# Balancing non-coplanar points

Année 2024 – 2025

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## 1 Introduction:

### 1.1 The problem:

Given n non-coplanar points ($n \geq 4$), we will say we have balanced them if we find a plane equidistant from each of the n points.

(a) Let A, B, C and D be four non-coplanar points. How many planes do these points balance?

(b) Let A, B, C, D and E be five non-coplanar points. How many planes do these points balance?

### 1.2 Results:

In this article we proved that the solution provided in the book from which this problem was selected was wrong due to oversimplification and a mistake in the generalization of point (a). Our solution took us through may areas of mathematics, such as space geometry, vectors and graphs, all combined into a creative and interesting proof.

## 2 Basic ideas:

1. It is obvious that if 2 points are situated on the same side of a plane and the distances between the points and the plane are equal, then the line between the two points will be parallel to the plane.
2. Similarly, if 3 or more points are situated on the same side of a plane, they will form a plane parallel to the one which balances them.

3. All of the points on the same side of the plane which balances them will be coplanar.
4. At least one point is located on the opposite side of the plane in order to satisfy the non-coplanarity condition in the hypothesis.
5. All of the points are situated on 2 planes which are parallel to the equidistant plane situated between them.
6. The equidistant plane splits the n given points into 2 groups, of sizes a and b, where:

   $$a+b=n \quad a\leq b \quad a,b \in \mathbb{N}$$

   We can refer to this kind of equidistant plane as a|b.

# 3 Solving question (a) :

$n=4 \rightarrow (a, b) \in \{(1, 3), (2, 2)\}$

Therefore, only the two following situations are possible:

A. Three points are on one side of the plane, while the fourth is on the other. (a, b) = (1, 3)
B. Two points are on each side of the plane. (a, b) = (2, 2)

## 3.1 Case 1|3:

Assume that points A, B and C are on the same side of plane Π, which is equidistant from the 4 points, while point D is on the opposite side of the plane. (fig.2)

$\rightarrow \Pi \parallel (ABC)$

$d(D, \Pi) = d(A, \Pi) \Leftrightarrow d(D, \Pi) = d((ABC), \Pi) = \frac{1}{2} d(D, (ABC))$

For the distance from point D to plane Π to be equal to the distance between plane Π and points A, B and C, it means that plane Π must pass through the middle of the overall distance from point D to plane ABC.

Let DP be a segment such that: $DP \perp (ABC)$, with $P \in (ABC)$. If M is the midpoint of DP, then $M \in \Pi$.

In a similar manner, we can determine the plane equidistant from the 4 given points, with point C on one side and the other 3 points on the opposite side. We repeat this process with points B and A, and determine that there are 4 planes satisfying the given conditions.

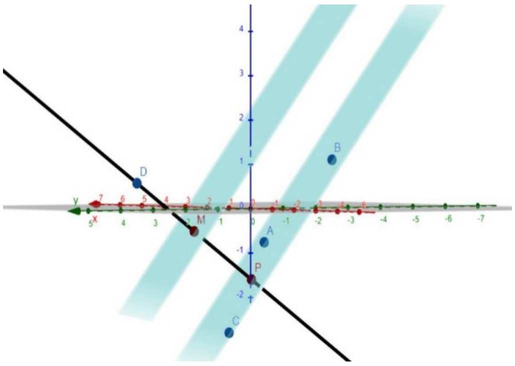


*Figure 1*

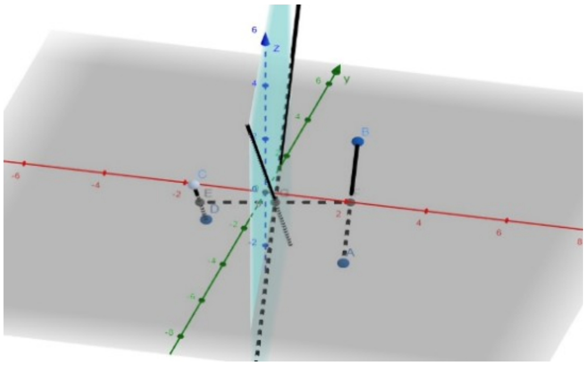


*Figure 2*

### 3.2 Case 2|2:

Assume that points A and B are on the same side of the plane Π, equidistant from the 4 points, while points C and D are on the other side. (fig.1)

→AB ∥ Π and CD ∥ Π

It is demonstrated in the same way that plane Π must be parallel to line CD. Since A, B, C and D are not included in the same plane, it follows that lines AB and CD must be non-parallel and non-concurrent.

Let E and F be the midpoints of AB and CD and G the midpoint of EF. → d(G, AB) = d(G, CD) → G ∈ Π

We will draw l1 ∥ AB with G ∈ l1 and l2 ∥ CD with G ∈ l2

→Plane Π is uniquely defined by lines l1, l2 concurrent in G.

Similarly, there exists a single plane that satisfies the conditions, so that on the same side as point A lies point C (or point D), while the other two points (B and D or B and C) are situated on the opposite side. Thus, in total, there are only three planes equidistant from the 4 given points, such that on one side of the plane are two of the points, and on the other side the remaining two.

So, the total number of planes that can balance any given 4 points in space is 7 (4+3).

If we consider a triangular pyramid (tetrahedron) with vertices A, B, C and D, then 4 of the 7 planes will be parallel to the faces of the pyramid and will pass through the midpoints of their respective heights (as resulted from the first case), while the other 3 planes will be parallel to a pair of opposite edges and equally distant from them (second case).

## 4 Solving question (b):

n=5 → (a, b) ∈{(1, 4), (2, 3)}

We have 2 possible cases: 1|4 and 2|3.

Unlike was the case with 4 points, for 5 points there are some arrangements in space for which there are no equidistant planes.

## 4.1 Case 1|4

For a point E and a plane defined by 4 points A, B, C, D there exists at most one plane Π that is equidistant from all 5 points. (fig.3)

→(ABCD) ∥ Π

There exists a unique plane Π parallel to (ABCD) that is equidistant to E and (ABCD).

Since E is not in (ABCD), the plane Π equidistant from all the 5 points must be positioned such that it simultaneously satisfies the distance constraint for E and the four points A, B, C, D

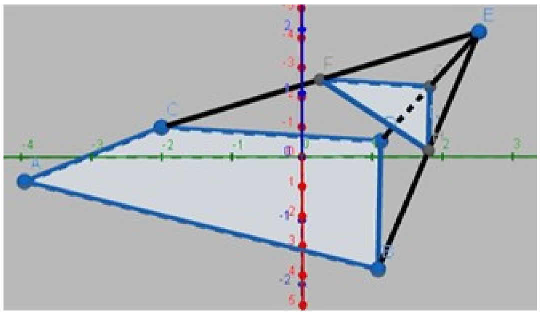


*Figure 3*

We can determine the equidistant plane Π by taking the midpoints of 3 segments connecting E with A, B, C, or D.

**Finding more 1|4 planes for the same set of points in space**

Suppose we have 2 1|4 planes:

(ABCD) | E→ 1 plane

(ABCE) | D→ 1 plane

A, B, C, D coplanar and A, B, C, E coplanar →

(ABC) =(ABCD) ∩ (ABCE) is either a plane or a line.

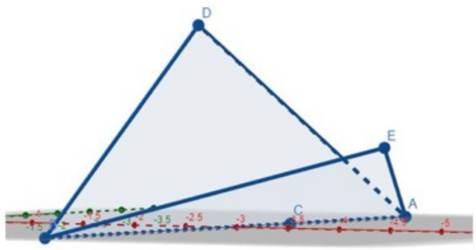


*Figure 4*

→ if A, B, C are not collinear ⇔ A, B, C, D, E coplanar (false)

→ if A, B, C are collinear ⇔ ∃ a set of points A, B, C, D, E which are balanced by 2 1|4 planes simultaneously. (fig.4)

In general, a set of 5 points in space can accommodate only a single 1|4 equidistant plane, the only exception being when 3 of the points are colinear (2 1|4 planes).

## 4.2 Case 2|3

Let's name the 3 coplanar points A, B and C, and the other two D and E. (fig 5)

→(ABC) ∥ Π and DE ∥ Π for the case to be solvable

The solution to this case would be the plane determined by the middles of any three lines drawn between any of the 3 coplanar points and the D or/and E points (any line that crosses the plane Π). For example, the middles of AD, BD, CE.

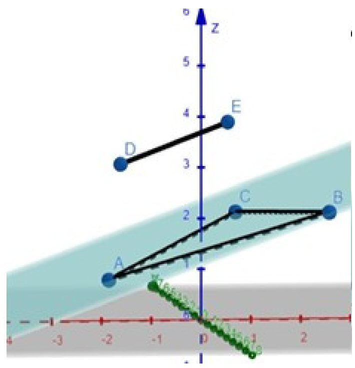


*Figure 5*

# 5 How to count the planes:

## 5.1 The book's solution

The number of possible equidistant planes for each case is equal to $\binom{5}{1}$ or $\binom{5}{4}$ for case 1|4 and $\binom{5}{2}$ or $\binom{5}{3}$ for case 2|3.

For case 1|4, the number of equidistant planes is $5! / (1! \times 4!) = 5$

For case 2|3 the number of equidistant planes is $5! / (2! \times 3!) = 10$

In total there should be 5+10=15 planes.

## 5.2 Our observations

Earlier we have proven that we cannot have more than 2 1|4 planes. Why did the calculations say there should be 5? What was overlooked?

These 15 planes cannot exist at the same time for the same set of points in space due to the properties imposed by the existence of each equidistant plane on the positioning of the 5 points.

It is obvious that we cannot find any arrangement of points in space in order to have all of the theoretically possible equidistant planes. By adding more and more properties (equidistant planes), the positioning of the points in space has to satisfy more and more criteria. At some point, the overlapping of properties will lead to a contradiction with the initial hypothesis. If we go past this point, adding even more properties, we can easily say that it leads to a contradiction because it contains a set of conflicting properties.

## 5.3 Creating a new notation system to facilitate our search for the correct answer

As previously mentioned, we can find a plane that is equidistant to 5 points situated 2 on one side and 3 on the other if and only if the line defined by the two points is parallel to the plane in which the other 3 are situated. We will refer to the following property - the line between 2 points (ex: A and B) are parallel to the plane which contains the other 3 (C, D, E) - as the notation of the line between the two points (AB). We will use "+" to signify the stacking of two or more such properties. For example: AB + AC means: AB || (CDE) and AC || (BDE). Each property is equivalent to the existence of an equidistant plane between the line and the plane created by the 3 points.

$\exists$ 2|3 plane(Π) (ex: AB | CDE) ⇔ AB || Π and (CDE) || Π ⇔ AB || (CDE) ⇔ property AB

## 5.4 Representing properties using graphs

We will use undirected graphs to represent a set of properties. Property AB is equivalent to a link between nodes A and B.

As the graph is undirected, property AB is the same as property BA.

Because the graph is an unordered set of points, set AB+BC is the same as AB+AC (2 edges incident in one node). For the previous example, the graph will look like this: 
If a set of properties (graph) is conflicting, any set(graph) containing it will also be conflicting.

In order to find the real number of possible coexisting planes we must find the maximum number of non-conflicting properties. This is equivalent to finding a graph with the maximum number of edges, for which we can find at least one possible arrangement of points in space. We will create a tree structure. Each node will represent a set of properties (a graph). The root of the tree will be the most general case: a set of points with property AB. If a node of the tree has non-conflicting properties, we will branch out from it by evaluating every case in which we add another property to the set. When we reach a node with a set of conflicting points, we will stop there and backtrack to the previous node. Black nodes are unchecked, green are possible cases and the red ones are impossible due to conflicting properties. The tree on the left is with graphs and the one on the right is with property examples.

**The tree structure**


*Tree structure 1*

We begin with the simplest case: property AB ( AB||(CDE) ). We previously proved that there is a set of points that satisfy the necessary criteria for this to happen.

**AB + AC (2 equidistant planes)**

→AB ∥ (CDE) and AC ∥ (BDE)

Let ABMC be a parallelogram such that AB ∥ MC and AC ∥ MB. ($\vec{AM} = \vec{AB} + \vec{AC}$) (fig.5)

AB ∥ MC and AB ∥ (CDE) → MC ∥ (CDE) → M ∈ (CDE)

$AC \parallel MB$ and $AC \parallel (BDE) \rightarrow MB \parallel (BDE) \rightarrow M \in (BDE)$

$\rightarrow M \in (CDE) \cap (BDE) \rightarrow M \in DE$

Therefore, this case exists only when M is located on the line DE

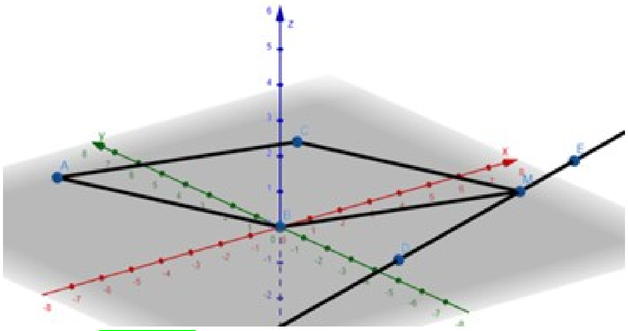
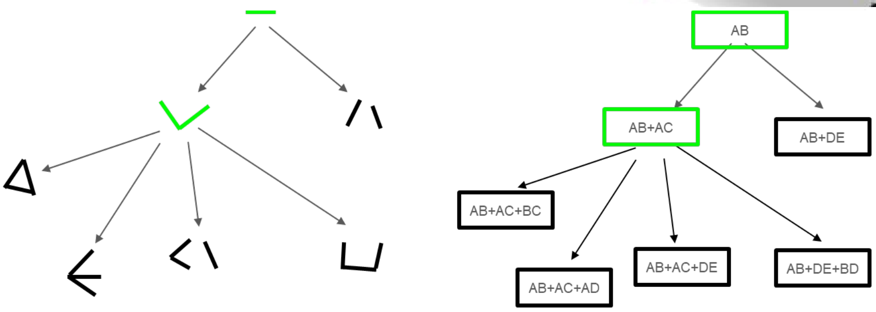


*Figure 6*




*Tree structure 2*

**AB+AC+DE (3 equidistant planes)**

From $AB + AC$, we know that $\exists\, M \in DE$, where ABMC is a parallelogram

Property DE → $DE \parallel (ABC)$

$M \in DE \rightarrow E \in DM \rightarrow DM \parallel (ABC)$

ABMC parallelogram $\rightarrow M \in (ABC)$

$\rightarrow DM \subset (ABC) \rightarrow DE \subset (ABC)$

$\rightarrow$ Contradiction with initial hypothesis (the 5 points are non-coplanar)

**AB+AC+BC (3 equidistant planes)**

From $AB + AC$, we know that $\exists\, M \in DE$, where ABMC is a parallelogram

Similarly, from $AB + BC$, we know that $\exists\, N \in DE$, where BANC is a parallelogram

$M \in DE$ and $N \in DE \rightarrow MN \Leftrightarrow DE$

ABMC parallelogram $\rightarrow M \in (ABC)$

BANC parallelogram $\rightarrow N \in (ABC) \rightarrow MN \subset (ABC) \rightarrow DE \subset (ABC)$

$\rightarrow$ Contradiction with initial hypothesis (the 5 points are non-coplanar)

**AB+DE+BD (3 equidistant planes)**

From BD+ DE, we know that ∃ M ∈ AC, where DBME is a parallelogram

Similarly, from AB + BD, we know that ∃ N ∈ CE, where BAND is a parallelogram

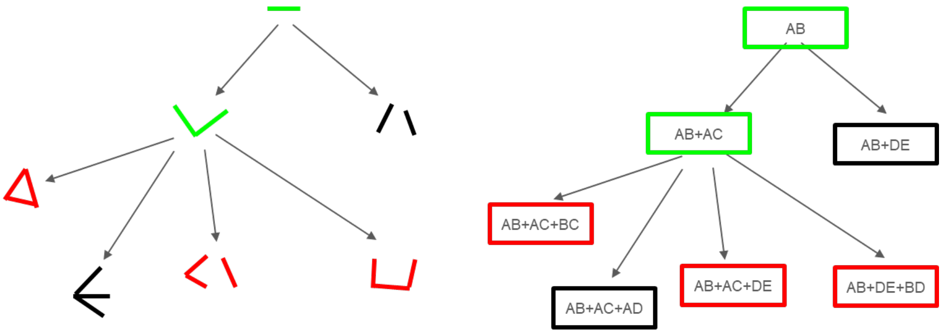
M ∈ AC → C ∈ AM

N ∈ CE → C ∈ EN → E = AM ∩ EN

DBME parallelogram → DB || ME and DB = ME

BAND is a parallelogram → BD || AN and BD = AN

→ ME = NA and ME || NA → MENA parallelogram → AM || EN → C does not exist (Contradiction)




*Tree structure 3*

**AB+AC+AD (3 equidistant planes)**

(fig.6 and fig.7)

AB || (CDE) and AC || (BDE) and AD || (BCE)

From AB + AC, we know that ∃ M ∈ DE, where ACMB is a parallelogram

Similarly, from AC + AD, we know that ∃ N ∈ BE, where ACND is a parallelogram

ACMB parallelogram → AC || MB and AC = MB

ACND parallelogram → AC || ND and AC = ND

→ MB || ND and MB = ND → MBDN parallelogram

M ∈ DE → E ∈ DM

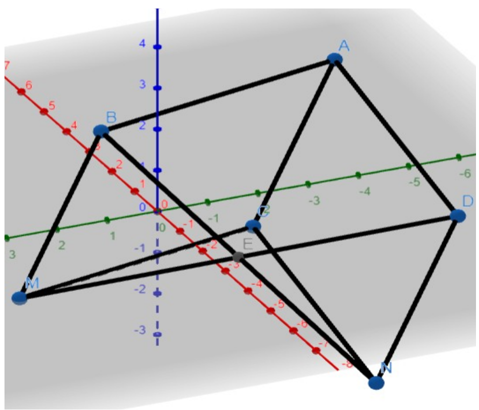



*Figure 6*

$N \in BE \rightarrow E \in BN \rightarrow E = DM \cap BNM \in DE \rightarrow E \in DM$

$E = DM \cap BN$ But DM, BN are diagonals of parallelogram MBDN → E is the middle of both diagonals DM and BN → DE = EM

$\rightarrow \overrightarrow{AE} = ½*(\overrightarrow{AD}+\overrightarrow{AM}) = ½*(\overrightarrow{AD}+\overrightarrow{AB}+\overrightarrow{AC})$

For a set of 5 points to have this set of three properties, they must satisfy the following criteria:

$\overrightarrow{AE} = ½\ (\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD})$

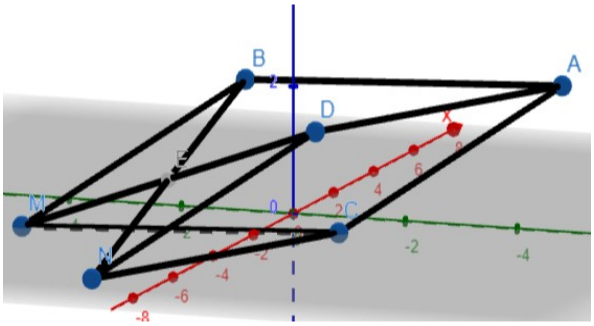
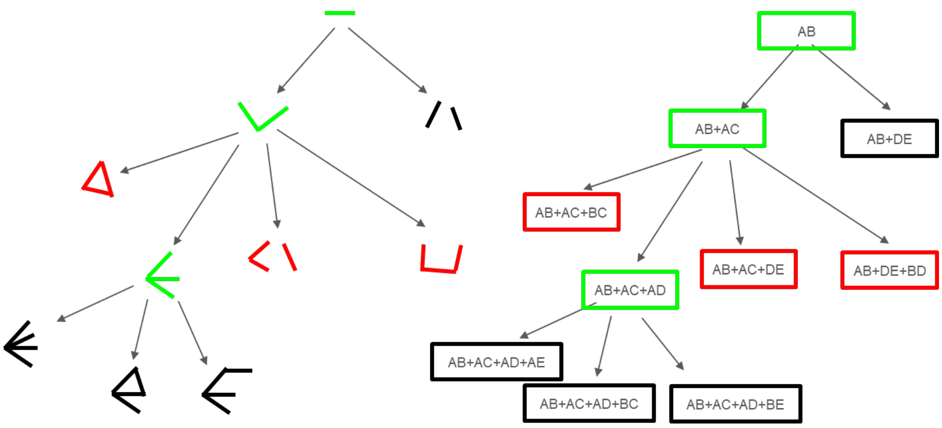



Figure 7




*Tree structure 4*

## AB+AC+AD+AE (4 equidistant planes)

Equivalent condition/property from AB+AC+AD → $\vec{AE} = ½(\vec{AB}+\vec{AC}+\vec{AD})$

property AE: AE ∥ (BCD) → ∃ XY ∈ (BCD) such as XY ∥ $\vec{AE}$ ⇔ $\vec{AE}$ = k * XY

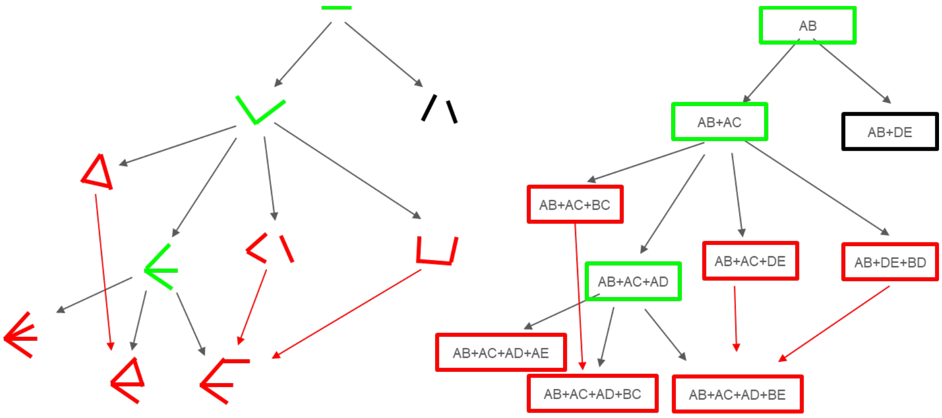
XY ∈ (BCD) ⇔ $\vec{XY} = a\vec{BC} + b\vec{CD} = a(-\vec{AB}+\vec{AC}) + b(-\vec{AC}+\vec{AD}) = -a\vec{AB}+(a-b)\vec{AC}+bAD$

$\vec{AE} = ½(\vec{AB}+\vec{AC}+\vec{AD}) = k\vec{XY}=k[-a\vec{AB}+(a-b)\vec{AC}+b\vec{AD}]$

½=-ak

½=bk → ak-bk=½-(-½)=1

½=ak-bk → Contradiction




*Tree structure 5*

## AB+DE (2 equidistant planes)

AB || (CDE)

let M be so that CM ∥ AB → M ∈ (CDE)

but if CM ∥ AB → M ∈ (ABC) → CM = (ABC) ∩ (CDE)

DE ∥ (ABC)

Let N be a point so that CN ∥ DE → N ∈ (ABC)

But if CN || DE → N ∈ (CDE) → CN = (ABC) ∩ (CDE) –> M-C-N colinear →

AB||CM||NC||DE → AB||DE

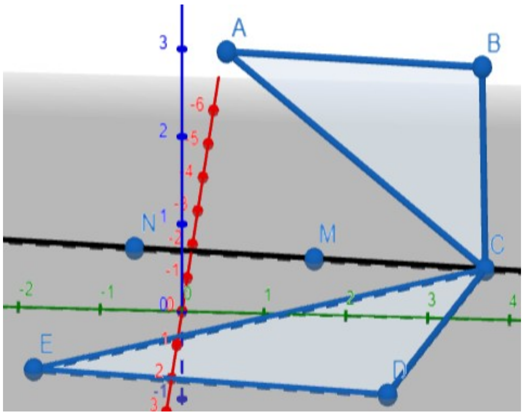
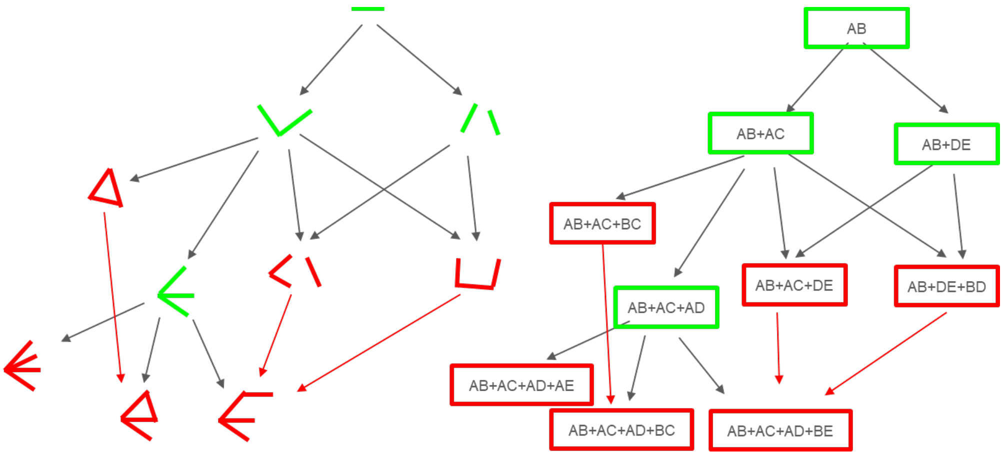



*Figure 8*

The points that have this set of properties must satisfy the condition that we can draw 2 parallel lines between 2 specific sets of 2 points each. (AB || DE)




*Tree structure 6 – the final tree structure*

### Bonus 1|4 plane for AB + DE

If AB ∥ DE → A, B, D, E coplanar → there is also a 1|4 equidistant plane, with A, B, D, E on one side and C on the other. Thus, there are actually 3 equidistant planes generated by this property set (fig.9)

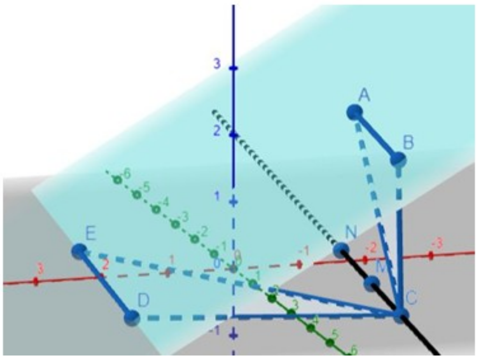


*Figure 9*

### Bonus planes for AB+AC

Continuing to search for possible extra 1|4 planes, we return to AB+AC, from which we determined that M ∈ DE, where ABMC is a parallelogram. (fig.10)

What happens if M = D? Then ABCD will become a parallelogram (and the 4th point, D, will be in the same plane), allowing for a 1|4 equidistant plane, with E on the other side.

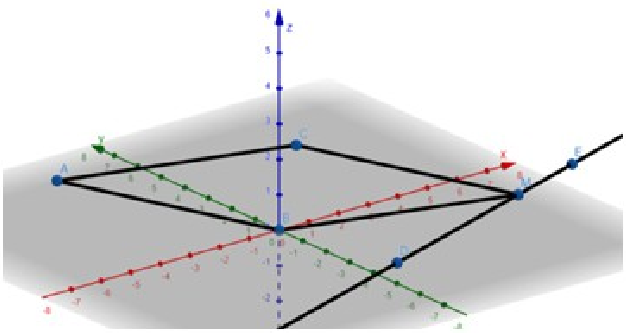


*Figure 10*

Because BD is now || with AC, BD ∥ (ACE) → +1 2|3 plane

Because CD is now || with AB, CD || (ABE) → +1 2|3 plane.

In total there will be 5 planes: 4 2|3 planes and 1 1|4 plane. For this, the points must be arranged so that they form a pyramid with a parallelogram as its base. (fig.11)

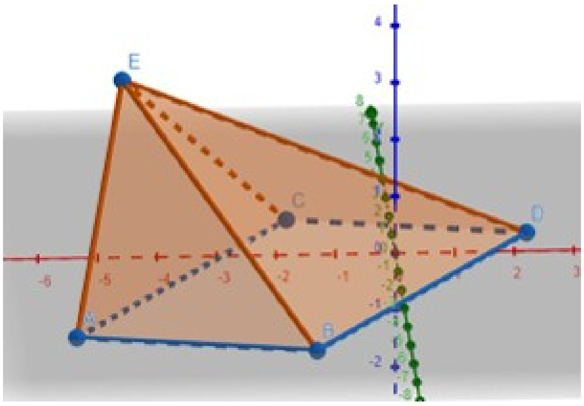



*Figure 11*

## Notes d'édition

(1) Il faut être prudent ici car les points M et D pourraient être confondus. Dans ce cas, on peut aussi conclure à une contradiction.

(2) Ici encore, les points M et N pourraient être confondus, auquel cas il n'y a pas de sens à parler de la droite (MN).

(3) Ici, aussi il faut vérifier que A est différent de M et E est différent de N.

(4) Il faut justifier que M est différent de D et N est différent de B.

(5) $\overrightarrow{AE}=\overrightarrow{AD}+\frac{1}{2}\overrightarrow{DM}=\overrightarrow{AD}+\frac{1}{2}\left(\overrightarrow{DB}+\overrightarrow{BM}\right)=\overrightarrow{AD}+\frac{1}{2}\left(\overrightarrow{DB}+\overrightarrow{AC}\right)=\overrightarrow{AD}+\frac{1}{2}\left(\overrightarrow{DA}+\overrightarrow{AB}+\overrightarrow{AC}\right)=\frac{1}{2}\left(\overrightarrow{AD}+\overrightarrow{AB}+\overrightarrow{AC}\right)$