# A strange staircase

Year 2024-2025

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## 1. Introduction

### 1.1. The problem

In the land of Matheopolis, there is a unique staircase that changes its structure every year in a specific manner.

Initially, in year 0, the staircase consists of a single step, measuring $1 \times 1 \times 1$ (length × width × height) (fig.1).

Each year, the staircase evolves according to the following rules:

- Each existing step is replaced by a scaled-down staircase made up of 9 sub-steps, resembling the one in fig.2.
- Each sub-step has $\frac{1}{3}$ of the length and width, and $\frac{1}{9}$ of the height of the original step.

Another way of viewing it is that each step divides its length and width into three equal strips, respectively its height into nine equal strips, forming 27 rectangular prisms. Out of those 27, all but 9 are deleted, which corresponds to the configuration of the staircase in fig.2.

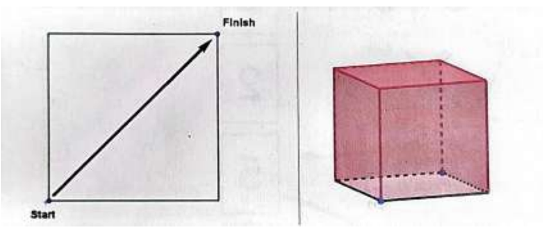


Figure 1: Year 0

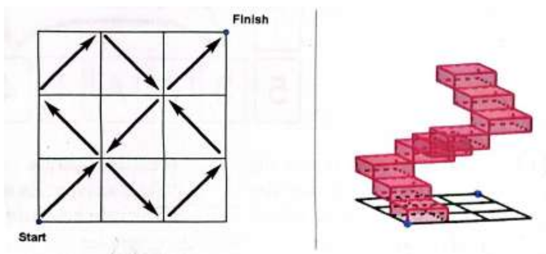


Figure 2: Year 1

Given this recursive construction, what can be determined about the structure of the staircase in year $n$? Specifically, determine:

- The total number of steps in the staircase,

- The number of turns (changes in direction) in the route taken, if someone were to walk up the staircase,
- Other emerging patterns or properties related to size, shape, or structure as $n$ increases.

### 1.2. Results

By induction, the dimensions of each step in year $n$ satisfy

$$l_n = w_n = 3^{-n}, \quad h_n = 9^{-n},$$

so that, over the fixed total height $h_0 = 1$, the number of steps is

$$s_n = \frac{1}{h_n} = 9^n.$$

Projecting the centers of the steps into the plane yields an L-system whose $n^{\text{th}}$ iterate $L_n$ is generated by either of two replacement rules $M$ (standard) and $M_2$ (reflected, yielding the Peano curve). Finally, denoting by $t(L_n)$ the number of turns in $L_n$, one finds:

- **Interpretation 1:** square turns $t_\square(L_n) = 5 \cdot 9^{n-1} - 1$, round turns $t_\circ(L_n) = 3 \cdot 9^{n-1} - 1$,
- **Interpretation 2:** square turns $t_\square(L_n) = \frac{9^n - 1}{2}$, round turns $t_\circ(L_n) = \frac{9^n - 1}{4}$,

  $\forall n \geq 1.$ $t(L_0) = 0$.

## 2. Analysis and Proof

*Notation:* Throughout this article, we will be using $n$ to refer to the number of years the staircase has, as well as how many iterations of change it undergoes since its initial state.

### 2.1. Step dimension and Number of steps

For any $n \geq 0$, let $P_n$ be the statement that

$$l_n = \tfrac{1}{3^n} \text{ and } w_n = \tfrac{1}{3^n} \text{ and } h_n = \tfrac{1}{9^n},$$

where $l_n$, $w_n$, and $h_n$ refer to the length, width and height of a step belonging to the staircase in its $n$th year.

**Base case:** The statement $P_0$ says that

$$l_1 = 1 \text{ and } w_1 = 1 \text{ and } h_1 = 1,$$

which is true.

**Inductive step:** For a given $k \in N$, $k \geq 0$, suppose that $P_k$ holds, that is,

$$l_k = \tfrac{1}{3^k} \text{ and } w_k = \tfrac{1}{3^k} \text{ and } h_k = \tfrac{1}{9^k}.$$

According to the problem statement,

$$\begin{cases} l_{k+1} = \frac{l_k}{3} = \frac{1}{3^{k+1}}, \\ w_{k+1} = \frac{w_k}{3} = \frac{1}{3^{k+1}}, \\ h_{k+1} = \frac{h_k}{9} = \frac{1}{9^{k+1}}. \end{cases}$$

proving $P_{k+1}$.

Therefore, by the principle of mathematical induction, for all $n \geq 0$, $P_n$ holds.

Since the height of the staircase remains constant, equal to $h_0 = 1$, the height of a step in year $n$ is $h_n = \frac{1}{9^n}$ and the steps do not overlap height-wise, it can be deduced that the number of steps in year $n$, $s_n$, is equal to the total height divided by the height of one step, meaning that

$$s_n = 9^n.$$

## 2.2. Evolution of the staircase

In order to answer further questions about the staircase, we need a mathematical description of its evolution. We will "flatten" the staircase—view it from above, projecting onto the x–y plane—and describe its route using an L-system–style notation (Lindenmayer 1968). Each step corresponds to a point $O$ at its center. Adjacent step-centers are connected by a straight segment, which we denote by the symbol $F$ (of length one step). To indicate a 90° change of direction, we use:

- − for a left turn,
- + for a right turn.

*Notation:* We will assign the symbols $L_n$ to the configurations of the staircase, written in the L-system.

Here are the first two.

**n** = **0**



Figure 3: $n = 0$

$$L_0 = O$$

**n** = **1** (fig.4)

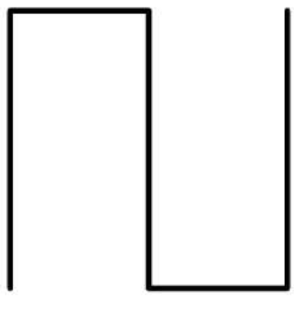

Figure 4: $n = 1$ [1]

$$L_1 = OFOFO + F + OFOFO - F - OFOFO$$

*Note:* At first, we are merely observing the pattern and attempting to transcribe it using the L-system notation. It makes no difference in the final image whether we swap $O$ and $+/-$ whenever they appear next to each other, so the base configuration $L_1$ can be written in many equivalent ways. Part of the aim of this paper is to select the most concise representation in our chosen formal language so that we can identify the recursive pattern.

We collectively decided that the staircase has two possible replacement rules:

- Pattern 1: Each step is replaced by an exact copy of the standard 9-step arrangement.
- Pattern 2: Certain steps are replaced by a flipped version of that 9-step arrangement (i.e., reflected along the diagonal).

#### 2.2.1 Pattern 1

According to this interpretation, with each iteration, we replace all the steps with the configuration of the staircase in fig.2, resembling the letter N. Since $L_0$ was one step, said configuration matches exactly $L_1$.

**n = 2** (fig.5)

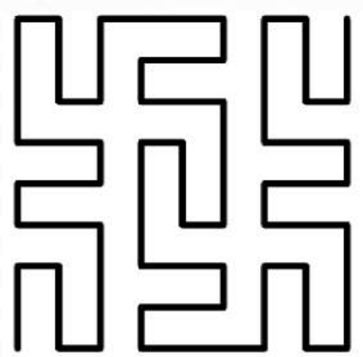

Figure 5: n=2

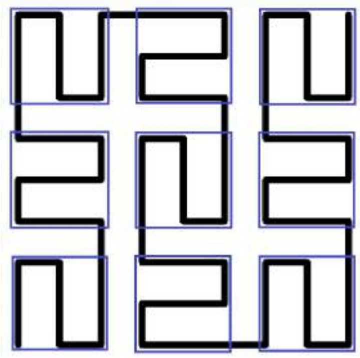

Figure 6: n=2

Each step has been replaced by the $L_1$ structure (fig.6). Analyzing the pattern after two iterations, we have the following configuration:

$$L_2 = L_1F - L_1 + FL_1 + FL_1 + FL_1F - L_1F - L_1F - L_1 + FL_1$$

*Note:* $L_1$ could be written similarly, which would help us simplify the problem. We adjust our way of writing it:

$$L_1 = OF - O + FO + FO + FOF - OF - OF - O + FO.$$

Let $A$ be the alphabet containing the symbols $F$, $O$, $+$ and $-$.

$$A = \{F, O, +, -\}$$

Let $A^*$ be the set that contains every finite expression that can be constructed using the elements in A, including the empty word $\varepsilon$. $A^*$ is the free monoid on A, the operation being that of concatenation (associative, with the identity element $\varepsilon$).

Let there be $M\colon A^* \to A^*$ such that

- $M(O) = OF - O + FO + FO + FOF - OF - OF - O + FO = L_1$,

- $M(F) = F$, $M(+) = +$, $M(-) = -$ and $M(\varepsilon) = \varepsilon$.

We will now prove that $M(w_1 w_2) = M(w_1)M(w_2), \forall w_1, w_2 \in A^*$.

Let $w_1 = a_1 a_2 ... a_m$ and $w_2 = b_1 b_2 ... b_n$, where $a_i, b_j \in A, \forall i \in \overline{1, m}, j \in \overline{1, n}$

From the properties of $M$,
$M(a_1 a_2 ... a_m b_1 b_2 ... b_n) = M(a_1)M(a_2)...M(a_m)M(b_1)M(b_2)...M(b_n) = M(a_1 a_2 ... a_m)M(b_1 b_2 ..._n)$,
since M applies to each character individually.

Since $M(w_1 w_2) = M(w_1)M(w_2)$ and $M(\varepsilon) = \varepsilon \implies M$ is an endomorphism over $A^*$.

By the definition of the problem, $L_{n+1}$ is formed by replacing each step $O$ in $L_n$ with the configuration of steps $L_1$.

Since $M$ is an endomorphism of the free monoid $A^*$ that only affects the $O$ symbols in the string, it follows that applying $M$ to $L_n$ performs exactly the same substitution: it replaces each $O$ with $L_1$. and leaves all other symbols unchanged.

Therefore, we can conclude that $M(L_n) = L_{n+1}$.

In conclusion, $\forall n \geq 0, n \in N$:

1. $M(L_n) = L_{n+1} \implies M^n(O) = L_n$
2. $M(L_n) = L_n F - L_n + F L_n + F L_n + F L_n F - L_n F - L_n F - L_n + F L_n$

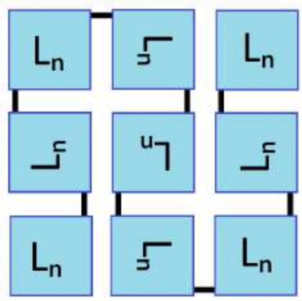

Figure 7: Visual aid

In fig.7 lies the visualisation of $L_{n+1} = L_n F - L_n + F L_n + F L_n + F L_n F - L_n F - L_n F - L_n + F L_n$.

#### 2.2.2 Pattern 2

According to this interpretation, all the steps are replaced with the configuration of the staircase in fig.2, but steps 2, 4, 6, 8 are associated to the flipped staircase, which results in the route corresponding to Peano's original space-filling curve (Peano 1890).

**n = 2**

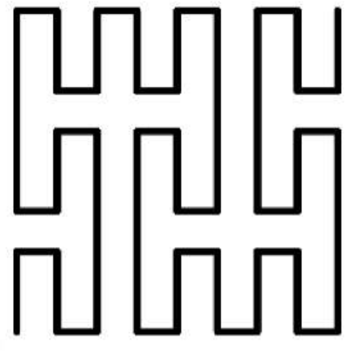

Figure 8: n=2

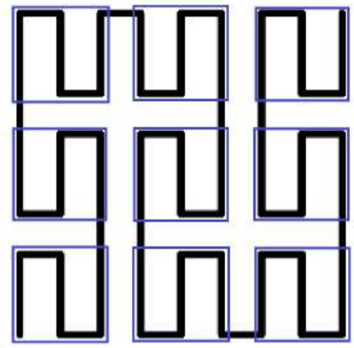

Figure 9: n=2

In order to analyse its shape, we will introduce some notation.

In this second interpretation, we will be using $X$ alongside $O$ to represent a point (step). They behave the same, but $X$ will expand in further iterations to a structure that is the reflection of $L_1$.

$$A_2 = \{F, O, X+, -\}$$

$A_2^*$ is the free monoid over $A_2$.

**Definition.** For $w \in A_2^*$, $(w)'$ is the *reflection* of $w$, corresponding to a reflection of $w$ across the line $x = y$ in the plane.

*Note:* $(+)' = -$, $(-)' = +$, $(F)' = F$, $(O)' = X$, $(X)' = O$.

For $w \in A_2^*$, $w = a_1 a_2..a_n$, $a_i \in A_2, \forall i \in \overline{1,n}$
$(w)' = (a_1 a_2..a_n)' = (a_1)'(a_2)'...(a_n)'$
Therefore, reflecting $w$ only impacts the $+/-$ and the $O/X$ signs.

It should be noted that $((w)')' = w, \forall w \in A_2^*$

Let there be $M_2 \colon A_2^* \to A_2^*$ such that:

- $M_2(O) = OFXFO+F+XFOFX-F-OFXFO$
- $M_2(X) = XFOFX-F-OFXFO+F+XFOFX$
- $M_2(F) = F$, $M_2(+) = +$, $M_2(-) = -$, $M_2(\varepsilon) = \varepsilon$.

We will now prove that $M_2(w_1 w_2) = M_2(w_1)M(w_2), \forall w_1, w_2 \in A_2^*$
Let $w_1 = a_1 a_2...a_m$ and $w_2 = b_1 b_2...b_n$, where $a_i, b_j \in A_2, \forall i \in \overline{1,m}, j \in \overline{1,n}$
From the definition of $M_2$ on each symbol in $A_2$,

$$M_2(a_1 a_2...a_m b_1 b_2...b_n) = M_2(a_1)M_2(a_2)...M_2(a_m)M_2(b_1)M_2(b_2)...M_2(b_n)$$

This expression can be grouped as:

$$M_2(a_1 a_2...a_m)M_2(b_1 b_2...b_n) = M_2(w_1)M_2(w_2)$$

Since $M_2(w_1 w_2) = M_2(w_1)M_2(w_2)$ and $M_2(\varepsilon) = \varepsilon \implies M_2$ is an endomorphism over $A_2^*$.

Define the reflection endomorphism $r \colon A_2^* \to A_2^*$ by $r(a) = a'$ on generators and extend to words. Since $M_2$ is also a monoid morphism, to prove $M_2 \circ r = r \circ M_2$ it suffices to check on generators:

$$M_2(+)' = + = (M_2(+))', \quad M_2(-)' = - = (M_2(-))', \quad M_2(F)' = F = (M_2(F))',$$

and by inspection of the two 9-letter blocks for $O$ and $X$, one sees

$$M_2(O)' = M_2(X), \quad M_2(X)' = M_2(O),$$

which matches $((M_2(O))' = M_2(O'))$ and $((M_2(X))' = M_2(X'))$. Because both functions are morphisms, this generator-level commutation lifts to all of $A_2^*$.

$M_2((w)') = (M_2(w))', \forall w \in A^*$, which is necessary, because it proves that reflection gets conserved after applying $M_2$ over $L_n$.

By the definition of the problem, $L_{n+1}$ is formed by replacing

$$\begin{cases} \text{Odd steps with } L_1 \\ \text{Even steps with } (L_1)' \end{cases}$$

Similarly to 2.2.1, because $M_2$ is an endomorphism that affects the $O$ and $X$ symbols the way it is posited in the problem, it follows that applying $M_2$ to $L_n$ leads to the same substitutions described above.

Therefore, we can conclude that $M_2(L_n) = L_{n+1}$. In conclusion, $\forall n \geq 0, n \in \mathbb{N}$:

1. $M_2(L_n) = L_{n+1} \implies M_2^n(O) = L_n$
2. $M_2(L_n) = L_nF(L_n)'FL_n + F + (L_n)'FL_nF(L_n)' - F - L_nF(L_n)'FL_n$

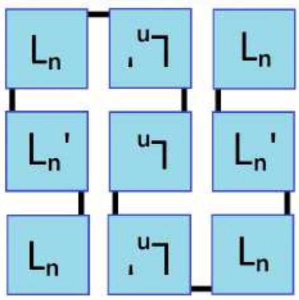


Figure 10: Visual aid

In fig.10 a visualisation of the relation $L_{n+1} = L_nF(L_n)'FL_n + F + (L_n)'FL_nF(L_n)' - F - L_nF(L_n)'FL_n$ is presented.

## 2.3. Number of turns

Let $t(L_n)$ be the number of turns the staircase takes after n iterations. We can define a turn in two ways:

1. *A square turn:* any 90° turn ($F + F$ or $F - F$). This way, an 180° turn ($F + F + F$ or $F - F - F$) is seen as a pair of two separate turns.
2. *A round turn:* an 180° turns are seen as an unique turn. All the unpaired 90° turns are also seen as one turn.

For each interpretation, we will find out both the number of round turns and the number of square turns. Regardless, $t(L_0) = 0$.

*Note:*

1. $t(F) = t(O) = t(X) = 0$

2. $t(+) = t(-) = 1 \iff$ there exists at least one $F$ on both sides of $+/-$, between it and the closest $+/-$ from the configuration, so that they do not cancel $(+O-)$.

*Note:* The following inductions can only start at $n = 1$ because in $L_1$ some + and − cancel each other and resulting in less turns than $+/-$ signs. By counting the turns in $L_1$ in each of the four situations, cases like $+O-$ get omitted and only the "real" turns are counted, for which, indeed, $t(+) = t(-) = 1$.

#### 2.3.1 Interpretation 1

**A. Square turns**

$t(L_1) = 4$
$L_{n+1} = L_nF - L_n + FL_n + FL_n + FL_nF - L_nF - L_nF - L_n + FL_n$
$t(w_1 w_2) = t(w_1) + t(w_2) \implies t(L_{n+1}) = 9 \times t(L_n) + 8, \forall n \geq 1$, since $L_{n+1}$ contains 9 $L_n$ and 8 $+/-$ signs.

Let $P_n^1$, $\forall n \in \mathbb{N}, n \geq 1$ be the statement that

$$t(L_n) = 5 \times 9^{n-1} - 1$$

**Base case:**
$P_1^1$ is true, as $t(L_1) = 4$.

**Inductive step:** For a given $k \in \mathbb{N}, k \geq 1$, suppose that $P_k^1$ holds.
$t(L_{k+1}) = 9 \times t(L_k) + 8$
$\implies t(L_{k+1}) = 9 \times (5 \times 9^{k-1} - 1) + 8$
$\implies t(L_{k+1}) = 5 \times 9^k - 1$, which proves $P_{k+1}^1$.
Therefore $t(L_n) = 5 \times 9^{n-1} - 1$, $\forall n \in \mathbb{N}, n \geq 1$

**B. Round turns**

$t(L_1) = 2$
*Note:* We observe that all 8 turns between the 9 iterations are unpaired, therefore they can be counted as sole turns.

Let $P_n^2$, $\forall n \in \mathbb{N}, n \geq 1$ be the statement that

$$t(L_n) = 3 \times 9^{n-1} - 1$$

**Base case:**
$P_1^2$ is true, as $t(L_1) = 2$.

**Inductive step:** For a given $k \in \mathbb{N}, k \geq 1$, suppose that $P_k^2$ holds.
$t(L_{k+1}) = 9 \times t(L_k) + 8$
$\implies t(L_{k+1}) = 9 \times (3 \times 9^{k-1} - 1) + 8$
$\implies t(L_{k+1}) = 3 \times 9^k - 1$, which proves $P_{k+1}^2$.
Therefore $t(L_n) = 3 \times 9^{n-1} - 1$, $\forall n \in \mathbb{N}, n \geq 1$

#### 2.3.2 Interpretation 2

**A. Square turns**

$t(L_1) = 4$
$L_{n+1} = L_nF(L_n)'FL_n + F + (L_n)'FL_nF(L_n)' - F - L_nF(L_n)'FL_n$
$t(+) = t(-) = 1 \implies t(w) = t((w)'), \forall w \in A_2^*$.
$\implies t(L_{n+1}) = 9 \times t(L_n) + 4$

Let $P_n^3$, $\forall n \in \mathbb{N}, n \geq 1$ be the statement that

$t(L_n) = \frac{1}{2} \times (9^n - 1)$

**Base case:**

$P_1^3$ is true, as $t(L_1) = 4$.

**Inductive step:** For a given $k \in \mathbb{N}, k \geq 1$, suppose that $P_k^3$ holds.

$t(L_{k+1}) = 9 \times t(L_k) + 4$

$\implies t(L_{k+1}) = 9 \times (\frac{1}{2} \times (9^k - 1)) + 4$

$\implies t(L_{k+1}) = \frac{1}{2} \times (9^{k+1} - 1)$, which proves $P_{k+1}^3$.

Therefore $t(L_n) = \frac{1}{2} \times (9^n - 1), \forall n \in \mathbb{N}, n \geq 1$.

**B. Round turns**

In the curve of Peano, all 180° turns are paired into 90° turns, therefore there are twice as many square turns as round turns.

Therefore $t(L_n) = \frac{1}{4} \times (9^n - 1), \forall n \in \mathbb{N}, n \geq 1$.

Figure 11: $n = 3$ (1)

Figure 12: $n = 3$ (2)

## 3. Auxiliary materials

In order to familiarise ourselves with the research topic at hand, particularly with the first interpretation of the staircase (that, as far as we know, doesn't have a proper name, but closely resembles a space-filling curve, known as the curve of Wunderlich), we have created the following materials.

### 3.1. 3D LEGO models

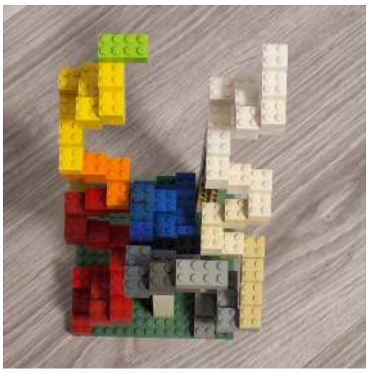

Figure 13: LEGO

Figure 14: LEGO

Figure 15: LEGO

Here is the most complete of our 3D models, with coloured bricks arranged in such a way as to suggest the continuity of the staircase.

### 3.2. Python code in Blender

We have created a virtual model in an app named Blender, which allows for the creation of objects through coding, as well as through manual addition.

The code employs a procedural approach to generate a 3D staircase structure using a context-free Lindenmayer system (L-system) implemented in Blender's Python API (bpy).

The system is initialized with an axiom, the string "$O$" and a single production rule:

$$O \rightarrow OF - O + FO + FO + FOF - OF - OF - O + FO$$

This rule determines the rewriting behavior applied iteratively to the axiom. The rule is applied to the original string $n$ times, where $n$ is input manually in the code, yielding a deterministic *string* of drawing and turning commands. The symbols used include:

- $F$: Move forward and create geometry (a rectangular prism)
- $+$: Rotate right by 90° about the Z-axis,
- $-$: Rotate left by 90° about the Z-axis,
- $O$: A placeholder that triggers further rewriting but produces no geometry.

The geometric form of each step is defined as a rectangular prism with the dimensions of each step being the ones determined in 2.1.

The script constructs geometry procedurally in Blender by interpreting the generated L-system string. At each "F" command, a rectangular prism is instantiated at the current position. Movement is constrained to a grid aligned with the local coordinate system, and vertical elevation is incrementally increased to simulate a stair-step effect.

Rotational commands (+ and −) alter the drawing direction by ±90° using Euler angles in the Z-axis.

A linear color gradient is applied to the generated steps using Blender's node-based material system. The red and blue components of the base color are interpolated across the total number of steps, resulting in a visual progression from red to blue along the staircase.

Link:Code

Figure 16: colourless, n=0[2]

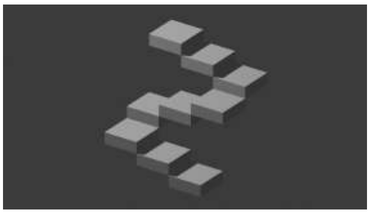

Figure 17: colourless, n=1

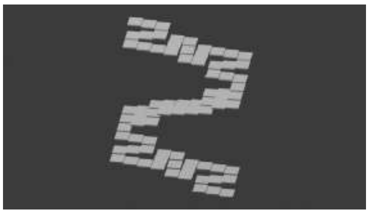

Figure 18: colourless, n=2

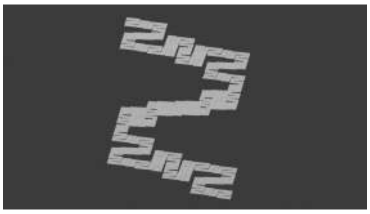

Figure 19: colourless, n=3

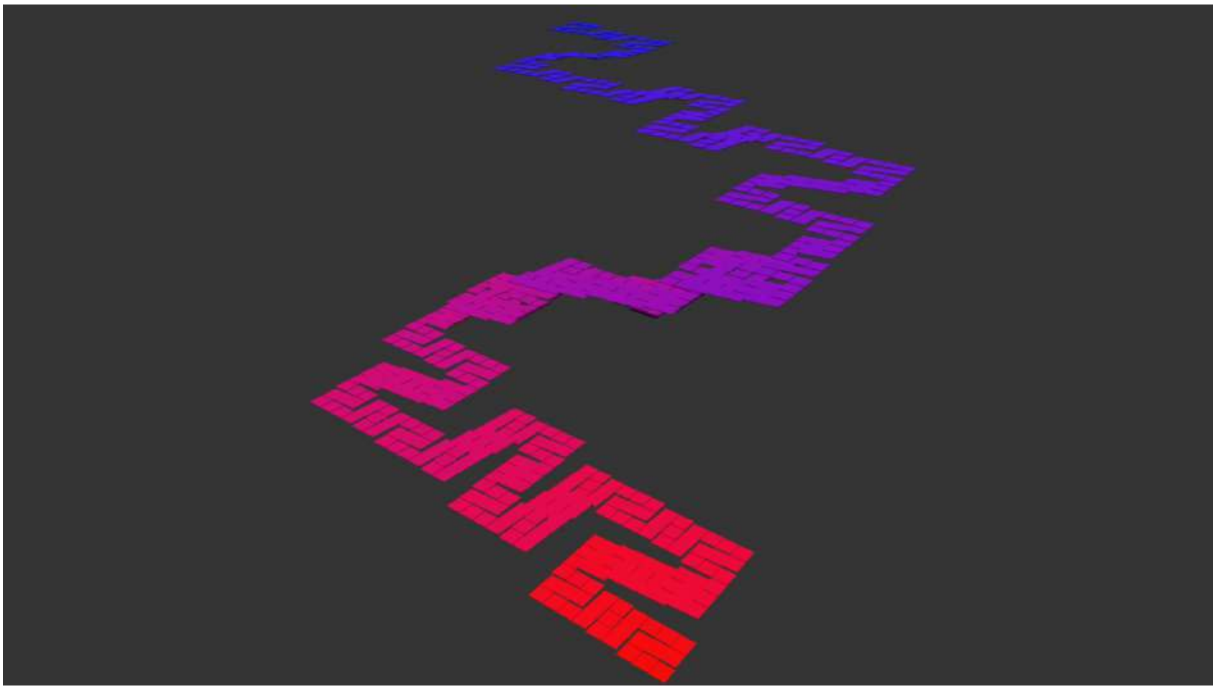

Figure 20: Colorful, n=3

## 4. Conclusion

This exploration revealed how a staircase, defined by shrinking step dimensions and recursive spatial rules, evolves into a structure of surprising geometric complexity. Each iteration multiplies its turns and steps, creating an exponential tapestry from a minimal seed. By formalizing its construction through induction and L-systems, we uncovered not only precise growth formulas but also a path to deeper structures: the emergence of the Peano curve under Pattern 2 points to the staircase's potential as a discrete model for space-filling phenomena.

These findings suggest broader implications in fractal geometry and symbolic dynamics, while raising further questions—can such recursive systems be extended to higher dimensions, curved spaces, or probabilistic rules?

In the end, we have not simply catalogued a staircase, but charted how simplicity, looped back on itself, becomes complexity.

## References

[1] Online Tools, *L-system Generator*, `https://onlinetools.com/math/l-system-generator`.

[2] Blender Foundation, *Blender 3D Software*, `https://www.blender.org/`.