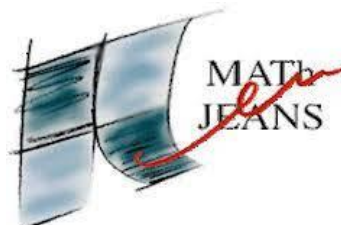


This article is written by students. It may include omissions and imperfections, which are reported as far as possible by our reviewers in the edition notes



## FORMING PUZZLES PATTERNS USING GEOMETRICAL SHAPES

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*School: Colegiul Național Iași, România*

**The main task:**

**This research project aims to find and demonstrate different ways we can cut and use squares in order to make different geometrical shapes.**

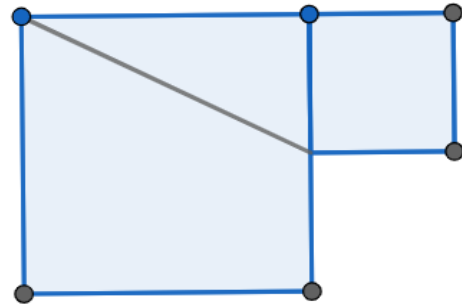
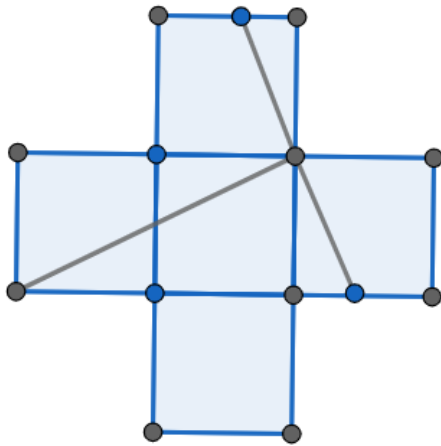
### Challenges:

- How to cut the cross of Saint Andrew (formed by five squares), so that a single square can be assembled from its parts?
- If one square is formed by five pieces, how can we choose their shapes in order to be able to obtain two identical squares?
- How do you make cuts in a square in order to obtain a regular octagon?
- How can we break down three squares to get bigger squares?

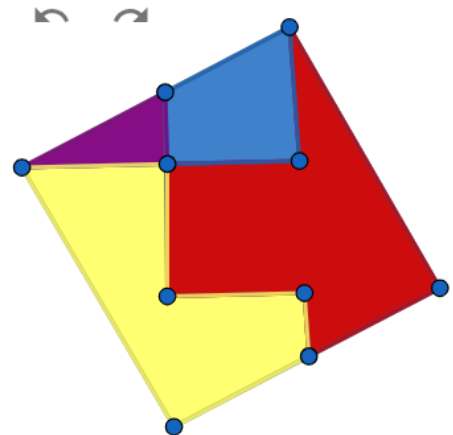
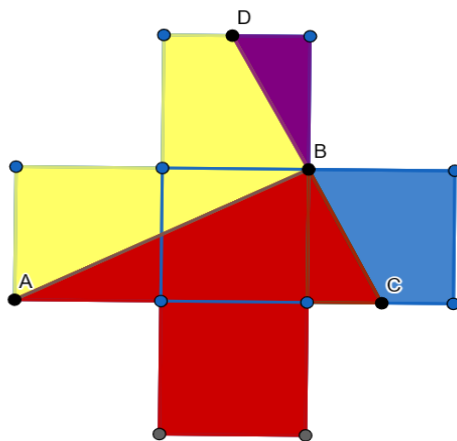
**1. First Challenge: How to cut the cross of Saint Andrew (formed by five squares), so that a single square can be assembled from its parts?**

Proof: [1] By moving one of the squares we can observe that its surface is the assembly of two squares built on the sides of a certain right angle. You can draw the segments AB and DC

( $AB \perp DC$ ), these segments will divide the cross in 4 parts that can form a square.



[2]



### Further research:

First, we create a grid on an infinite surface by putting the crosses one next to the other. The cross shape tiles the plane in a regular way.

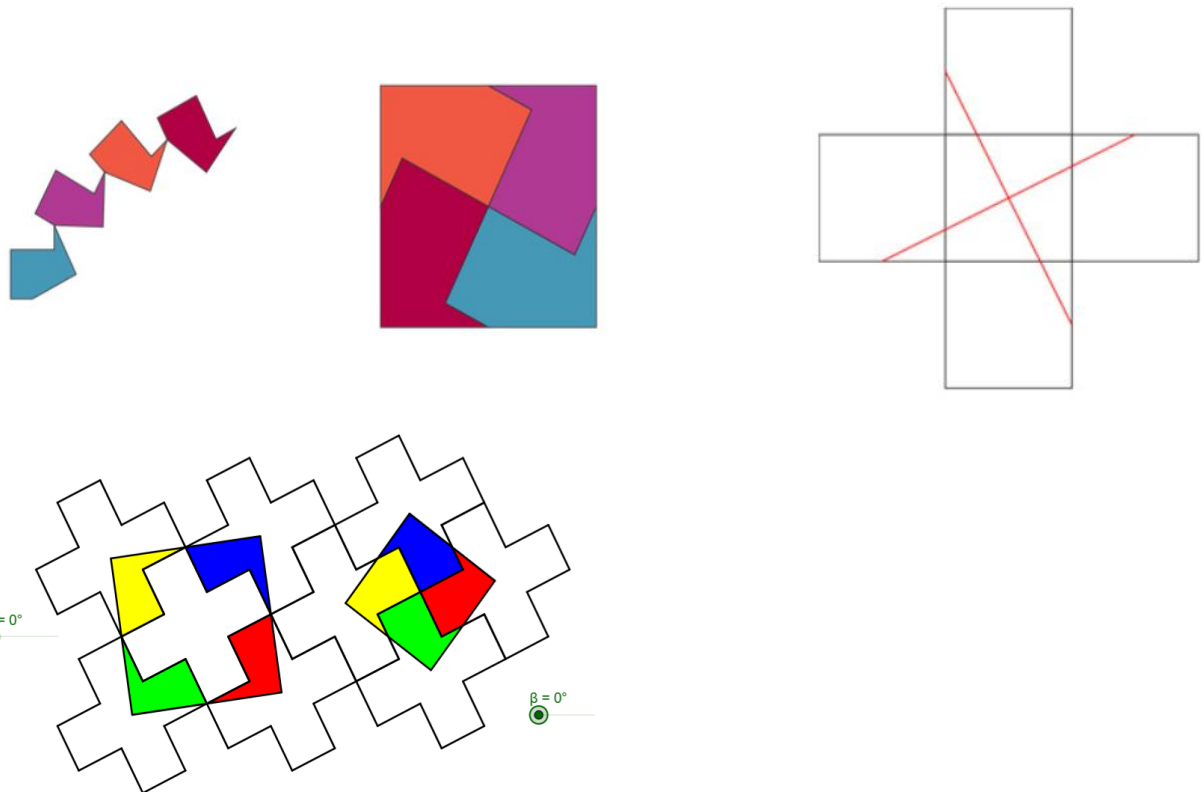
If you pick any point inside a cross and mark the same point in all the crosses of the tiling, you get a grid of points that can be connected to form a grid of squares.

Those grid-lines split up the crosses into pieces which also form the squares in that grid. There will only be four pieces if the grid intersection is located anywhere in the cross's central square, also you will get five or even six pieces, if you put the grid intersection too far into one of the

arms of the cross. [3]

### Further research:

(overlapping two grids on an infinite surface with pieces that have the same area). The only way to make the four pieces have the same shape is to put the grid intersection in the center of the cross. The cuts then go through the midpoint of the sides of the arms of the cross, not through the vertices.

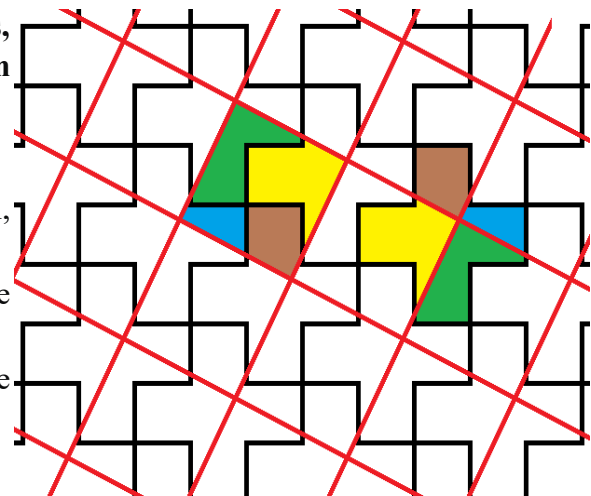


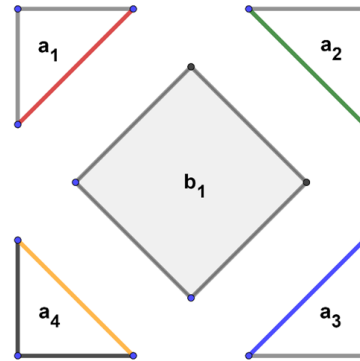
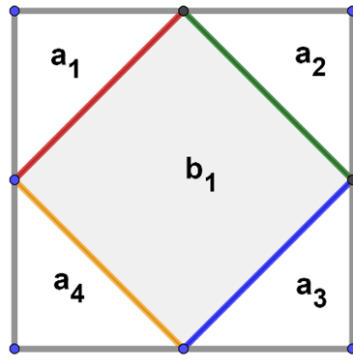
**2. Second challenge: If one square is formed by five pieces, how can we choose their shapes in order to be able to obtain two identical squares?**

#### First proof:

We will strategically name the side of the main square with  $2 \cdot 1$ , in order to have easier notations throughout the proof

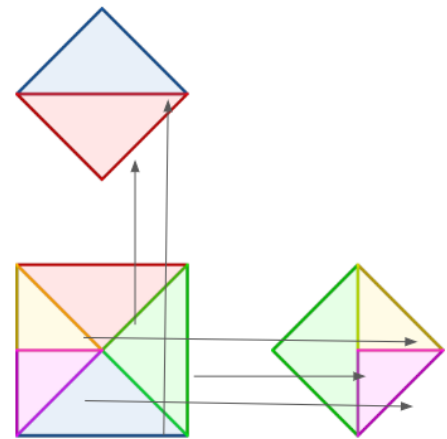
- The first step is to find the midpoints of the square, therefore obtaining half of a segment with the length 1
- The second step is to trace a square by connecting the midpoints
- The third step consists in cutting out the four triangles



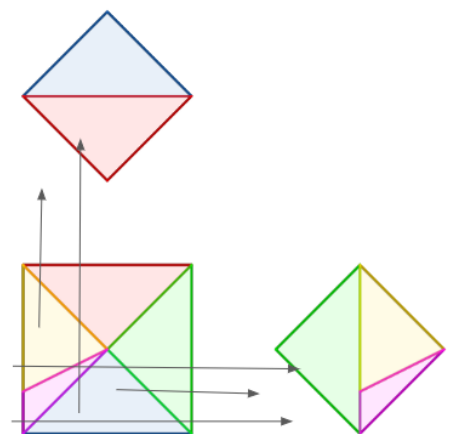


**Second proof:**

- We can draw the diagonals of a square to create 4 identical right isosceles triangles. These can be joined in 2 pairs of 2 along their hypotenuse to form 2 identical squares.
- Side of original square=1  
Area of original square= $l^2$   
Area of each triangle= $l^2/4$   
Area of smaller squares= $l^2/2$   
Side of smaller square= $l/\sqrt{2}$
- Pythagorean theorem:  
 $l^2=(1/\sqrt{2})^2+(1/\sqrt{2})^2$   
 $l^2=l^2/2+l^2/2$   
 $1=1/2+1/2$



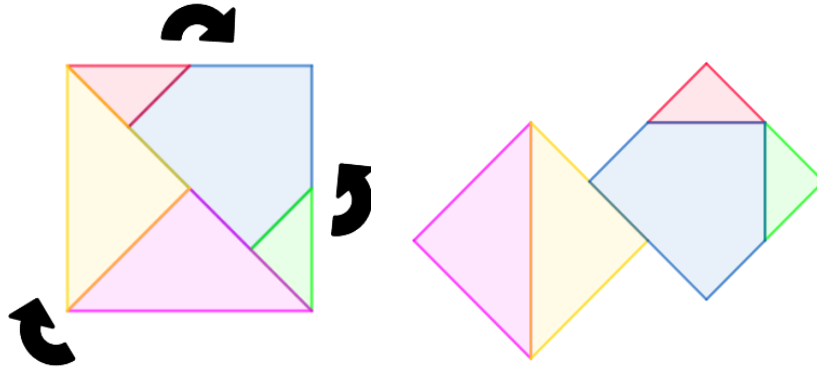
We still have to use a 5th shape, so we can cut one of the triangles into 2 shapes. We can choose to make the cut wherever we want. As there are infinitely many points along the sides of the triangle between which to make the cut, there are infinitely many ways to cut, thus resulting in infinitely many solutions.



**Third proof:**

- By drawing the diagonal, we split the big square into 2 identical triangles with equal areas.
- We can cut one such triangle in 2 ways in order to obtain squares.
  - Cut it along the median towards the hypotenuse, obtaining 2 right isosceles

- triangles. We can join these two along the hypotenuse, forming a square.
- Draw lines from the middles of the legs perpendicular to the hypotenuse, creating two triangles and a pentagon. We can rotate these triangles around a point and join them with the pentagon in order to obtain another square.
- Because the two squares are created from shapes that formed two identical triangles, their area will be the same, so the squares will also be identical.



### 3. Third challenge: How do you make cuts in a square in order to obtain a regular octagon?

Main Proof:

Knowing that a regular octagon's angle values are equal to:

$$\frac{n-2}{n} 180^\circ = \frac{6 \cdot 180^\circ}{8} = 135^\circ,$$

where  $n$  represents the number of sides, respectively 8, it is clear that a regular octagon can be obtained simply by cutting isosceles triangles off the edges of the square.

However, this is just a preliminary conclusion, because in order for the octagon obtained to also be regular, the sides of the triangles must have the right equal length. Otherwise, the result would be just an octagon with  $135^\circ$  angles, like **EFGHIJKL** in the figure on the right.

In order to determine the length of the sides of the isosceles triangle that need to be cut off, we apply the Pythagorean Theorem in the isosceles triangle  $\triangle BEF$ :

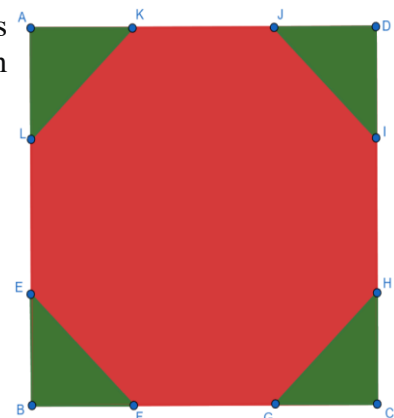
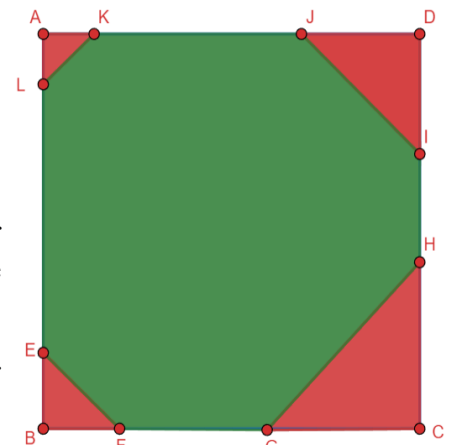
$$EF^2 = BE^2 + BF^2 \Rightarrow EF = BF\sqrt{2}$$

But since the octagon is regular,  $FG = EF = BF\sqrt{2}$

$$\text{Also, } FG = GH \Rightarrow GC = BF$$

$$\text{Since } BF + FG + GC = BC = l,$$

$$BF(2 + \sqrt{2}) = l \Rightarrow BF = l \frac{2 - \sqrt{2}}{2}$$



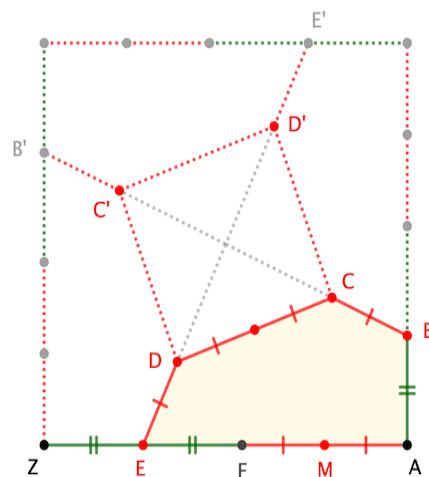
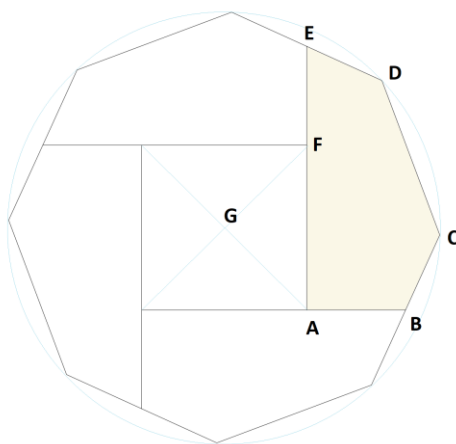
This way, the length of the sides of the isosceles triangles was determined precisely, depending just on the side length of the square from which they are cut off.

Alternative solution: For the square on the left, locate E along the side of ZA (and corresponding points B, E', B' along the other sides), such that:

$$\frac{ZE}{ZA} = \frac{AE - AF}{AE + AE - AF} \cong 0,2725$$

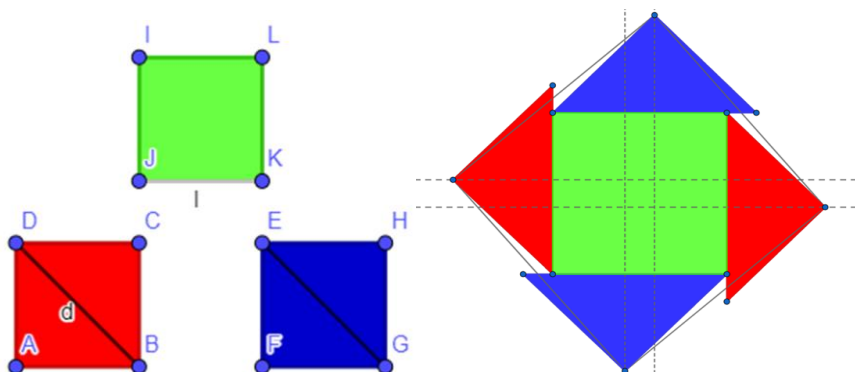
[4] Check the measurements by verifying that lines BB' and EE' pass through the center of the square. Reflect Z in E to locate F.

Locate M, the midpoint of the segment FA and afterwards locate C and D along BB' and EE' at distance AM, likewise for C' and D'.

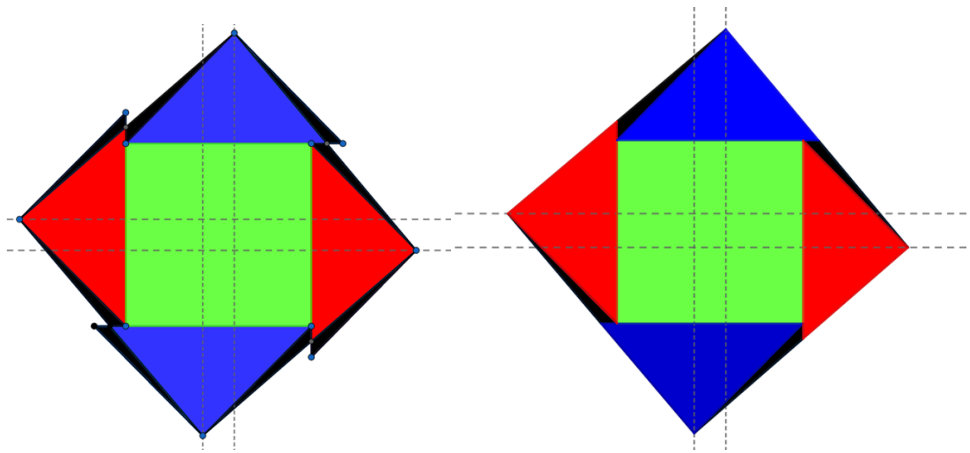


#### 4. Fourth challenge: How can you break down three squares to get bigger squares?

First proof: Two of the three squares are dissected through their diagonals into triangles which then will be placed alongside the remaining square. There will now be 4 empty spaces formed.



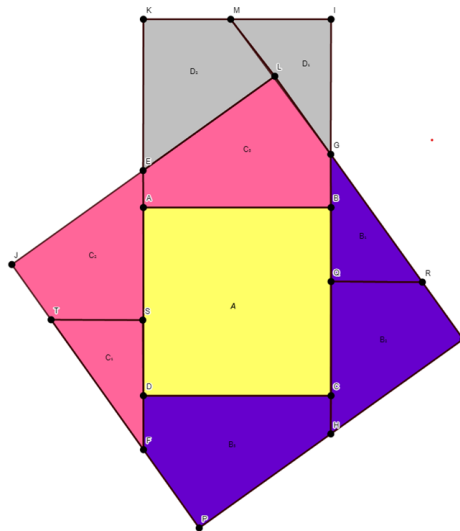
As the black triangles below are all equal, they can be put inside in order to form the big square. Now, the square is filled



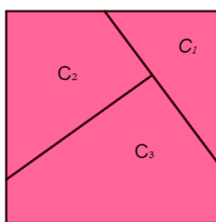
[5]

Second proof: This proof starts from using the first proof's place of the centre square. The lines of the centre square are elongated in order to reach the borders of the big square. Then, perpendiculars are traced, as to follow the small square's measurements. The grey forms are put to exemplify the placing of the forms

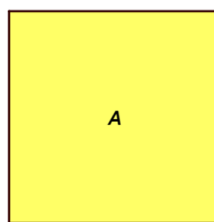
$QR=TS$	$QR+RN=L$
$RN=IT$	$TS+IT=L$
$EA=CH$	$EA+JE=L$
$JE=NH$	$CH+NH=L$
$GB=DF$	$GB+SF=L$
$SF=GQ$	$DF+QG=L$



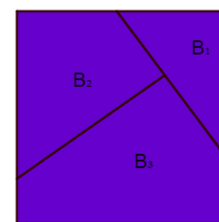
Second proof:



$$\begin{aligned} C_1 &= B_1 \\ C_2 &= B_2 \\ C_3 &= B_3 \end{aligned}$$



$$\begin{aligned} C_1 + C_2 + C_3 &= A \\ B_1 + B_2 + B_3 &= A \end{aligned}$$



[6]

Third proof:

Each starting square has the area  $l^2$  so at the end will be obtained a square with the surface area equal to  $3l^2$ , having The side length .

We align the three squares horizontally and draw a point at the distance  $l\sqrt{3}$  on each side (I and J, respectively).

We also draw a perpendicular line from J on BG, which intersects IH in K. Since  $\angle JHK = \angle HIG$ ,

$$JK = JH \tan \angle HIG = JH \frac{1}{\sqrt{3}} = l(3 - \sqrt{3}) \frac{1}{\sqrt{3}} = l(\sqrt{3} - 1)$$

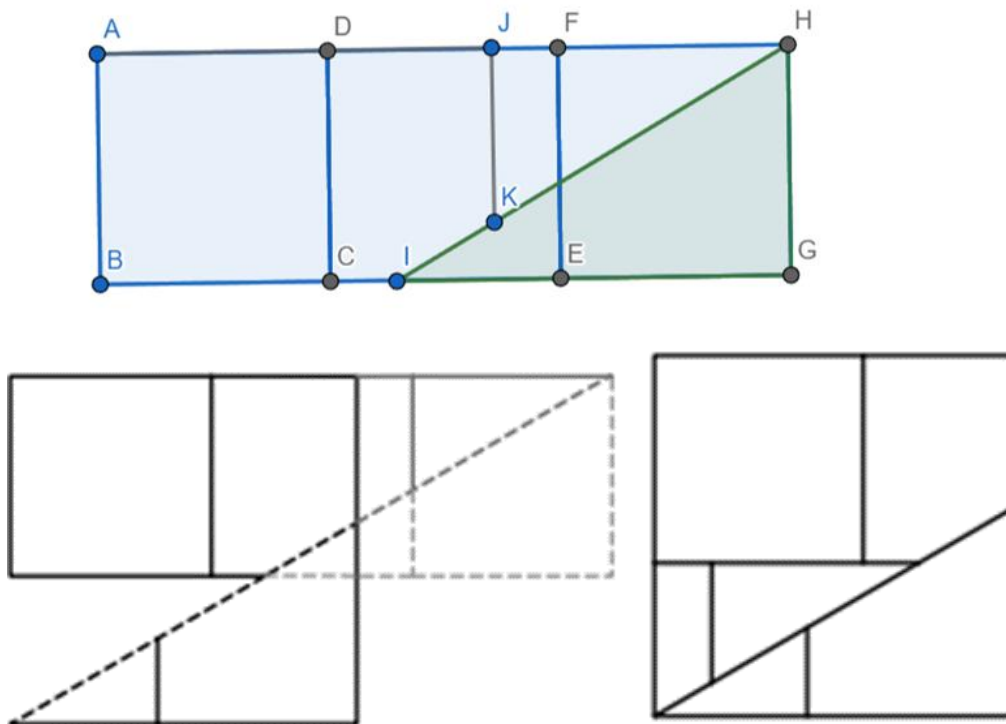
Sliding the triangle HIG along the HI line until J, K, H and G are collinear, we obtain the second side of the final square, JG, which has the correct length:

$$JG = JK + HG = l(\sqrt{3} - 1) + l = l\sqrt{3}$$

The area of the missing triangle is

$$\frac{l(\sqrt{3} - 1)l(3 - \sqrt{3})}{2} = \frac{l^2(4\sqrt{3} - 6)}{2}$$

Same as the one of  $\triangle JKH$ , so if we move it there, we obtain the square.



## EDITION NOTES

[1] Throughout the paper, the word “*proof*” should be replaced with something like “*solution to the problem.*” A proof must be based on rigorous mathematical arguments, which, as noted in the following comments, are not always provided in the work.

[2] It is not clear why the two figures above have been included. The two figures below are sufficient to illustrate the construction, although a bit more mathematical explanation would have been desirable. For instance, it should have been proved that the points C and D in the figure on the right are the midpoints of the segments to which they belong. Moreover, the hypotenuse of the violet triangle and the oblique side of the blue trapezoid are implicitly assumed to be aligned. This, too, would have required a proof. The same remark applies to the sides of the yellow and red polygons.

[3] Some illustrative examples would be necessary.

[4] Where does this formula come from? Moreover, it would have been necessary to explain why the parts of the following figure on the right actually form the octagon on the left.

[5] Although this is fairly evident, an explanation should have been provided as to why the black triangles are all congruent.

[6] It would also be necessary to include (at least in sketch form) a proof showing that the pink and violet figures can be rearranged to form two squares.