and diagonal symmetry to create mirrored components within the shape (Figure 26). Applying the stamp on all the parallelograms, we obtain the result (Figure 27).

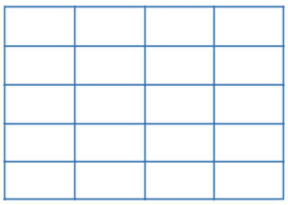


Figure 25 - Base

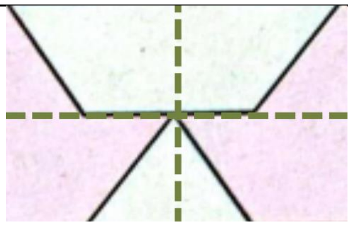


Figure 26 - Design

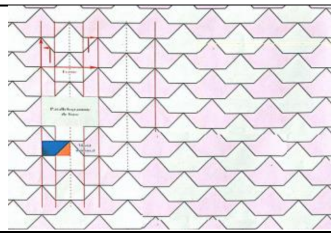


Figure 27 - Result

4.10. Rectangle with glide-reflection symmetry

We start by covering the base with rectangles, using only slipping (Fig. 28). The design is formed by using a motif that is aligned with the vertical axis y (Fig. 29). Applying the model on the base, using both translation and symmetry, we achieve the result (Fig. 30).

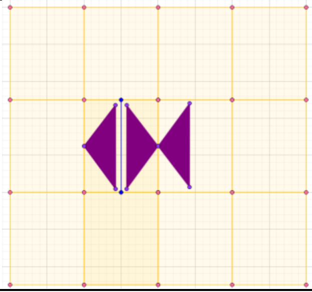
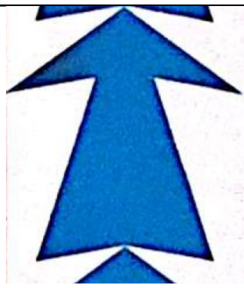
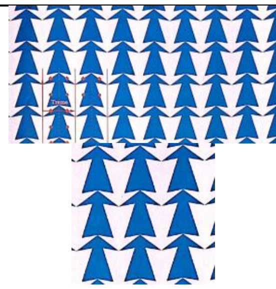
		

Figure 28- Base	Figure 29- Design	Figure 30- Result

4.11. Rectangular bisymmetry

We start by covering the base with rectangles, using slipping (Fig. 31) and each rectangle has been divided in 4 smaller, recurring motifs (Fig. 32). The motifs are repeated with the help of rotational symmetry and the symmetry with the vertical axis and horizontal axis. The result can be achieved with a combination of symmetry with OX and OY axes and translation. (Fig.33)

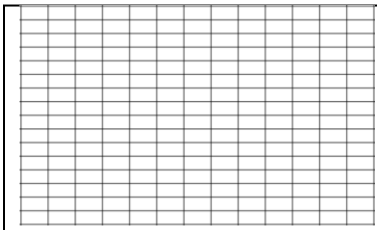
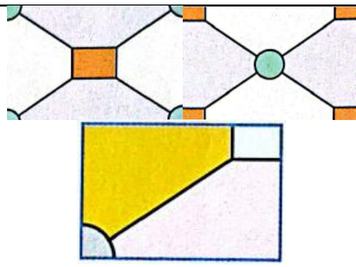
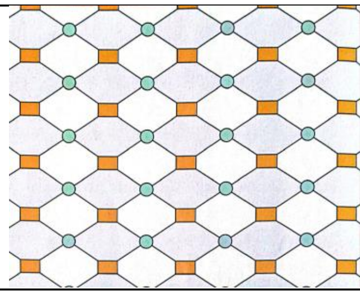
		

Figure 31- Base	Figure 32- Design	Figure 33- Result

4.12. Rhombic bisymmetry

We start by covering our base with rectangles (Fig. 34). The design has a recurring motif that is duplicated and rotated 180° around the middle point of its edge. The transformations used are rotation and symmetry (Fig. 35). The result can be achieved by using symmetry along the OX and OY axes on the new formed shapes and sliding them (Fig. 36).

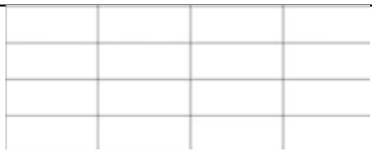
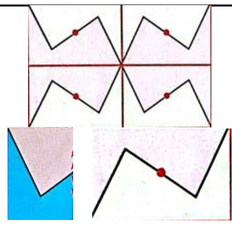
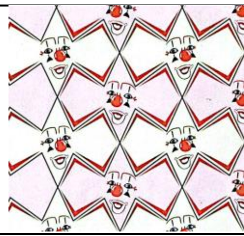
		

Figure 34- Base	Figure 35- Design	Figure 36- Result

4.13. Square with four rotating slides

We start by covering the plane with squares (Figure 37). The design uses both horizontal and diagonal symmetry to create mirrored components within the shape (Figure 38). The stamp printer — a cone rolling over a square then under each neighbor — imprints the design onto each parallelogram (Figure 39).

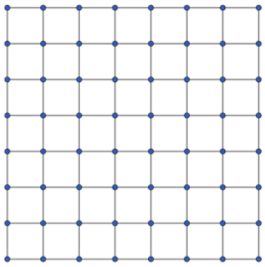
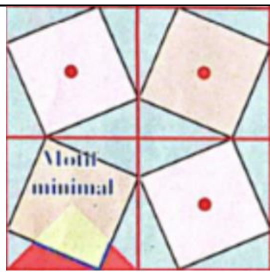
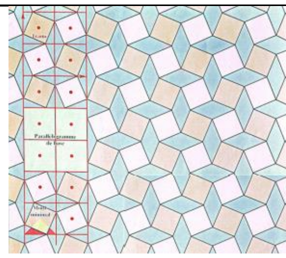
		

Figure 37 - Base	Figure 38 - Design	Figure 39 - Result

4.14. Hexagonal trisymmetric

We start by covering the plane with parallelograms arranged on a triangular base (Figure 40). The printing stamp (Figure 41) — an equilateral triangle alternately struck on and under the plane, describing a regular lattice — is applied to each triangle, covering the plane (Figure 42).

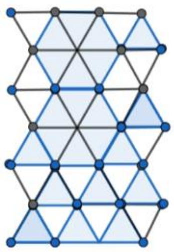
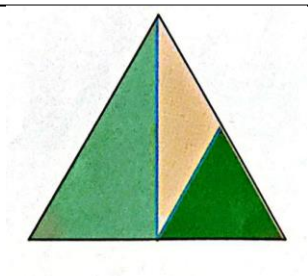
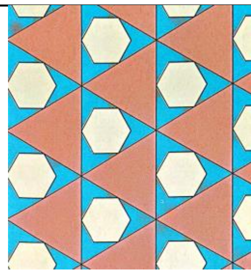
		

Figure 40 - Base	Figure 41 - Design	Figure 42 – Result

4.15. Mirrored hexagons with rotational symmetry

We start by covering the plane with parallelograms arranged on a diamond-shaped base (Figure 43). The printing stamp (Figure 44) — an equilateral triangular cone rolling on the plane — is applied to each unit, generating the design. This can be achieved through a combination of symmetry with respect to the OX and OY axes and translation (Figure 45).

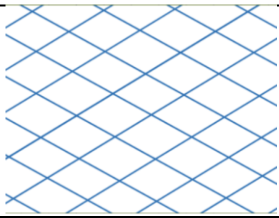
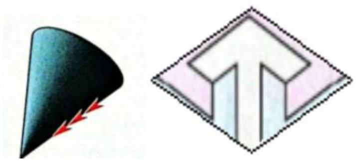
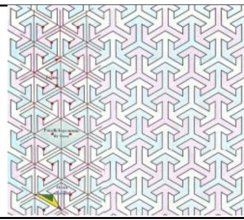
		

Figure 43 - Base	Figure 44 – Design	Figure 45 - Result

4.16. Square with full symmetry

A typical square grid serves as the foundation (Figure 46). The design features a motif that is translated along the OX and OY axes, respectively, after being reflected across a vertical and horizontal axis (Figure 47). This procedure creates a glide-reflection by fusing translation and line symmetry. A regular, alternating tiling pattern that preserves mirrored continuity across rows, columns, and any diagonal at a 45° angle is the result (Figure 48).

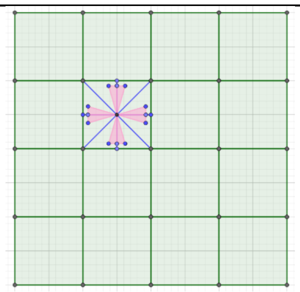
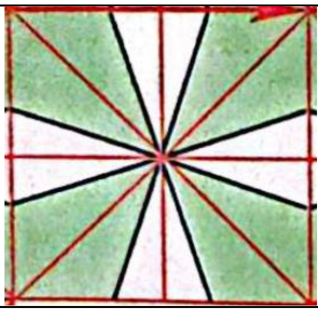
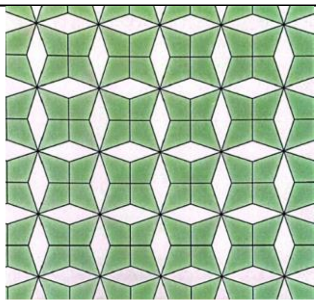
		

Figure 46 - Base	Figure 47 - Design	Figure 48 - Result

4.17. Hexagonal total symmetry

A typical parallelogram grid set up in the shape of a diamond serves as the base (Figure 49). To create a hexagonal shape, the base parallelogram is split up into smaller motifs, which are then rotated around one of its obtuse vertices or symmetrically reflected in relation to one of the adjacent edges of the obtuse vertices (Figure 50). As a result, rows and columns have mirrored continuity and a regular, alternating tiling pattern (Figure 51).

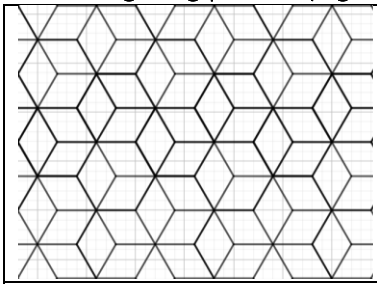
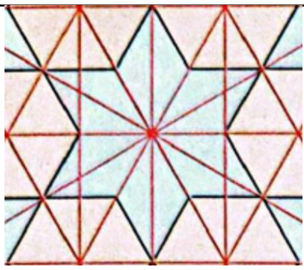
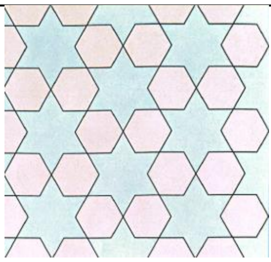
		

Figure 49 - Base	Figure 50 - Design	Figure 51 - Result

5. Conclusion

We obtained 17 ways of paving the plane with parallelograms using 4 main types of transformations (rotation, translation, symmetry and sliding).