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Gears

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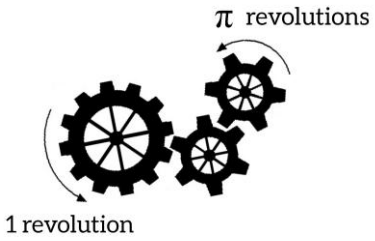
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Problem statement

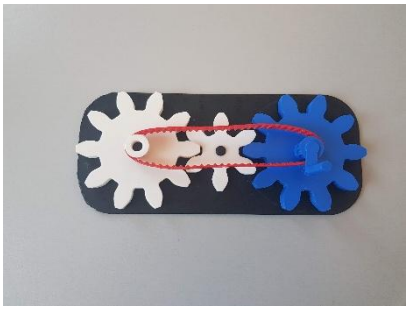
We have gears with between 15 and 30 teeth. We want to create an assembly with a ratio as close as possible to π. In other words, when the first gear makes one revolution, the last gear must make 3.141592… revolutions (figure 1).



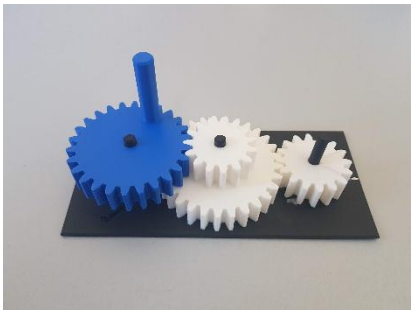
1. Problem statement

General findings

The problem can be approached from two very different perspectives: multiple gears can be mounted on a single shaft, or not.



2. The non-overlapping model



3. The overlapping model

Case A - non-overlapping gears

1.1. General observations

- ☼ each gear is supported by a single axle
- ☼ each one interacts directly with two other gears

1.2. The number of gears does not affect the number of rotations, but only the direction of rotation

1.2.1. Mathematical demonstration

In order to simplify the notation, we abbreviate the number of teeth t_i of the first gears by a, b, c, …

- ☼ **For two gears:**

  For one rotation of the first gear (a), the second gear (b) rotates $\frac{a}{b}$ times.

  Example: a=4, b=8 ⇒ b rotates $\frac{a}{b} = \frac{4}{8} = \frac{1}{2}$ times for one rotation of gear a.

- ☼ **For three gears:**

  The second gear rotates by $\frac{a}{b}$ times the number of rotations of the first gear.

  The third gear rotates by $\frac{b}{c}$ times the number of rotations of the second gear.

  Example: a=4, b=5, c=6 ⇒ gear a rotates once, gear *b* rotates $\frac{a}{b}$ times, and gear *c* rotates $\frac{b}{c}$ of $\frac{a}{b}$, that is $\frac{a}{c}$.

- ☼ **For n gears:**

  The second gear rotates by $\frac{a}{b}$ times the number of rotations of the first gear.

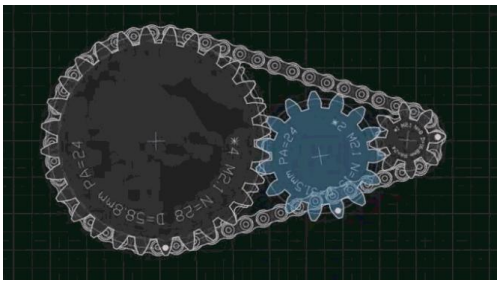
  The third gear rotates by $\frac{b}{c}$ times the number of rotations of the second gear.

  The fourth gear rotates by $\frac{c}{d}$ times the number of rotations of the third gear, and so on.

  Example: a, b, c, …, n ⇒ gear *a* rotates once, gear *b* rotates $\frac{a}{b}$ times, gear *c* rotates $\frac{b}{c} \cdot \frac{a}{b}$ times, and gear *n* rotates $\frac{t_{n-1}}{t_n} \cdot \frac{t_{n-2}}{t_{n-1}} \cdot \ldots \cdot \frac{c}{d} \cdot \frac{b}{c} \cdot \frac{a}{b} = \frac{t_1}{t_n}$ times.

1.2.2. Visual demonstration

In this case (figure 2 and 3), whether the last gear is directly connected to the first one by a chain or whether there are intermediate gears between them, the transmission functions correctly, meaning that the last gear rotates the same number of times.



4. Non-overlapping gears

5. QR code – visual demonstration

1.3. Finding the solution

Pairs of numbers (a, b), with a > b, $\frac{a}{b} = \pi$, where a and b are natural numbers in the interval [15,30]. (1)

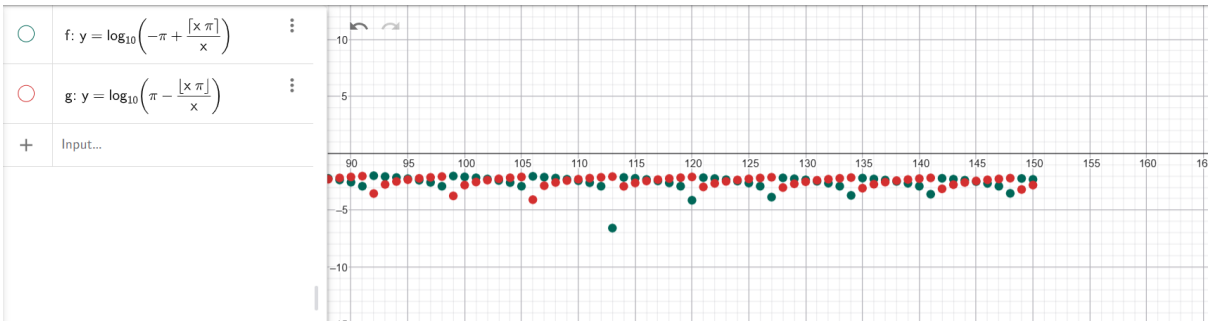
1.3.1. Visual function approach

Based on the definition: $error = \pi - \frac{\text{numerator}}{\text{denominator}}$ we define the function

$$f(x) = \min(log_{10}|\pi - \frac{\lceil x\pi \rceil}{x}|, log_{10}(\pi - \frac{\lfloor x\pi \rfloor}{x}))$$

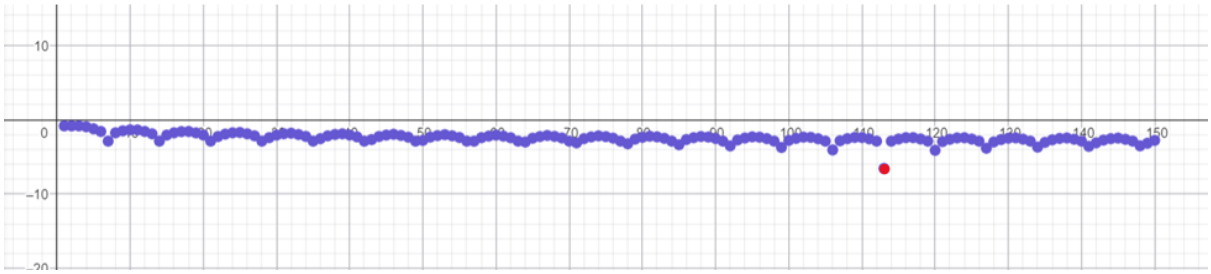
Explanation:

☼ For each denominator *x*, there is a unique numerator such that the error is minimal. This is one of the two integers between which the product of x and π lies ($\lfloor x\pi \rfloor \leq \pi x \leq \lceil x\pi \rceil$), leading to a positive ($\pi - \frac{\lfloor x\pi \rfloor}{x}$) and a negative error ($\pi - \frac{\lceil x\pi \rceil}{x}$), for which we use the absolute value in order to obtain a correct representation. In addition, we used a logarithmic representation in order to show the precision of the error at the decimal level. From these two possible errors, we choose the minimal one. Therefore, if the denominator is known, the numerator is also determined. Knowing the denominator (and implicitly the numerator), one can determine the minimum error corresponding to each denominator.



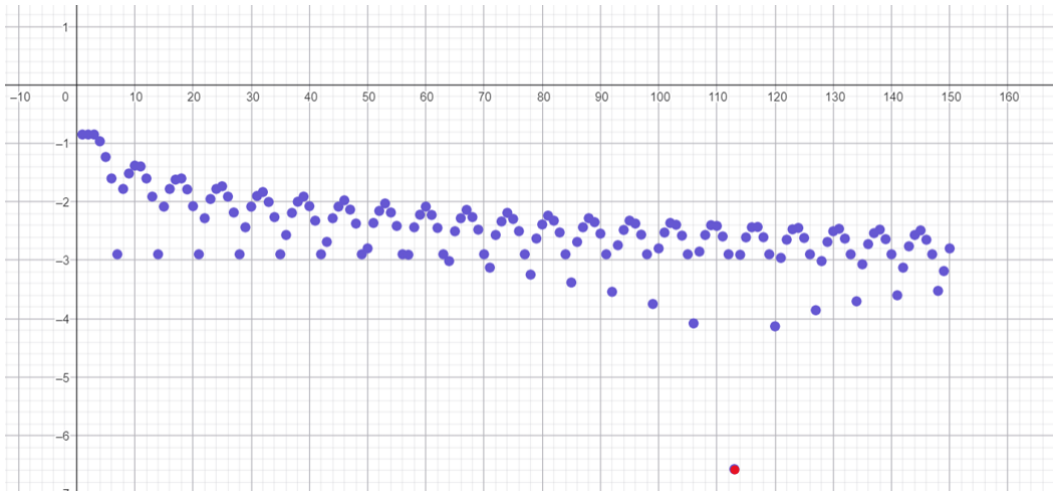
6. The two functions that determine the error, by approximating the numerator first upward, then downward

☼ Represented in the XOY coordinate system, the function is as follows (X – denominator, Y – logarithmic error):



7. The minimum of the two functions, that is the exact error for each natural x

☼ The function exhibits a generally decreasing trend, with its curve flattening out (as the numerator increases, the error decreases, but only up to a point beyond which the decrease becomes negligible), while on small intervals it shows local increases. Thus, the region of interest is reduced to the first few hundred values of *x*.



8. The optimal point

☼ If we map the values on the x-axis to values symmetric with those on the y-axis (taking into account that these are logarithmically scaled), we can easily observe that the error decreases relatively proportionally as the number of teeth increases up to x=113, where the error becomes very small compared to previous values. Beyond this point, the decrease in error is negligible relative to the increase in the denominator (number of teeth). Thus, the red point appears to be an optimal one, where the number of teeth is relatively small (113 — applicable in real-life practice) and the error has a relatively small value (10^{-7}).

1.3.2. Formal approach

In this section, no lower bound on the number of teeth of gears is imposed.
The problem becomes equivalent to finding the best ratio *p/q* to approximate π, where *p* and *q* are integers, $1 \leq p, q \leq u$. The ordered set of all fractions with $0 \leq p \leq q \leq u$ is called the *Farey sequence* F_u of order *u*. It can be generated recursively from $\frac{0}{1}, \frac{1}{1}$ by applying the *mediant* operation $\oplus$, defined as $\frac{a}{b} \oplus \frac{c}{d} = (a+c)/(b+d)$. First, we list some properties of Farey sequences.

Proposition 1. If $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a}{b} \oplus \frac{c}{d} < \frac{c}{d}$.
Proof. $\frac{a}{b} < \frac{c}{d}$ <=> ad < bc. $\frac{a}{b} \oplus \frac{c}{d} - \frac{a}{b} = \frac{(a+c)}{(b+d)} - \frac{a}{b} = \frac{b(a+c) - a(b+d)}{b(b+d)} = \frac{bc-ad}{b(b+d)} > 0$. Similarly, $\frac{c}{d} - \frac{a+c}{b+d} = \frac{c(b+d) - d(a+c)}{d(b+d)} = \frac{bc-ad}{d(b+d)} > 0$ q. e. d.

Observation: This means the Farey sequence always generates simple fractions. (2)

Proposition 2. Two irreducible fractions $\frac{a}{b} < \frac{c}{d}$ are Farey neighbours if and only if *bc - ad = 1*.
Furthermore, let $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$ be consecutive neighbours in a Farey sequence. Then *p/q* is the mediant of *a/b* and *c/d.* Conversely, given Farey neighbours $\frac{a}{b} < \frac{c}{d}$, their mediant is a Farey neigbour to both.

Proof of the second assertion. Thanks to the first assertion (which is admitted), *bp - aq = cq - dp = 1*, thus *p(b+d) = q(a + c)*, hence $\frac{p}{q} = \frac{a+c}{b+d}$. Conversely, suppose *bc – ad = 1*. Then *b(a+c) – a(b+d) = bc – ad = 1* and similarly, *(b+d)c – (a+c)d = bc – ad = 1*, so the mediant *(a+c)/b+d)* is a Farey neighbour to *a/b* and *c/d*. q. e. d.

This implies the Farey fraction always generates the best possible approximation. (3)

Further explanations and details are given in Ainsworth et. al., *The Farey Sequence*.

The algorithm, when applied to $x = \frac{1}{\pi}$, gives the following sequence of fractions (we record only the lower bounds of the intervals): 0/1, 1/4, 2/7, 3/10, 4/13, 5/16, 6/19, 7/22, 113/355. Then, we take the result and simply invert it to get an approximation for π. (4)
We can stop here, since we observed above that decreasing the error further would require gears with more than 10000 teeth.

Introducing a lower bound (5)
Let us add a constraint to the problem: a lower bound *l* on numerators and denominators.
Hence for *l* = 1, the problem is solved. We can simply generate the Farey sequence and at each point choose the fractions between which $\frac{1}{\pi}$, can be found. Then, we take the result and simply invert it to get an approximation for π. Let us now treat the case *l* > 1. Letting *l* vary, we observe that, assuming the Farey approximation for *l* = 1 is $\frac{n}{m}$, then until *l* = n + 1 we have the same approximation. Afterwards, we simply search for equivalent fractions (for example, $\frac{44}{14}$ instead of $\frac{22}{7}$). When we run out of options, we apply the mediant property using previous approximations (for small enough numbers, this will generally be $\frac{22}{7}$). At some point we may have to resort to equivalent fractions of previous approximations.

Eventually, we reach a point where the best approximation is simply $\frac{u}{1}$. The process simply amounts to generating Farey fractions as long as they meet the bounds, before switching to $\frac{u}{1}$.

1.4. Intermediate conclusion for non-overlapping gears

In conclusion, the ideal mechanism would consist of two gears, one with 113 teeth and the other with 355 teeth. (6)

Case B - overlapping gears

2.1. First approach

2.1.1. Terms

- t_i = number of teeth (of a specific i gear)
- *nt* = total number of teeth
- *ng* = total number of gears
- *na* = total number of axles
- r_i = number of rotations of a specific i axle
- *e* = error

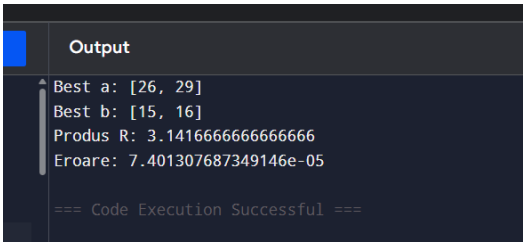
*Because two gears on the same shaft have identical rotation count, the first and last axles will each have only one gear ($ng = 2 \cdot na - 2$).

2.1.2. The linking formula between the number of rotations of the first and last axles

- *for two axles:* $\frac{r_2}{r_1} = -\frac{t_1}{t_2}$.
- *for na=k axles*: $\frac{r_k}{r_1} = (-1)^{k-1} \frac{t_1}{t_2} \cdot \frac{t_3}{t_4} \cdot \ldots \cdot \frac{t_{2k-3}}{t_{2k-2}}$.

Let Pn_k(numerator product) = $\prod_{i=2}^{k} t_{2i-3}$ and Pd_k(denominator product) = $\prod_{i=2}^{k} t_{2i-2}$.
With the new notation, the formula from above can be compacted to $\frac{r_k}{r_1} = (-1)^{k-1} \frac{Pn_k}{Pd_k}$.

*By applying the formula, we can observe that for a system with 4 gears and 3 axles (e.g., 15, 16, 26, 29), the error is of the order of 10^{-5}, which is very small (figure 9). Therefore, for a real gear system, 4 gears are sufficient. Also, if the number of axles is odd, the rotation direction of the last gear relative to the first gear remains the same. For an even number of axles, the rotation direction is reversed.



9. Code result

10. QR code: a real system with 4 gears and 3 axles (15, 16, 26, 29); the last gear rotates 3 full turns and 50° (which corresponds to approximately to 0.14 of a full rotation)

2.1.3. The relationship between the most important variables

The most important variables of the problem are the error, the number of axles, and the total number of teeth. By finding a direct relationship between them, it is possible to represent the three-dimensional function that characterizes it, and thus approximate the point at which all three have minimal values.

1. *Error and number of teeth (the numerator Pn_k)*

As required by the problem statement, t_i belongs to the interval [15, 30]; r_1=1, the direction of rotation does not matter, and r_k must be as close to π as possible.

$r_k = \frac{Pn_k}{Pd_k}$ *(1)*

$r_k = \pi + e$ *(2)*

from(1), (2) $\Rightarrow \frac{Pn_k}{Pd_k} = \pi + e$ *(3)*

$e \to 0$ *(4)*

from(3), (4) $\Rightarrow \quad Pd_k = \lfloor \frac{Pn_k}{\pi} + 0.5 \rfloor$ *(5)*

from(3), (5) $\Rightarrow \quad e = \frac{Pn_k}{\lfloor \frac{Pn_k}{\pi} + 0.5 \rfloor} - \pi$ (7)

2. *Number of gears and number of teeth (the numerator Pn_k)*

Although a mathematical formula has not yet been found, by decomposing the numerator (number of teeth) into factors within the range [15, 30] using a computer, multiple ways to build the gear system can be obtained. Even though the number of teeth of the gears can differ, the number of gears is always the same. (8)

Example:

$600\ (= 5 \cdot 6 \cdot 4 \cdot 5) = 30 \cdot 20$ (one gear with 30 teeth and one with 20)

$600\ (= 5 \cdot 5 \cdot 4 \cdot 6) = 25 \cdot 24$ (one gear with 25 teeth and one with 24)

The number of teeth can vary, but the number of gears is constant: 2 in this case. (9)

3. *Error and number of gears*

So far, no direct relationship has been found; only through the number of teeth can it be approached (the error is known, the number of teeth is determined, and then the number of gears is found). Therefore, at the moment, the error as a function of the number of gears (and vice versa) can only be generated through computer calculations.

2.1.4. Results

Finally, based on the computer calculations the exact relationship between the variables can be determined.

2.2. Second approach

2.2.1. Overlapping gears case generates infinite approximations

As regards the overlapping gears case, we will prove there is always a better approximation available as long as we increase the number of gears.

Lemma. $\frac{log(u+1)}{log(u)}$ and hence $\frac{log(u)}{log(u+1)}$ is irrational for any positive integer $u > 1$.
Proof: Suppose there exists a *u* such that the fraction is rational. This is equivalent to the existence of positive integers *a* and *b* such that the fraction $= \frac{a}{b} <=> (u+1)^b = u^a$. This is impossible because the two sides have opposite parity. q. e. d.

Proposition 3. The overlapping gears case can always generate better approximations by increasing the number of gears sufficiently if $u \geq 3$ or $l \geq 2$.
Proof: For $l = 1$, we may simply set $l = 2$. Let us assume $u = l + 1$, from which we will obtain that any u > *l* > 1 will work. This is equivalent to finding integers n_1 and n_2 such that $u^{n_1} \cdot (u+1)^{n_2}$ approximates a positive real target T (in our case π). Taking logs, we get that $n_1 \cdot log(u) + n_2 \cdot log(u+1)$ approximates $log(T)$. Dividing by $log(u+1)$, we obtain that $n_1 \cdot \frac{log(u)}{log(u+1)} + n_2$ should approximate $\frac{log(T)}{log(u+1)}$. By Kronecker's Theorem, we are done. q. e. d. (10)

2.3. Third approach

2.3.1. Domain expansion

The domain [15,30] can be expanded to [1,30]. It suffices to express each prime number less than 15 as a product of fractions with numerators and denominators in [15,30]. While playing with some numbers, here is what we found:

$$2 = \frac{30}{15}$$
$$3 = \frac{30}{15} \cdot \frac{30}{20}$$
$$5 = \frac{30}{15} \cdot \frac{30}{15} \cdot \frac{30}{24}$$

$$7 = \frac{30}{15} \cdot \frac{30}{15} \cdot \frac{28}{16}$$
$$11 = \frac{30}{15} \cdot \frac{30}{15} \cdot \frac{30}{15} \cdot \frac{22}{16}$$
$$13 = \frac{30}{15} \cdot \frac{30}{15} \cdot \frac{30}{15} \cdot \frac{26}{16}$$

From now on, we'll work with primes from 2 to 29 so keep the substitutions in mind.

2.3.2. Brute force

If we rely on the usage of brute force (exhaustion method), which essentially means checking every possible arrangement of gears, it will have the worst possible time complexity and it would take us years to find new approximations. Its complexity is:
$O(n)=10^n$, where n is the total number of gears and 10 because there are 10 prime numbers available to us (from 2 to 29).
That escalates too fast, we estimate that for 14 gears it would take the algorithm 1 month to finish, so we definitely need a better algorithm than pure brute force.

2.3.3. The idea

What if instead of checking fractions to see if they are equal to π, we know the numerator, which is a product of any prime up to 29, and see what denominators can approximate π the best, the denominator needs to be a valid number (it is not a multiple of any prime greater than 29), we'll start by declaring:

Π = some product of primes up to 29

π_{inf} = "inferior π" = an approximation less than π

π_{sup} = "superior π" = an approximation greater than π

Given Π, we know that there exists some numbers β (not necessarily natural, but the naturals are the ones we look for) that satisfy the relation:

$$\pi_{inf} \leq \frac{\Pi}{\beta} < \pi_{sup}$$

We need to find β so let's rearrange the terms:

$$\frac{1}{\pi_{inf}} \geq \frac{\beta}{\Pi} > \frac{1}{\pi_{sup}} \iff \frac{\Pi}{\pi_{inf}} \geq \beta > \frac{\Pi}{\pi_{sup}}$$

But β has to be a natural number so let's adjust a bit the relation:

$$\lceil \frac{\Pi}{\pi_{sup}} \rceil \leq \beta \leq \lfloor \frac{\Pi}{\pi_{inf}} \rfloor$$

Remember β needs to be not just a natural number, but also a product of primes less or equal to 29.
We need to make a code that checks combination of products of primes up to 29, applies the formula $\lceil \frac{\Pi}{\pi_{sup}} \rceil \leq \beta \leq \lfloor \frac{\Pi}{\pi_{inf}} \rfloor$ for every combination and then checks in the interval if there are any numbers β that satisfy the condition on prime factors using that code.

This method is called heuristical greedy in computer science and it is significantly more efficient than brute force.

2.3.4. Final result

We plan to aim high so the upper limit we'll restrict the product is the limit of "unsigned long long" in C++, basically the limit of the application we work in (code:blocks), which is equal to 2^{65}-1, that is a really big number, in contrast, it is 37 times larger than the number of sand grains on planet Earth. Using our heavily optimised code, it took us only 140.000 seconds, which is roughly 2 days of non-stop work for our computer to find the best approximation, therefore, the number of correct digits we found is: 9.

Let's see the fraction:

$$\frac{19930728263064896}{6344147844359325} = \mathbf{3.14159265}89\ldots$$

$$= \frac{29^3 \cdot 23^2 \cdot 17^6 \cdot 2^6}{19^1 \cdot 13^1 \cdot 11^3 \cdot 7^6 \cdot 5^2 \cdot 3^8}$$

9. *QR code: Visualisation of the best approximation found*

To generalize, we can do the same thing with every irrational number, not just π, here are some quick examples:

$$e = \frac{23871125}{8781696} = \frac{23^2 \cdot 19^2 \cdot 5^3}{11^2 \cdot 7^1 \cdot 3^4 \cdot 2^7} = \mathbf{2.718281}87\ldots$$

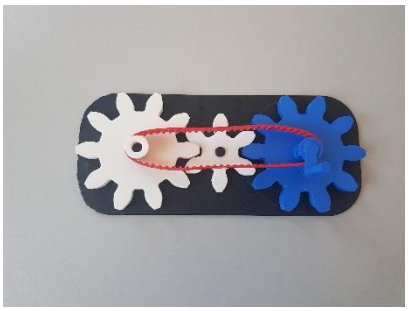
$$\sqrt{2} = \frac{25222981855}{17835341097} = \frac{29^3 \cdot 23^3 \cdot 17^1 \cdot 5^1}{19^5 \cdot 7^4 \cdot 3^1} = \mathbf{1.414213}95\ldots \text{(11)}$$

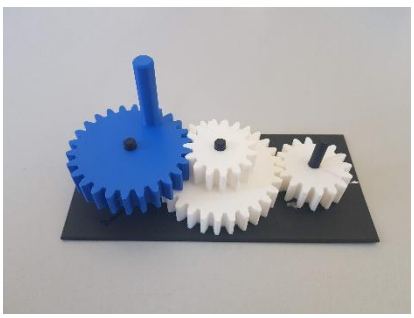
2.4. Intermediate conclusion for overlapping gears

Theoretically, we can obtain better approximations by increasing the number of gears. However, in a real system, where the gears do not mesh perfectly, a larger number of gears results in greater inaccuracy, which ultimately neutralizes the theoretical mathematical precision. Therefore, in real-life applications, the 4-gear approximation (15, 16, 26, 29) is the most effective solution.

Real prototype

In order to better understand the problem, we 3D-printed two mechanisms representing the two general cases (overlapping and non-overlapping gears). Using the non-overlapping model (figure 10), we demonstrated visually that the number of gears does not affect the number of rotations, but only the direction of rotation. With the overlapping model, we showed that a system with 4 gears and 3 axles (e.g., 15, 16, 26, 29) is suitable for a real-life mechanism (figure 3).





Real life applications/practical side

Gear systems have been used for centuries and have numerous applications. They played a fundamental role in industrial development and modern mechanics, enabling the efficient transmission of power from engines and motors. Examples of the use of wheel mechanisms in real life:

Transport

Cars, motorcycles

- allow smooth acceleration;
- maintain efficient engine speed;
- enable reverse motion;
- increased torque when starting.

Bicycles

- make climbing hills easier;
- increased speed;
- reduce pedaling effort.

Energy production

Wind turbines use large gearboxes to convert the slow rotational speed of turbine blades into the higher rotational speeds required by generators.

Many turbines use planetary gear systems because they:

- handle high torque;
- reduce gearbox size;
- improve efficiency.

Precision mechanisms

Small gear systems are found in:

- washing machines;
- hair clippers;
- drills and other power tools;
- kitchen mixers;
- watches.

Although there is no practical need for a gear mechanism to transmit a rotation ratio equal to π, the same design approach can be generalized to approximate any real-valued transmission ratio. In this sense, the methodology developed for approximating π still applies, the only change being the target value, which can be replaced with any other desired constant.

General conclusion

It can be observed that in the case of overlapping gears, a better approximation can be achieved, as gears with a relatively small number of teeth are used, whereas in the case of non-overlapping gears, since only two gears are used, the required number of teeth is higher. Therefore, for small mechanisms, where the gears must have small dimensions (and implicitly a small number of teeth), the solution is the

overlapping of gears. For large mechanisms, where the gears can have a high number of teeth, in order to avoid possible errors or failures (resulting from a large number of components), non-overlapping gears can be used.

Sources:

Ainsworth, J., Dawson, M., Pianta, J., Warwick, J., Year 4 Project. (2012). The Farey Sequence. https://webhomes.maths.ed.ac.uk/~v1ranick/fareyproject.pdf

Editor's notes

(1) Presumably, what is meant is to look for a/b close to π, with a variable upper bound on the denominator b. The constraint on a and b belonging to the interval [15,30] will be ignored throughout Case A.

(2) Does 'simple' mean irreducible (i.e. a and b have no common divisor but 1)? Well, this is true but does not quite follow from Proposition 1. It rather follows from Proposition 2, where the following definition is needed: two fractions are called *Farey neighbours* if they arise as consecutive elements in some Farey sequence F_u.

(3) What is meant is an approximation algorithm. Let x be an irrational number in [0,1]. A sequence of intervals $[a/b,c/d]$, containing x and bounded by Farey neighbours, is constructed inductively: the mediant p/q divides the interval in two subintervals, one retains the subinterval which contains x. Proposition 2 guarantees that the endpoints of the subinterval are again Farey neighbours. In addition, an upper bound on the error follows, $|x- a/b| < |c/d -a/b|=|(bc-ad)/db| < 1/db$.

(4) When dealing with approximations, taking reciprocals requires some care. If x approximates A better than y, it does not necessarily follow that $1/x$ approximates $1/A$ better than 1/y. For example, take A=1, x=0.5, and y=1.6.

(5) The following paragraph is very difficult to understand, and some parts of it are definitely unintelligible. It was presumably written in a hurry. In a future version of the work, if any, this paragraph should be completely rewritten.

(6) However, this does not meet the requirement that the number of teeth per gear be less than 30.

(7) $e \to 0$ means that we are looking for values that make e smallest. This forces Pd_k to be the integer which is closest to Pn_k/π. Whence the expression $\lfloor Pn_k/\pi + 0.5 \rfloor$.

(8) Presumably, the idea is to pick an integer Pn_k that can be expressed as a product of l integers between 15 and 30, compute $Pd_k = \lfloor Pn_k/\pi + 0.5 \rfloor$ and hope that Pd_k can also be expressed as a product of the same number l of integers between 15 and 30. An example is provided on Fig. 9: $Pn_2 = 754 = 26 \cdot 29$, $Pn_2/\pi = 240.005654$, the closest integer is $Pd_2 = 240 = 15 \cdot 16$.

(9) Indeed, the same number can sometimes be expressed in several different manners as a product of smallish integers. However, the number of factors may change. For instance, 810 000 = 30·30·30·30 = 15·15·15·15·16.

(10) Let us interpret the statement as: the error can be made arbitrarily small. Fixing an integer $u > 1$, it is true that any positive real number T is the limit of a sequence of rational numbers of the form $v_j = u^{nj} \cdot (u+1)^{mj}$, with n_j and m_j positive or negative integers. The key ingredient is Dirichlet's Theorem, which states that when x is irrational, for every integer N, there exist integers p and q such that $|px-q|<1/N$. By the Lemma, this applies to $x = (log(u+1))/(log(u))$, yielding integers p and q such that $-log(u)/N < p\ log(u+1)-q\ log(u) < log(u)/N$. Passing to exponentials shows that $(u+1)^p u^{-q}$ can be made arbitrarily close to 1. In other words, for every fixed decreasing sequence of numbers $z_j > 1$ tending to 1, we can pick a number $w_j = (u+1)^p\ u^{-q}$ or $(u+1)^{-p}\ u^q$ which lies between 1 and z_j. This proves the claim for $T=1$.

For the general case, we can assume that $T>1$ and $z_1 < T$. We shall construct inductively numbers $v_j = u^{nj} \cdot (u+1)^{mj}$ such that $v_j < Tz_j$ and $v_j > Tz_j^{-1}$. Let v_1 be the smallest power of w_1 that passes above T. Then we divide v_1 by powers of w_2 until we pass below T, getting v_2 such that $v_2 < Tz_2$ and $v_2 > Tz_2^{-1}$. Then we multiply v_2 by powers of w_3 until we pass above T, getting v_3, and so on.

To apply this approximation result to gears, we take $T= \pi,\ u=2$ and observe that 2=30/15 and 3=(30 ·30)/(20 ·15), thus every number $v_j = 2^{nj} \cdot 3^{mj}$ is a product of fractions whose numerators and denominators lie between 15 and 30, hence can be realized by a system of overlapping gears.

(11) In order for the obtained fraction to be achievable by gears with at least 15 teeth, one must pair factors in [15,30] in the numerator with similar factors in the denominator, leaving only factors less than 15. These small factors are then converted into fractions as in § 2.3.1. For the approximation of π, a possibility is

$(29^3 \cdot 23^2 \cdot 17^6 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)/(19 \cdot 13 \cdot 11^3 \cdot 21^6 \cdot 5 \cdot 5 \cdot 3 \cdot 3) = (29/19) \cdot (29/13) \cdot (29/11) \cdot (23/11)^2 \cdot (17/21)^6 \cdot (30/15)^6 \cdot (15/30)^4 \cdot (24/30)^2 \cdot (15/30)^2 \cdot (20/30)^2$.

For the approximation of e,

$(23^2 \cdot 19^2 \cdot 5 \cdot 5 \cdot 5)/(11^2 \cdot 21 \cdot 27 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)$.

For the approximation of √2, one first multiplies numerator and denominator by 3, so that

$(29^3 \cdot 23^3 \cdot 17 \cdot 5)/(19^5 \cdot 7^4 \cdot 3) = (29^3 \cdot 23^3 \cdot 17 \cdot 5 \cdot 3)/(19^5 \cdot 21^2 \cdot 7 \cdot 7)$.