Kirkman's Schoolgirls

2024.-2025.

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1. Topic presentation

Fifteen young schoolgirls walk every day of the week, from Monday to Sunday, in an orderly way, forming five rows of three schoolgirls each. How should we organize them every day of the week so that no pair of schoolgirls shares the same row for more than one day?

2. Presentation of conjectures and final results

2-1 With the simplify version

First, we can simplify the topic by considering nine schoolgirls. If we perform a calculation based on the set {9;8} (where 9 represents all the girls, and 8 represents all the girls excluding one), we can make the simplified calculation: 8÷2=4. Therefore, we can suppose that the nine girls can walk for four different days without being next to the same neighbors. Then, if we represent this scenario using tables, the result is also four days.
Then, we can deduce that nine schoolgirls can walk for four days under these conditions.

2-2 With fifteen girls

Afterward, we can apply the same reasoning to fifteen schoolgirls. If we perform the calculation based on the set {15;14}, we find: 14÷2=7. Therefore, we can suppose that the fifteen girls can walk for seven different days without being next to the same person.
And, if we model this with circles and triangles that constitute the circle, the result is also seven. Consequently, we can deduce that fifteen schoolgirls can walk for seven days.

To solve this problem, we can also use a matrix method with the Sarrus rules and we find the same previous result.

3. Article text

3-1 History and enunciation

Kirkman's schoolgirl problem is a problem in combinatorics proposed by Thomas Penyngton Kirkma in 1850 as Query VI in The Lady's and Gentleman's Diary. The problem states:
Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast.

3-2 Topic with a table

We can model the rows using a table where the rows correspond to the lines of schoolgirls. Since there are fifteen schoolgirls, the table has fifteen squares. As the schoolgirls are lined up three by three in five rows, the table includes five rows and three columns.
Furthermore, the schoolgirls are represented by numbers from 1 to 15. Then, we know that the squares called {1;2}, {2;3} and {1;3} (forming a triplet), can't appear in the same row twice. (We identify three such sets within each row that can't be repeated in the same row). Once we have this table, we can proceed to find the result...

1	2	3
4	5	6
7	8	9
10	11	12
13	14	15

3-3 A simplified version of the topic with nine schoolgirls

First, we create a table with three columns and three rows. (This table is similar to the table created previously for the fifteen schoolgirls.) So, to fill the table, we can place any of the nine schoolgirls in the first square, giving us nine possibilities. And in the second square, we have 9-1=8 possibilities because we exclude the schoolgirl already placed in the first square. These two cells form a set of schoolgirls: {9;8}.

1	2	3
4	5	6
7	8	9

We suppose that, to find the number of days where they can walk, we make this calculation:

- first, we multiply the two numbers in the identified pair: $9 \times 8 = 72$;
- then, we divide the result by two because the pair {1;2} is the same as the pair {2;1} (order doesn't matter for a pair of neighbors). And the result is the number of the total possible sets which aren't repeated twice: $72 \div 2 = 36$;
- finally, we divide the result by nine because in each arrangement (represented by a day's walk) we find nine new sets that we substract from the total possible sets found earlier. The result is equivalent to the number of days where they can walk: $36 \div 9 = 4$. So, we suppose that the nine girls can walk four different days without being next to the same person.

And to prove this, we make the tables. We can see that, to find the next table, we can use the diagonal shift. Specifically, the diagonal of the first table becomes a row in the subsequent table. After, we see that we can find four different tables without a first number being a set with the same second number more than once. As one table is equivalent to one day and as we get four tables, we suppose that the schoolgirls can walk four different days without being next to the same person. So, the previous calculation is confirmed.

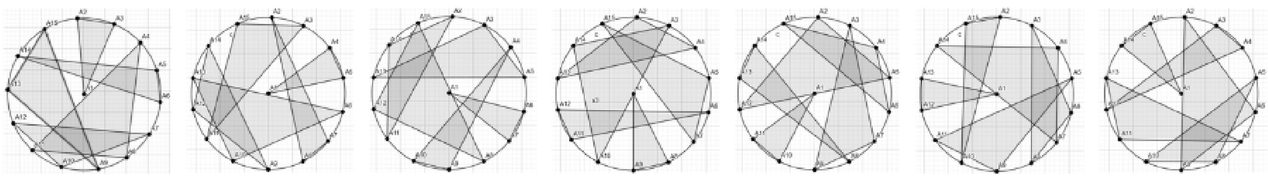
1	2	3	4	1	7
4	5	6	2	5	8
7	8	9	3	6	9
1	5	9	1	6	8
2	6	7	5	7	3
3	4	8	2	9	4

3-4 The version of the topic with the fifteen schoolgirls

We can apply the same method. So the initial set is {15;14} and the calculation is as follows:

- The multiplication of the two numbers in the set: $15\times 14=210$
- The first division to find the total number of unique sets: $210\div 2=105$
- The second division by the total number of schoolgirls to find the number of days: $105\div 15=7$. So, we suppose that the fifteen girls can walk seven diferent days without being next to the same person.

And to prove this, we can use another method because using tables can be dificult...
So, we can model this with circles. Indeed, in one circle, we place fourteen points on the circumference and one point at the center. This gives us fifteen points representing the fifteen girls. Next, we connect three points on the circumference to form a triangle. This triangle represents one row of three girls, with the vertices of the triangle (the points on the circumference) representing the girls in that row. If we connect all the points on the circumference three by three to create triangles, we obtain five triangles, representing the five rows of three schoolgirls. And we know that one circle is equivalent to one day.
To make a new arrangement, we can consider i the vertices of a triangle. So, we keep the same configuration of the first circle but we move all the triangles on the circle towards the right so that i+2.
Therefore, we find seven different circles so seven days. So, the previous calculation is confirmed.



1	2	15
3	7	10
4	5	13
6	9	11
8	12	14

3	4	15
5	9	12
6	7	1
8	11	13
10	14	2

5	6	15
7	11	14
3	8	9
10	13	1
12	2	4

7	8	15
9	13	2
10	11	5
12	1	3
14	4	6

11	12	15
13	3	6
9	14	1
2	5	7
4	8	10

9	10	15
11	1	4
12	13	7
14	3	5
6	8	2

13	14	15
1	5	8
2	3	11
4	7	9
6	10	12

3-5 Now, we can generalise and simplify the calculation

- First, we establish a set of numbers based on the total number of girls walking. In this set, the first number is the total number of girls, which we can note as n. The second number is the previous number minus one, to avoid having the same girl in the same relative position across different arrangements... so it can be noted as $n-1$. Therefore, the set is: $\{n;n-1\}$.
- Then, we multiply these two numbers together, and then divide the product by two to avoid counting identical pairs (where the order of the two girls is reversed). This gives us a number, let's call it x, which represents the total number of the different sets: $n(n-1)\div 2=x$
- Finally, we divide the previous result (x) by n (the total number of girls) because each day's arrangement , we substract n possibilities of sets. So, we find the total number of days where they can walk: x:n.

And, we can simplify this calculation, because we observe that it is equivalent to dividing the second number of the initial set by two. For example:

- for nine girls, we make 8÷2=4
- for fifteen girls: 14÷2=7.

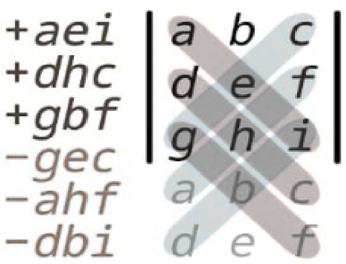
3-6 Another way with matrix

We found out a way to organize the girls every day of the week so that no set of schoolgirls shares the same row for more than one day.

Number the girls 1 through 15 , then divide the girls into 3 groups: 1-7, 8-14, and 15.
You will always keep these numbers in order remembering to 'wrap-around' when you get to 14 ie: 8 will follow 14. You only have to solve the top row.

1	2	8	3	5	9	4	7	10	6	15	12	11	14	13
2	3	9	4	6	10	5	1	11	7	15	13	12	8	14
3	4	10	5	7	11	6	2	12	1	15	14	13	9	8
4	5	11	6	1	12	7	3	13	2	15	8	14	10	9
5	6	12	7	2	13	1	4	14	3	15	9	8	11	10
6	7	13	1	3	14	2	5	8	4	15	10	9	12	11
7	1	14	2	4	8	3	6	9	5	15	11	10	13	12

After we arranged the 7 columns with numbers from 1 to 7 that are shown in the borders (except the one where all numbers are equal to 15), we will place a column with numbers from 8 to 14. More precisely, the first column that starts with 8 is the same as the first column from those placed with the first group of numbers from 1-7. The second column that needs to be placed is the same as the second one that was placed with the group from 1-7, maintaining the correspondence.
For nine schoolgirls to walk for four days, we will recognise the rows, the columns, and the positive and negative terms in the formula for the determinant of the matrix.
In matrix theory, the rule of Sarrus is a mnemonic device for computing the determinant of a matrix named after the French mathematician Pierre Frédéric Sarrus. Consider a matrix then its determinant can be computed by the following scheme.



The rule for nine schoolgirls to walk in four days is here:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \quad \begin{pmatrix} a & e & i \\ d & h & c \\ g & b & f \end{pmatrix} \quad \begin{pmatrix} g & e & c \\ a & h & f \\ d & b & i \end{pmatrix}$$

(the matrix, the transpose of the matrix and the products of Sarrus rule)

4. Conclusion

Therefore, we can suppose that this calculation works only with multiples of three and specifically for odd multiples. This is because when we have an even number of girls, the second number in our set (n−1) becomes odd and when we divide an odd number by two we find a decimal number but we can't divide a day.
For example with twelve girls, we make: 11÷2=5.5. So, it's not possible.

If the calculation is keeping with our conclusion, we can conjecture about other numbers of girls:

- with twenty-one girls, we have the set {21;20}: 20:2=10.

So we can suppose that twenty-one girls walk ten days.

- with twenty-seven girls, we have the set {27;26}: 26:2=13.

So we can suppose that twenty-seven girls walk thirteen days.

- with thirty-three girls, we have the set {33;32}: 32:2=16.

So we can suppose that thirty-three girls walk sixteen days.

- with thirty-nine girls, we have the set {39;38}: 38:2=19.

So we can suppose that thirty-nine girls walk nineteen days.

- …

So we can observe that we add three days to find the number of days for the next odd multiple of three. And, we can express this as a recurrence relation. Indeed, if we note d(n) as the number of days n girls can walk, where n is an odd multiple of three, we have the arithmetic progression:

$d(n)=(n-1)/2 \quad \text{if } n=3(2k+1)$

$d(n)=((6k+3)-1)/2$

$d(n)=3k+1 \quad \text{where } k\in N$

...

The problem can be generalised to the case where the number of schoolgirls is 6m+3 and the number of days is 3m+1, for any natural number m. In this form the problem attracted a lot of attention, but was not solved until 1969 when Dijen Ray-Chaudhuri and Richard Wilson showed that the arrangement is possible for any value of m.

The proof is in this article: https://people.math.ethz.ch/~halorenz/publications/pdf/flu.pdf