Modelling and constructing safe streets

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Problem

Vehicles while moving follow specific trajectories, more or less complex according to their purpose, which present particular and interesting aspects when studied with mathematical tools. The goal of this article is to find the safest way to construct parts of streets, considering different cases according to their relative position, and minimizing the risk of losing control while driving them. For such goal we will be using a curve called the "Euler Spiral".

Contents

1. Introduction

In this chapter, we moved to the field of designing and constructing roads. In particular, we focused on finding a way to build streets such that they are the safest possible, where by "safe" we mean that their curvature (1) doesn't change too quickly, in order to reduce the risk of losing control while driving them.
The curve that best fits our request is the Euler spiral, which is defined as the curve whose

curvature changes linearly with its curve length (2). Given $a \in \mathbb{R}$, its parametric equation can be written as:

$$\begin{cases} x(t) = a\displaystyle\int_0^t \cos\left(\frac{u^2}{2}\right)du = aC(t) \\ y(t) = a\displaystyle\int_0^t \sin\left(\frac{u^2}{2}\right)du = aS(t) \end{cases} \quad t \in (-\infty, +\infty) \tag{1}$$

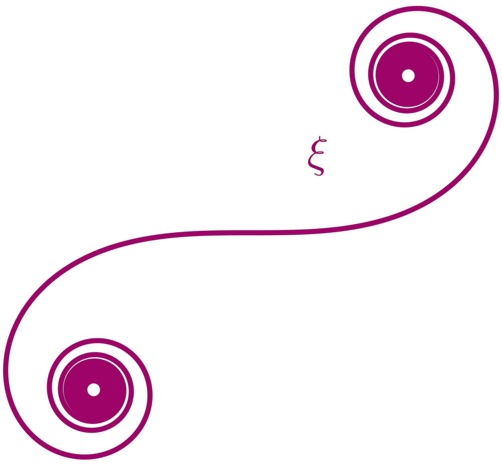


Figure 1: The Euler spiral

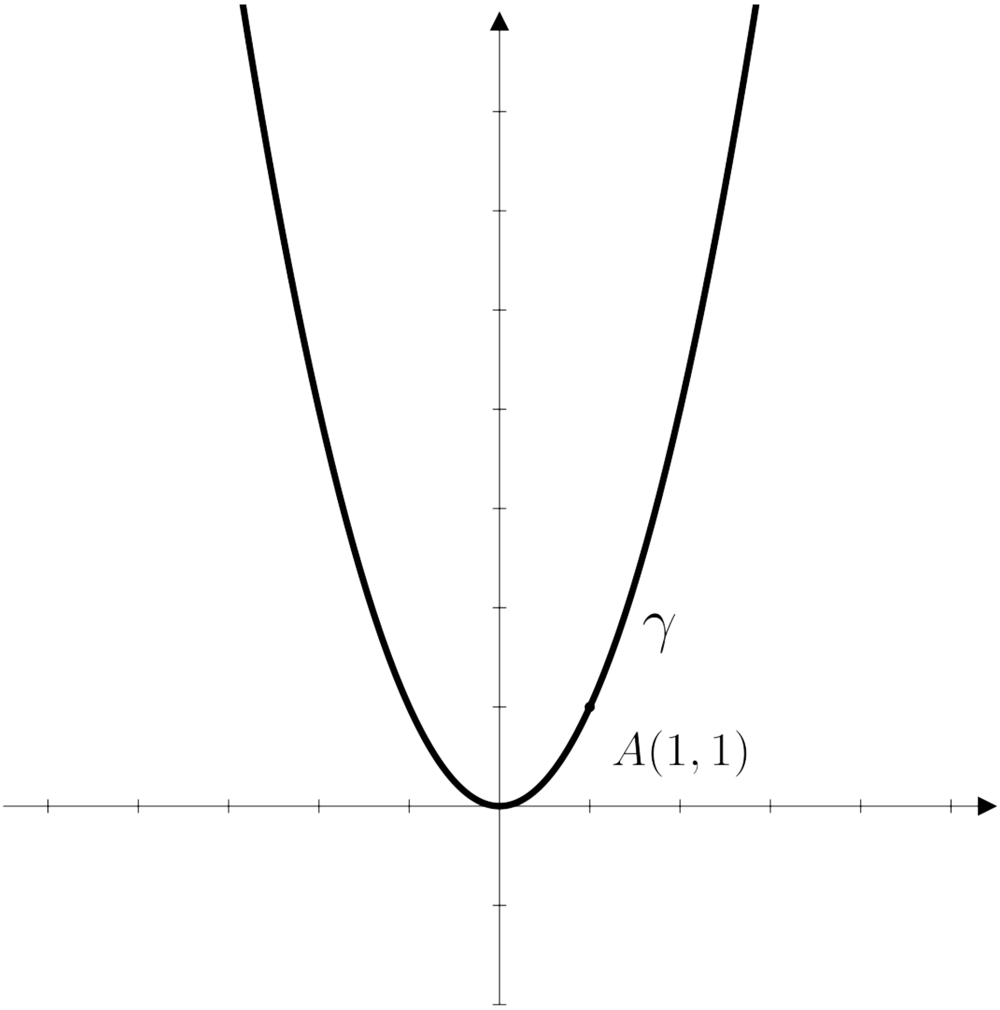
where C and S are the *Fresnel integrals*, defined as(3):

$$C(x) = \int_0^x \cos\left(\frac{u^2}{2}\right)du$$
$$S(x) = \int_0^x \sin\left(\frac{u^2}{2}\right)du$$

Throughout the several different cases of our analysis, we are going to be using both the cartesian plane and the complex one. In such way we will be able to catch a glimpse of the differences and similarities between the two methods.

2. Approximating a curve with an Euler spiral

The aim of this section is to find a way to approximate a generic curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ using an Euler spiral. We will first approximate the curve using different non rotated spirals, moving then on to considering a single rotated one.



2.1. Non rotated spirals

Let us consider to this purpose the following parametric equation defining γ:

$$\gamma : \begin{cases} x(t) = t \\ y(t) = f(t) \end{cases} \quad t \in (t_i, t_f),$$

where $t_i, t_f \in \mathbb{R} \cup \{-\infty, +\infty\}$ and $f : \mathbb{R} \to \mathbb{R}$ is a real valued function of class C^2 with $f''(t) \neq 0$ for all $t \in \mathbb{R}$.
Recalling the parametric definition of the Euler spiral, we have:

$$\begin{cases} x(t) = aC(t) \\ y(t) = aS(t) \end{cases} \quad t \in (-\infty, +\infty),$$

with $a \in \mathbb{R} - \{0\}$. Such curve is an Euler spiral centered on $(0, 0)$. We can obtain a more generic form of an Euler spiral with a translation by the vector $\vec{b} = (x_0, y_0)$:

$$\begin{cases} x(t) = x_0 + aC(t) \\ y(t) = y_0 + aS(t) \end{cases} \quad t \in (-\infty, +\infty). \tag{2}$$

To approximate γ in one of its points P, we will be looking for the Euler spiral ξ that satisfies the following statements:

- ξ is tangent to γ in P, meaning the tangent line of the two curves in P is the same;
- The curvature of ξ in P is the same as γ.

Let us first find the tangent line to γ at $(w, f(w))$. Given the vectorial equation of a line:

$$\mathbf{P}(t) = \mathbf{P_0} + t\mathbf{v}$$

The parametric equation of the tangent line can be written as:

$$\begin{cases} x(t) = x_T + tv_x \\ y(t) = y_T + tv_y \end{cases} \quad t \in (-\infty, +\infty)$$

Thus, since $x_T = w$, $y_T = f(w)$, $v_x = \dfrac{d}{dw}w = 1$ and $v_y = \dfrac{d}{dw}f(w) = f'(w)$ we get:

$$\begin{cases} x(t) = w + t \\ y(t) = f(w) + tf'(w) \end{cases} \quad t \in (-\infty, +\infty)$$

(4)The tangent line to the Euler spiral at the point $(x(\varphi_0), y(\varphi_0))$, on the other hand, can be expressed as:

$$\begin{cases} x(t) = x_0 + aC(\varphi_0) + t \cdot a \cdot \cos\left(\dfrac{\varphi_0^2}{2}\right) \\ y(t) = y_0 + aS(\varphi_0) + t \cdot a \cdot \sin\left(\dfrac{\varphi_0^2}{2}\right) \end{cases} \quad t \in (-\infty, +\infty)$$

Since $(w, f(w)) \equiv (x(\varphi_0), y(\varphi_0))$, we get:

$$\begin{cases} w = x_0 + aC(\varphi_0) \\ f(w) = y_0 + aS(\varphi_0) \end{cases},$$

from which we can find x_0 and y_0:

$$\begin{cases} x_0 = w - aC(\varphi_0) \\ y_0 = f(w) - aS(\varphi_0) \end{cases}$$

Since the slopes of the tangent lines must be the same, we have that:

$$\frac{\Delta y_\gamma}{\Delta x_\gamma} = \frac{\Delta y_\xi}{\Delta x_\xi} \to f'(w) = \frac{a \cdot \sin\left(\dfrac{\varphi_0^2}{2}\right)}{a \cdot \cos\left(\dfrac{\varphi_0^2}{2}\right)} = \tan\left(\frac{\varphi_0^2}{2}\right)$$

Hence we can find φ_0 as (5):

$$\frac{\varphi_0^2}{2} = \arctan\left(f'(w)\right) \quad \vee \quad \frac{\varphi_0^2}{2} = \pi + \arctan\left(f'(w)\right)$$

$$\varphi_0 = \pm\sqrt{2\arctan\left(f'(w)\right)} \quad \vee \quad \varphi_0 = \pm\sqrt{2\pi + 2\arctan\left(f'(w)\right)}$$

These values of φ_0 allow us to write the following expressions of x_0 and y_0:

$$\begin{cases} x_0 = w - aC\left(\pm\sqrt{2\arctan\left(f'(w)\right)}\right) \\ y_0 = f(w) - aS\left(\pm\sqrt{2\arctan\left(f'(w)\right)}\right) \end{cases} \vee \begin{cases} x_0 = w - aC\left(\pm\sqrt{2\pi + 2\arctan\left(f'(w)\right)}\right) \\ y_0 = f(w) - aS\left(\pm\sqrt{2\pi + 2\arctan\left(f'(w)\right)}\right) \end{cases} \tag{3}$$

Moving on to the second condition, the curvature of a generic parametric curve can be computed as (6):

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}$$

Therefore the curvature of the Euler spiral is:

$$\kappa(t) = \frac{t}{a} \text{(7)}$$

Hence the value of a is:

$$a = \frac{\varphi_0}{\kappa_\gamma(w)} = \varphi_0 \frac{\left(1 + f'(w)^2\right)^{\frac{3}{2}}}{|f''(w)|}$$

Since the two functions

$$g(\varphi) = \varphi \int_0^\varphi \cos\left(\frac{u^2}{2}\right) du$$
$$h(\varphi) = \varphi \int_0^\varphi \sin\left(\frac{u^2}{2}\right) du$$

are both odd functions, the four parametric equations (3) come down to the following two:

$$\begin{cases} x_0 = w - \sqrt{2\arctan\left(f'(w)\right)}\dfrac{\left(1 + f'(w)^2\right)^{\frac{3}{2}}}{|f''(w)|} C\left(\sqrt{2\arctan\left(f'(w)\right)}\right) \\ y_0 = f(w) - \sqrt{2\arctan\left(f'(w)\right)}\dfrac{\left(1 + f'(w)^2\right)^{\frac{3}{2}}}{|f''(w)|} S\left(\sqrt{2\arctan\left(f'(w)\right)}\right) \end{cases}$$

$$\vee$$

$$\begin{cases} x_0 = w - \sqrt{2\pi + 2\arctan\left(f'(w)\right)}\dfrac{\left(1 + f'(w)^2\right)^{\frac{3}{2}}}{|f''(w)|} C\left(\sqrt{2\pi + 2\arctan\left(f'(w)\right)}\right) \\ y_0 = f(w) - \sqrt{2\pi + 2\arctan\left(f'(w)\right)}\dfrac{\left(1 + f'(w)^2\right)^{\frac{3}{2}}}{|f''(w)|} S\left(\sqrt{2\pi + 2\arctan\left(f'(w)\right)}\right) \end{cases}$$

Finally, substituiting these values inside the parametric equations of the Euler spiral (2) gives us the equations of ξ.

$$\begin{cases} x(t) = w + \sqrt{2\arctan\left(f'(w)\right)}\dfrac{\left(1+f'(w)^2\right)^{\frac{3}{2}}}{|f''(w)|}\left(-C\left(\sqrt{2\arctan\left(f'(w)\right)}\right)+C(t)\right) \\ y(t) = f(w) + \sqrt{2\arctan\left(f'(w)\right)}\dfrac{\left(1+f'(w)^2\right)^{\frac{3}{2}}}{|f''(w)|}\left(-S\left(\sqrt{2\arctan\left(f'(w)\right)}\right)+S(t)\right) \end{cases}$$

$$\vee$$

$$\begin{cases} x(t) = w + \sqrt{2\pi + 2\arctan\left(f'(w)\right)}\dfrac{\left(1+f'(w)^2\right)^{\frac{3}{2}}}{|f''(w)|}\left(-C\left(\sqrt{2\pi + 2\arctan\left(f'(w)\right)}\right)+C(t)\right) \\ y(t) = f(w) + \sqrt{2\pi + 2\arctan\left(f'(w)\right)}\dfrac{\left(1+f'(w)^2\right)^{\frac{3}{2}}}{|f''(w)|}\left(-S\left(\sqrt{2\pi + 2\arctan\left(f'(w)\right)}\right)+S(t)\right) \end{cases}$$

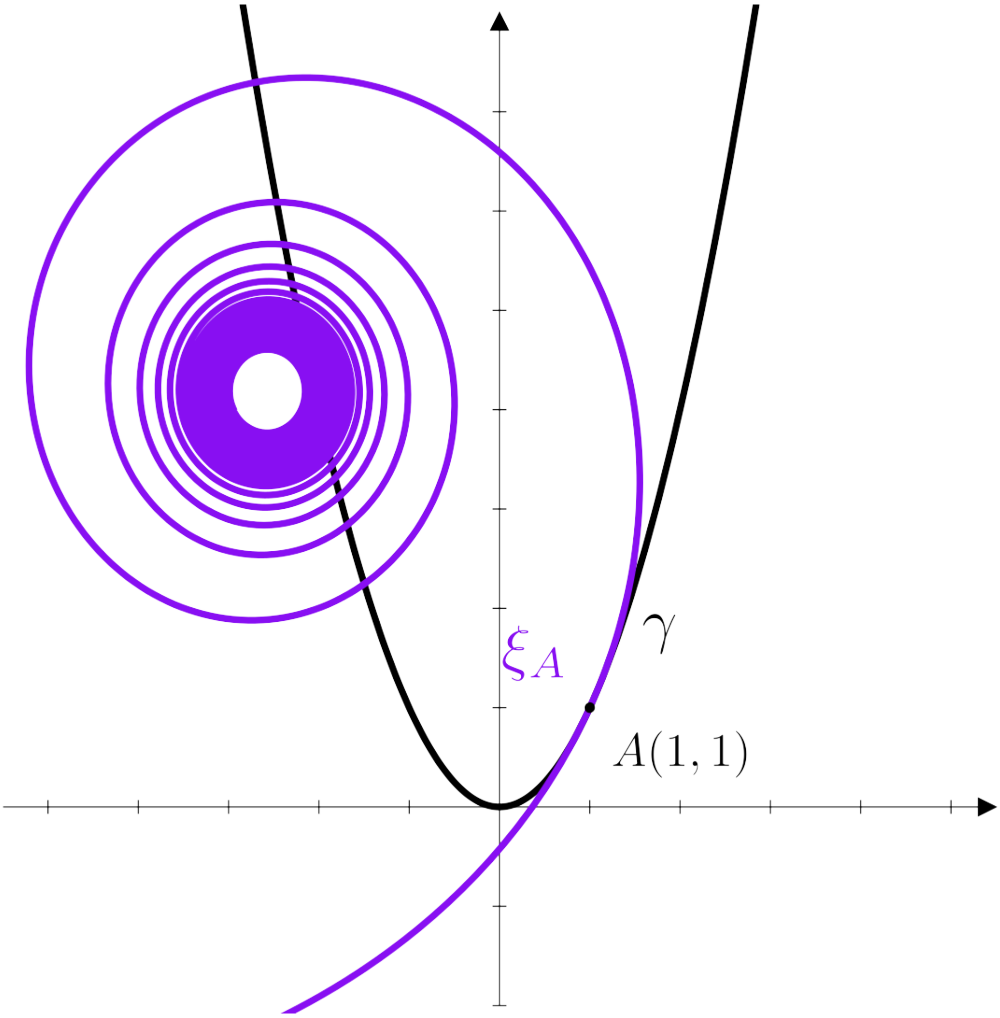
Chasles' relation gives:

$$\int_a^b \alpha(x)dx - \int_a^c \alpha(x)dx = \int_c^b \alpha(x)dx$$

we can rewrite the equations as:

$$\boldsymbol{\xi_1(w) :} \begin{cases} \boldsymbol{x(t) = w + \sqrt{2\arctan\left(f'(w)\right)}\dfrac{\left(1+f'(w)^2\right)^{\frac{3}{2}}}{|f''(w)|}\left(\int_{\sqrt{2\arctan\left(f'(w)\right)}}^{t}\cos\left(\dfrac{u^2}{2}\right)du\right)} \\ \boldsymbol{y(t) = f(w) + \sqrt{2\arctan\left(f'(w)\right)}\dfrac{\left(1+f'(w)^2\right)^{\frac{3}{2}}}{|f''(w)|}\left(\int_{\sqrt{2\arctan\left(f'(w)\right)}}^{t}\sin\left(\dfrac{u^2}{2}\right)du\right)} \end{cases}$$

$$\boldsymbol{\xi_2(w) :} \begin{cases} \boldsymbol{x(t) = w + \sqrt{2\pi + 2\arctan\left(f'(w)\right)}\dfrac{\left(1+f'(w)^2\right)^{\frac{3}{2}}}{|f''(w)|}\left(\int_{\sqrt{2\pi+2\arctan\left(f'(w)\right)}}^{t}\cos\left(\dfrac{u^2}{2}\right)du\right)} \\ \boldsymbol{y(t) = f(w) + \sqrt{2\pi + 2\arctan\left(f'(w)\right)}\dfrac{\left(1+f'(w)^2\right)^{\frac{3}{2}}}{|f''(w)|}\left(\int_{\sqrt{2\pi+2\arctan\left(f'(w)\right)}}^{t}\sin\left(\dfrac{u^2}{2}\right)du\right)} \end{cases}$$



2.2. Fixed rotated spiral

Let us move on considering now a rotated spiral with fixed amplitude. In order to do that, we may as well describe the curve γ using the complex plane $\mathbb{C}$. This will allow us to handle with more ease the rotation of the spiral. Moreover we can define $\gamma : \mathbb{R} \to \mathbb{C}$ as the complex valued function:

$$\gamma(t) = t + if(t)$$

where $f : \mathbb{R} \to \mathbb{R}$ is a real valued function of class C^2, with $f''(t) \neq 0$ for every $t \in \mathbb{R}$.
We can define the Euler spiral rotated by an angle ϑ and stretched of a factor $r \in \mathbb{R} - \{0\}$ as:

$$\xi_0(t) = \left(\int_0^t \cos\left(\frac{u^2}{2}\right) du + i \int_0^t \sin\left(\frac{u^2}{2}\right) du \right) \cdot re^{i\vartheta} = re^{i\vartheta} \cdot (C(t) + iS(t)) \quad t \in (-\infty, +\infty)$$

Using the Euler equation:

$$re^{i\vartheta} = r(\cos\vartheta + i \sin\vartheta)$$

we can write the equation of the spiral, translated of the vector (x_0, y_0) as:

$$\xi : \begin{cases} \Re(\xi(t)) = x_0 + r\left(C(t)\cos\vartheta - S(t)\sin\vartheta\right) \\ \Im(\xi(t)) = y_0 + r\left(C(t)\sin\vartheta + S(t)\cos\vartheta\right) \end{cases} \quad t \in (-\infty, +\infty)$$

Next we can write the equation of the lines respectively tangent to γ and to ξ as:

$$\tau_\gamma : \begin{cases} \Re(\tau_\gamma(t)) = w + t \\ \Im(\tau_\gamma(t)) = f(w) + tf'(w) \end{cases} \quad t \in (-\infty, +\infty)$$

$$\tau_\xi : \begin{cases} \Re(\tau_\xi(t)) = q_x + t\left(r\cos\vartheta\cos\left(\frac{\varphi_0^2}{2}\right) - r\sin\vartheta\sin\left(\frac{\varphi_0^2}{2}\right)\right) \\ \Im(\tau_\xi(t)) = q_y + t\left(r\sin\vartheta\cos\left(\frac{\varphi_0^2}{2}\right) + r\cos\vartheta\sin\left(\frac{\varphi_0^2}{2}\right)\right) \end{cases} \quad t \in (-\infty, +\infty),$$

where:

$$\begin{aligned} q_x &= x_0 + rC(\varphi_0)\cos\vartheta - rS(\varphi_0)\sin\vartheta \\ q_y &= y_0 + rC(\varphi_0)\sin\vartheta + rS(\varphi_0)\cos\vartheta \end{aligned}$$

Since the two lines must be tangent on the same point, we have that:

$$\begin{cases} q_x = w \Rightarrow x_0 = w - rC(\varphi_0)\cos\vartheta + rS(\varphi_0)\sin\vartheta \\ q_y = f(w) \Rightarrow y_0 = f(w) - rC(\varphi_0)\sin\vartheta - rS(\varphi_0)\cos\vartheta \end{cases}$$

And since the slope of the tangent must be equal in the two cases, we get:

$$f'(w) = \frac{r\sin\vartheta\cos\left(\frac{\varphi_0^2}{2}\right) + r\cos\vartheta\sin\left(\frac{\varphi_0^2}{2}\right)}{r\cos\vartheta\cos\left(\frac{\varphi_0^2}{2}\right) - r\sin\vartheta\sin\left(\frac{\varphi_0^2}{2}\right)} \Rightarrow f'(w) = \tan\left(\vartheta + \frac{\varphi_0^2}{2}\right),$$

where in the second step we have used the reverse addition formulas of sine and cosine.
In addition to that we have that the curvature of γ and ξ in the point of the approximation must be equal:

$$\frac{\varphi_0}{r} = \frac{|f''(w)|}{(1 + f'(w)^2)^{\frac{3}{2}}} \Rightarrow \varphi_0 = r\frac{|f''(w)|}{(1 + f'(w)^2)^{\frac{3}{2}}}$$

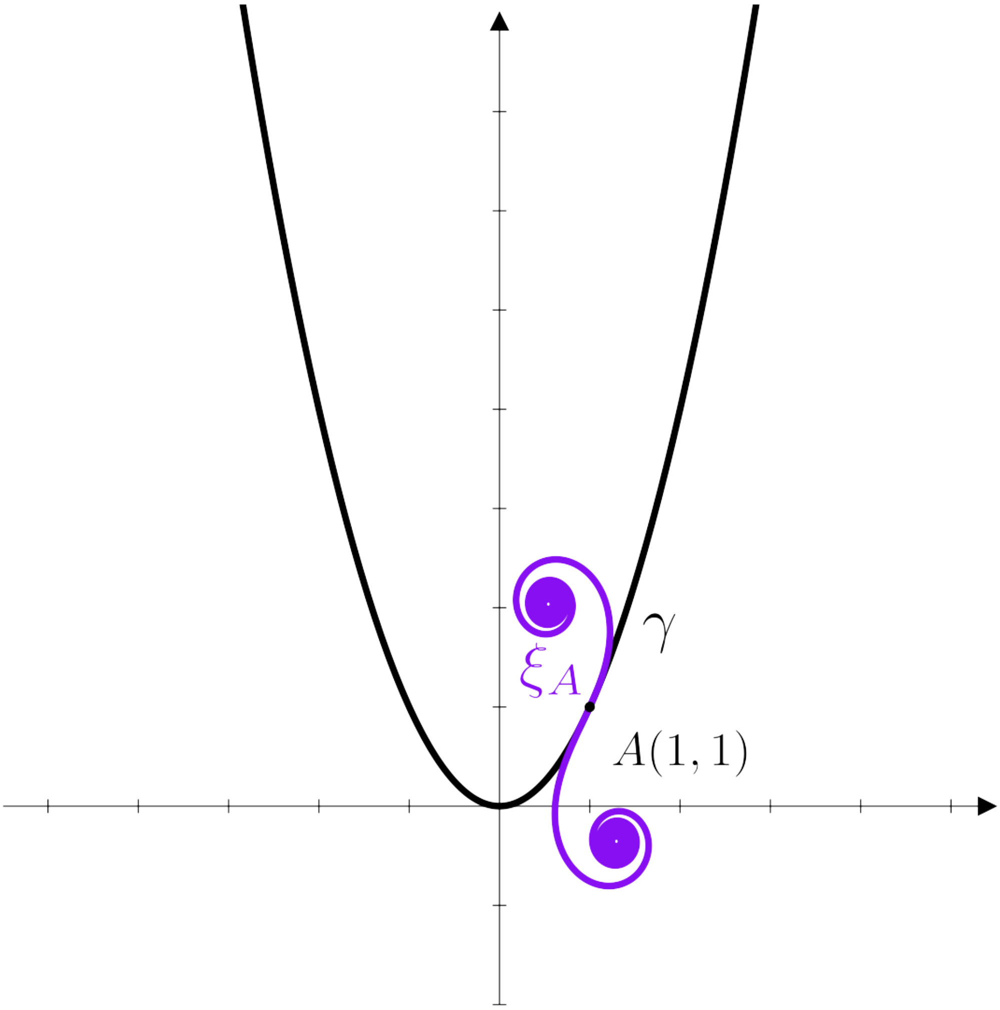
Therefore we can express ϑ as:

$$\begin{aligned} \vartheta + \frac{\varphi_0^2}{2} &= \arctan\left(f'(w)\right) \qquad \vee \qquad \vartheta + \frac{\varphi_0^2}{2} = \pi + \arctan\left(f'(w)\right) \\ \vartheta &= \arctan\left(f'(w)\right) - \frac{\varphi_0^2}{2} \qquad \vee \qquad \vartheta = \pi + \arctan\left(f'(w)\right) - \frac{\varphi_0^2}{2} \end{aligned}$$

Eventually, we can write the equation of the approximating spiral as:

$$
\xi_{1w}: \begin{cases}
\Re(\xi_w(t)) = w \\
\qquad\qquad - r\cos\left(\arctan\left(f'(w)\right) - r^2 \dfrac{\left(f''(w)\right)^2}{2\left(1+f'(w)^2\right)^3}\right)\left[C\left(\dfrac{r|f''(w)|}{\left(1+f'(w)^2\right)^{\frac{3}{2}}}\right) - C(t)\right] \\
\qquad\qquad - r\sin\left(\arctan\left(f'(w)\right) - r^2 \dfrac{\left(f''(w)\right)^2}{2\left(1+f'(w)^2\right)^3}\right)\left[S(t) - S\left(\dfrac{r|f''(w)|}{\left(1+f'(w)^2\right)^{\frac{3}{2}}}\right)\right] \\
\Im(\xi_w(t)) = f(w) \\
\qquad\qquad - r\sin\left(\arctan\left(f'(w)\right) - r^2 \dfrac{\left(f''(w)\right)^2}{2\left(1+f'(w)^2\right)^3}\right)\left[C\left(\dfrac{r|f''(w)|}{\left(1+f'(w)^2\right)^{\frac{3}{2}}}\right) - C(t)\right] \\
\qquad\qquad - r\cos\left(\arctan\left(f'(w)\right) - r^2 \dfrac{\left(f''(w)\right)^2}{2\left(1+f'(w)^2\right)^3}\right)\left[S\left(\dfrac{r|f''(w)|}{\left(1+f'(w)^2\right)^{\frac{3}{2}}}\right) - S(t)\right]
\end{cases}
$$

$$
\xi_{2w}: \begin{cases}
\Re(\xi_w(t)) = w \\
\qquad\qquad - r\cos\left(\pi + \arctan\left(f'(w)\right) - r^2 \dfrac{\left(f''(w)\right)^2}{2\left(1+f'(w)^2\right)^3}\right)\left[C\left(\dfrac{r|f''(w)|}{\left(1+f'(w)^2\right)^{\frac{3}{2}}}\right) - C(t)\right] \\
\qquad\qquad - r\sin\left(\pi + \arctan\left(f'(w)\right) - r^2 \dfrac{\left(f''(w)\right)^2}{2\left(1+f'(w)^2\right)^3}\right)\left[S(t) - S\left(\dfrac{r|f''(w)|}{\left(1+f'(w)^2\right)^{\frac{3}{2}}}\right)\right] \\
\Im(\xi_w(t)) = f(w) \\
\qquad\qquad - r\sin\left(\pi + \arctan\left(f'(w)\right) - r^2 \dfrac{\left(f''(w)\right)^2}{2\left(1+f'(w)^2\right)^3}\right)\left[C\left(\dfrac{r|f''(w)|}{\left(1+f'(w)^2\right)^{\frac{3}{2}}}\right) - C(t)\right] \\
\qquad\qquad - r\cos\left(\pi + \arctan\left(f'(w)\right) - r^2 \dfrac{\left(f''(w)\right)^2}{2\left(1+f'(w)^2\right)^3}\right)\left[S\left(\dfrac{r|f''(w)|}{\left(1+f'(w)^2\right)^{\frac{3}{2}}}\right) - S(t)\right]
\end{cases}
$$

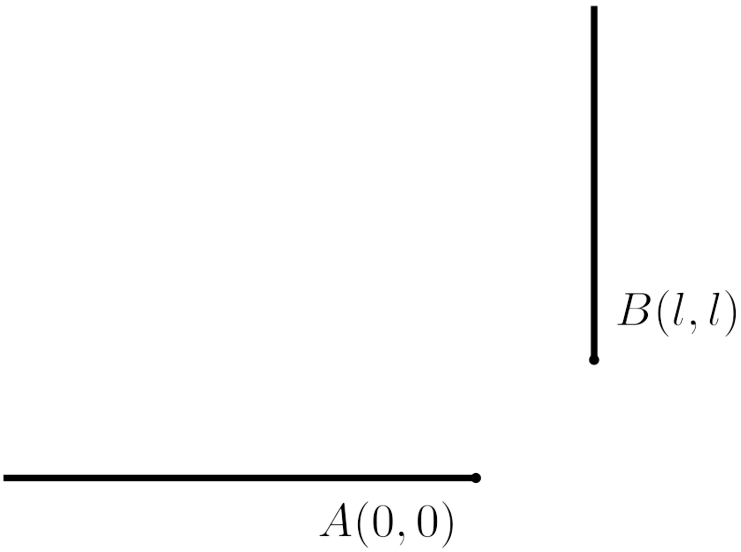


3. Connecting two streets

After finding how to approximate a generic curve with an Euler spiral, in this next section we are going to search for a way to connect two streets using two Euler spirals.

3.1. Perpendicular symmetric roads

We will first begin by finding how to join two orthogonal roads, symmetric with respect to a line.
Without loss of generality, we can place our coordinate system on the endpoint of one of the two streets:



Thus, the point B will have coordinates $B(l, l)$, with $l \in \mathbb{R} - \{0\}$
The two spirals ξ_A and ξ_B we are looking for will have to satisfy the following restrictions:

- $A \in \xi_A$ and $B \in \xi_B$;
- Both the curvature of ξ_A in A and the one of ξ_B in B must be zero;(8)
- The two spirals must be symmetric with respect to the same line of the streets;
- The two spirals must only meet once at the point T, where they also must have the same tangent.

Therefore we can write the equation of ξ_A as an Euler spiral centered on the origin:

$$\xi_A \begin{cases} x(t) = aC(t) \\ y(t) = aS(t) \end{cases} \quad t \in (-\infty, +\infty)$$

The spiral ξ_B, instead, will be translated of the vector (l, l). Furthermore, since it must be symmetric to ξ_A, the parametric equation of ξ_B will have the sine and cosine integrals inverted:

$$\xi_B \begin{cases} x(t) = l + aS(t) \\ y(t) = l + aC(t) \end{cases} \quad t \in (-\infty, +\infty)$$

We can write the tangent lines to the two spirals as:

$$\tau_A \begin{cases} x(t) = aC(\varphi) + t \cdot a\cos\left(\dfrac{\varphi_0^2}{2}\right) \\ y(t) = aS(\varphi) + t \cdot a\sin\left(\dfrac{\varphi_0^2}{2}\right) \end{cases} \quad t \in (-\infty, +\infty)$$

$$\tau_B \begin{cases} x(t) = l + aS(-\varphi) + t \cdot a\sin\left(\dfrac{\varphi_0^2}{2}\right) \\ y(t) = l + aC(-\varphi) + t \cdot a\cos\left(\dfrac{\varphi_0^2}{2}\right) \end{cases} \quad t \in (-\infty, +\infty)$$

Given that, for the fourth condition, the two spirals must have the same tangent at the intersection point, the slopes of the tangents themselves must be the same:

$$\frac{a\cos\left(\frac{\varphi_0^2}{2}\right)}{a\sin\left(\frac{\varphi_0^2}{2}\right)} = \frac{a\sin\left(\frac{\varphi_0^2}{2}\right)}{a\cos\left(\frac{\varphi_0^2}{2}\right)} \Rightarrow \frac{\sin^2\left(\frac{\varphi_0^2}{2}\right)}{\cos^2\left(\frac{\varphi_0^2}{2}\right)} = 1 \Rightarrow \tan\left(\frac{\varphi_0^2}{2}\right) = 1$$

Such equation leads us to the solutions:

$$\frac{\varphi_0^2}{2} = \frac{\pi}{4} \quad \vee \quad \frac{\varphi_0^2}{2} = \pi + \frac{\pi}{4}$$

$$\varphi_0 = \pm\sqrt{\frac{\pi}{2}} \quad \vee \quad \varphi_0 = \pm\sqrt{\frac{5\pi}{2}}$$

Once again, for the fourth condition, we have that the two tangents must be the same line:

$$\begin{cases} l + aS(-\varphi) = aC(\varphi) \\ l + aC(-\varphi) = aS(\varphi) \end{cases}$$

The two equations are clearly equivalent and they both have the following solution:

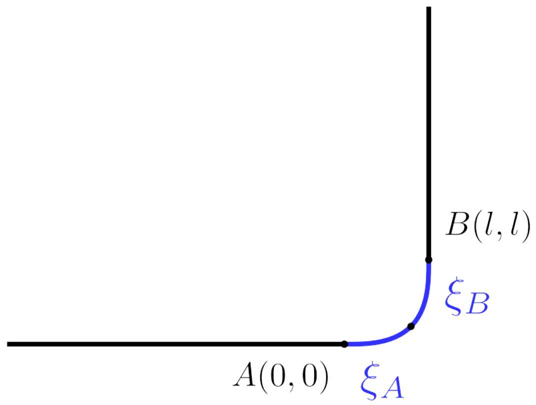
$$a = \frac{l}{C(\varphi) - S(-\varphi)} = \frac{l}{C(\varphi) + S(\varphi)}$$

The values of a will eventually be:

$$a = \frac{\pm l}{C\left(\sqrt{\frac{\pi}{2}}\right) + S\left(\sqrt{\frac{\pi}{2}}\right)}$$

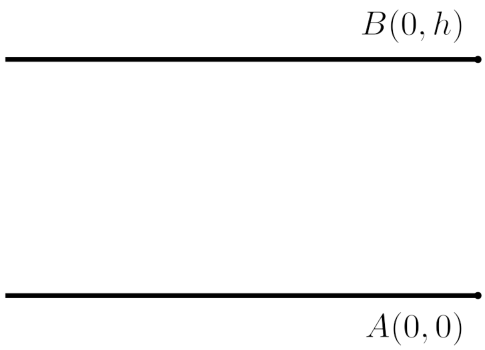
$$\vee$$

$$a = \frac{\pm l}{C\left(\sqrt{\frac{5\pi}{2}}\right) + S\left(\sqrt{\frac{5\pi}{2}}\right)}$$



3.2. Parallel roads

Let us now move on considering two parallel roads:



We can write the equations of the two arcs of spiral as:

$$\xi_A \begin{cases} x(t) = aC(t) \\ y(t) = aS(t) \end{cases} \quad t \in (-\infty, +\infty)$$

$$\xi_B \begin{cases} x(t) = aC(t) \\ y(t) = h - aS(t) \end{cases} \quad t \in (-\infty, +\infty)$$

The tangents, instead, can be expressed as:

$$\tau_A \begin{cases} x(t) = aC(\varphi_0) + t \cdot a\cos\left(\frac{\varphi_0^2}{2}\right) \\ y(t) = aS(\varphi_0) + t \cdot a\sin\left(\frac{\varphi_0^2}{2}\right) \end{cases} \quad t \in (-\infty, +\infty)$$

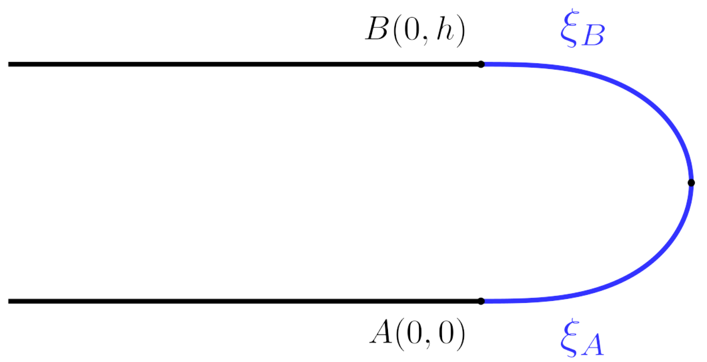
$$\tau_B \begin{cases} x(t) = aC(\varphi_0) + t \cdot a\cos\left(\frac{\varphi_0^2}{2}\right) \\ y(t) = h - aS(\varphi_0) - t \cdot a\sin\left(\frac{\varphi_0^2}{2}\right) \end{cases} \quad t \in (-\infty, +\infty)$$

Since the tangent must be vertical on the joining point of the arcs:

$$\cos\left(\frac{\varphi_0^2}{2}\right) = 0 \Rightarrow \frac{\varphi_0^2}{2} = \frac{\pi}{2} \quad \vee \quad \frac{\varphi_0^2}{2} = \frac{3\pi}{2} \Rightarrow \varphi_0 = \pm\sqrt{\pi} \quad \vee \quad \varphi_0 = \pm\sqrt{3\pi}$$

And since it must be the same line, we get:

$$h - aS(\pm\sqrt{\pi}) = aS(\pm\sqrt{\pi}) \to a = \frac{\pm h}{2S(\sqrt{\pi})}$$



3.3. Symmetric roads

Let us proceed by considering now two roads where the second one is obtained by rotating the first one by an angle $\vartheta \in \mathbb{R}^+$ such that $0 < \vartheta < \frac{\pi}{2}$ around a point.
To this aim we can place our coordinate system so that the first street starts at the point $(-l, 0)$, with $l \in \mathbb{R}^+$, as shown in figure (2):

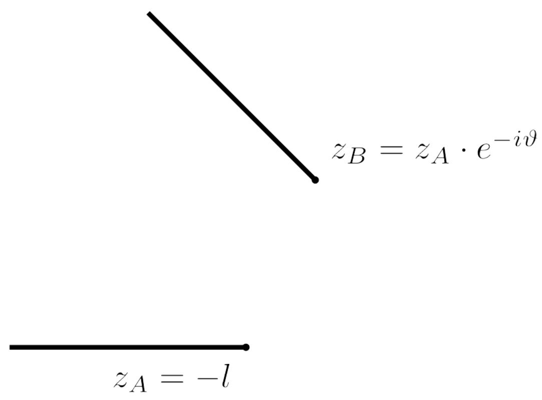


Figure 2: Symmetric roads

The conditions of the joint will be the same as before.
To be able to better handle the rotations, we can once again consider the curves in the complex plane $\mathbb{C}$. The equation of the Euler spiral can thus be written as:

$$\xi_A(t) = -l + aC(t) + iaS(t) \qquad t \in (-\infty, +\infty)$$

To express ξ_B, instead, we have to first flip ξ_A over the real axis and then rotate it of $-\vartheta$. The second spiral therefore becomes:

$$\xi_B(t) = \overline{\xi_A}(t) \cdot e^{-i\vartheta} \qquad t \in (-\infty, +\infty)$$

$$\xi_B : \begin{cases} \Re(\xi_B(t)) = \cos\vartheta\,(-l + aC(t)) - aS(t)\sin\vartheta \\ \Im(\xi_B(t)) = -\sin\vartheta\,(-l + aC(t)) - aS(t)\cos\vartheta \end{cases} \qquad t \in (-\infty, +\infty)$$

The two tangents, as usual, can be computed as:

$$\tau_A : \begin{cases} \Re(\tau_A(t)) = -l + aC(\varphi_0) + ta\cos\left(\frac{\varphi_0^2}{2}\right) \\ \Im(\tau_A(t)) = aS(\varphi_0) + ta\sin\left(\frac{\varphi_0^2}{2}\right) \end{cases} \qquad t \in (-\infty, +\infty)$$

$$\tau_B : \begin{cases} \Re(\tau_B(t)) = q_x + ta\left(\cos\vartheta\cos\left(\frac{\varphi_0^2}{2}\right) - \sin\vartheta\sin\left(\frac{\varphi_0^2}{2}\right)\right) \\ \Im(\tau_B(t)) = q_y + ta\left(-\sin\vartheta\cos\left(\frac{\varphi_0^2}{2}\right) - \cos\vartheta\sin\left(\frac{\varphi_0^2}{2}\right)\right) \end{cases} \quad t \in (-\infty, +\infty)$$

Where:

$$q_x = -l\cos\vartheta + aC(\varphi_0)\cos\vartheta + aS(\varphi_0)\sin\vartheta$$
$$q_y = l\sin\vartheta - aC(\varphi_0)\sin\vartheta - aS(\varphi_0)\cos\vartheta$$

Since the slopes must be equal, we have that:

$$\frac{\Delta y_A}{\Delta x_A} = \frac{\Delta y_B}{\Delta x_B} \Rightarrow \frac{a\sin\left(\frac{\varphi_0^2}{2}\right)}{a\cos\left(\frac{\varphi_0^2}{2}\right)} = \frac{-a\sin\vartheta\cos\left(\frac{\varphi_0^2}{2}\right) - a\cos\vartheta\sin\left(\frac{\varphi_0^2}{2}\right)}{a\cos\vartheta\cos\left(\frac{\varphi_0^2}{2}\right) - a\sin\vartheta\sin\left(\frac{\varphi_0^2}{2}\right)}$$
$$\tan\left(\frac{\varphi_0^2}{2}\right) = -\tan\left(\vartheta + \frac{\varphi_0^2}{2}\right)$$

Solving for φ_0, we get:

$$\varphi_0 = \pm\sqrt{-\vartheta} \qquad \vee \qquad \varphi_0 = \pm\sqrt{\pi - \vartheta}$$

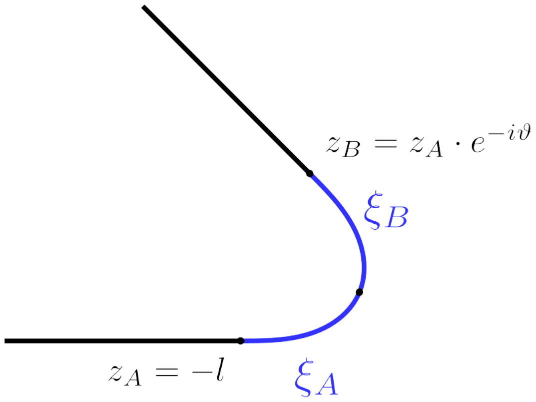
Considering now that the real part of the two curves at the junction point must be the same, we get:

$$-l + aC(\varphi_0) = -l\cos\vartheta + aC(\varphi_0)\cos\vartheta - aS(\varphi_0)\sin\vartheta$$

$$a = \frac{l(\cos\vartheta - 1)}{C(\varphi_0)\cos\vartheta - S(\varphi_0)\sin\vartheta - C(\varphi_0)}$$

Finally, replacing φ_0:

$$a = \frac{l(\cos\vartheta - 1)}{(\cos\vartheta - 1)C(\pm\sqrt{\pi - \vartheta}) - S(\pm\sqrt{\pi - \vartheta})\sin\vartheta}$$


$z_B = z_A \cdot e^{-i\vartheta}$
ξ_B
$z_A = -l$
ξ_A

Notes d'édition

(1) Dans le plan, cherchons à approcher des courbes par des lignes simples au voisinage d'un point. En première approximation, une droite, la tangente, fait l'affaire. On peut obtenir une meilleure approximation en remplaçant la tangente par un cercle, voir Figure 3. Comment choisir son rayon ? La direction de la tangente d'une courbe est donnée par son angle polaire φ, qui varie au cours du temps. La particularité du cercle, c'est que si il est parcouru à vitesse constante 1, alors φ est une fonction affine du temps t, et la dérivée $\frac{d\varphi}{dt}$ est l'inverse du rayon. En général, pour une courbe parcourue à vitesse constante 1, la fonction $\frac{d\varphi}{dt}$ s'appelle la courbure, son inverse (en valeur absolue) est le rayon du cercle osculateur, celui qui approxime le mieux la courbe en un point. Le signe de la courbure indique si la courbe tourne vers la droite ou vers la gauche. Lorsque la courbe est exprimée en coordonnées cartésiennes $(x(t); y(t))$, avec un paramétrage quelconque, la courbure est donnée par la formule suivante

$$\text{courbure} = (x'(t)y"(t) - x"(t)y'(t))/(x'(t)^2 + y'(t)^2)^{3/2}.$$

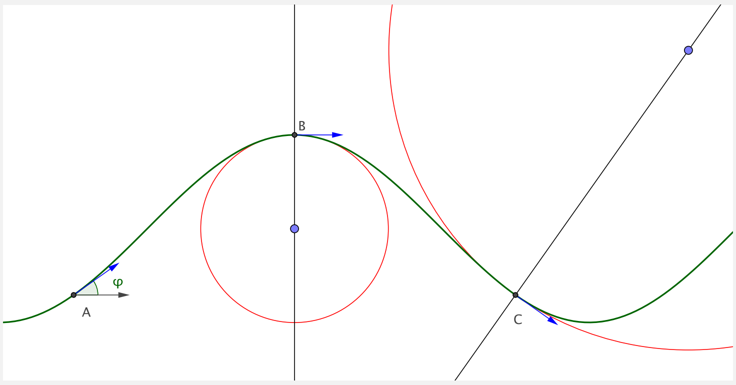


Figure 3: Courbure

(2) Concrètement, dans le cadre d'un trajet en voiture, la notion de courbure est très facile à comprendre. Cela signifie pour le conducteur que sur une ligne droite (courbure nulle), le volant est droit. Sur un arc de cercle (courbure constante), le volant est tourné d'un angle fixe. Passer directement d'une droite à un cercle obligerait le conducteur à tourner le volant de façon instantanée, provoquant une force centrifuge brusque due au changement de direction brutal. La spirale d'Euler permet de tourner le volant à une vitesse constante.

(3) Ces intégrales ne peuvent pas être calculées simplement à l'aide de fonctions usuelles mais peuvent s'approcher par des approximations numériques. On ne peut pas les "calculer" à la main.

(4) Il faut comprendre ici que φ_0 la valeur du paramètre au point précis de tangence entre la spirale et la courbe.

(5) Formellement, il y a une infinité de solutions : $\frac{\varphi_0^2}{2} = k\pi + \arctan(f'(w))$. Cependant, les auteurs n'en retiennent que deux car physiquement les autres se superposent géométriquement sur les deux sélectionnées. En effet, l'angle qui se trouve à l'intérieur de la tangente représente l'inclinaison de la route à un instant donné et les deux solutions sélectionnées correspondent aux deux sens de la route. Cette remarque est reproductible en plusieurs endroits dans l'article.

(6) La notation $\dot{x}$ est une autre façon classique pour noter la dérivée $x'(t)$.

(7) Le calcul de la courbure peut être détaillé. En dérivant par rapport à t, on obtient avec le Théorème Fondamental de l'Analyse :

$$\dot{x}(t) = a\cos\left(\frac{t^2}{2}\right) \text{ et } \dot{y}(t) = a\sin\left(\frac{t^2}{2}\right)$$

En dérivant une seconde fois, on obtient :

$$\ddot{x}(t) = -at\sin\left(\frac{t^2}{2}\right) \text{ et } \ddot{y}(t) = at\cos\left(\frac{t^2}{2}\right)$$

Le terme du numérateur de la courbure donne alors (après factorisation par a^2t) :

$$\dot{x}\ddot{y} - \dot{y}\ddot{x} = a^2t\left[\cos^2\left(\frac{t^2}{2}\right) + \sin^2\left(\frac{t^2}{2}\right)\right].$$

De plus, puisque $\cos^2(\theta) + \sin^2(\theta) = 1$, le numérateur (en valeur absolue) se simplifie en :

$$|\dot{x}\ddot{y} - \dot{y}\ddot{x}| = a^2|t|$$

En injectant les dérivées dans le dénominateur et en utilisant la même identité trigonométrique, on obtient alors :

$$(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}} = (a^2)^{\frac{3}{2}} = |a^3|$$

Enfin, en regroupant ces deux termes, on trouve la courbure :

$$\kappa(t) = \left|\frac{t}{a}\right|.$$

(8) Les courbures sont souhaitées nulles en A et B, les jonctions, car l'objectif est de raccorder deux routes qui ont des courbures nulles (droites).