A VERY SHORT ALPHABET

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Druhus Daria Elena (student, 9th grade), Manole Maria (student, 9th grade), Rotariu Diana Gabriela (student, 9th grade)

School: Colegiul National “Costache Negruzzi” Iași, Romania

Teacher: Ph.D. Ioana Catalina Anton

Researcher: Ph.D. Iulian Stoleriu, University “Alexandru Ioan Cuza” Iasi, Romania

Abstract

We choose a complex problem which it’s very useful in everyday life for creating and decoding passwords (1). How many times were you uncertain about your security level on social media or your smartphone and your computer? The problem is in the way we create passwords. But the real problem is not for us, ordinary people, but for big companies because it is very important and even essential to them and perhaps for the national security to make a very strong password. This problem may give us a short insight on how word passwords could be created when specific requirements for the letter arrangements have to be satisfied.

The problem

We are given a tiny alphabet, which is formed by only four letters, $\alpha, \beta, \gamma, \delta$. The tasks are:

a) Find the number of seven distinct letter words containing an even number of α.

b) How many seven distinct letter words in this alphabet do not contain two consecutive α?

c) Find the number of all seven distinct letter words that do not contain any of the sequences $\gamma\delta$ and $\delta\gamma$.

d) How many seven distinct letter words can be formed considering all the above conditions?

Method. We shall solve the given tasks using suitable recurrence relations for C_n, which is the number of ways to construct n-letter words using the letters of the given short alphabet.

First of all we decided to work in a classical way: with trials, then we realized that we can solve the problem easier with the principles of recurrence and multiplication (2). We use them as rules for the words formed with n letters respecting the conditions mentioned above.

We know, thanks to the principle of multiplication that we have a total of 4^n words. Using unknown quantities for the number of the words with even number of letters of α and odd number of letters of α, we find recurrence equations.

We analyse according to the first letter, which can be either α, either one of the three ones remaining, and we create recurrences.

Solution of the problem

a) At the beginning we tried to count the words grouping them depending on the number of letters α. It can be 2, 4 or 6. For the case of 6 appearances of α, it is really easy to count them because there are only 6 words. But, for the other cases one might to forget to analyse some of them. We tried to calculate at a time depending on the first letter of α, the second and so on. Because it was not a certain result, we tried a different way. Let

$P_n =$ number of *n*-letter words that contain an ***even number of α,***
$I_n =$ number of distinct words with n letters, which contain an ***odd number of α.***

Let $a_1, a_{2,} a_3, \dots, a_n$ be the letters of the words which have n letters. So a_1 is the first letter of the words with n letters, a_2 the second letter and so on. All of them can have the values mentioned in the problem: $\alpha, \beta, \gamma, \delta$.

A particular case

For $n = 4$, let $P_4 = P_{(=\alpha)} + P_{(\neq\alpha)}$, where $P_{(=\alpha)}$ is the number of words with the first letter α, and $P_{(\neq\alpha)}$ is the number of words with the first letter different from α, so β, γ or δ.

For $a_1 = \alpha$, $P_{(=\alpha)} = I_3$, as the rest of the word contains an odd number of α and has length 3.
For $a_1 \neq \alpha$, $P_{(\neq\alpha)} = 3P_3$, as a_1 can have 3 values, and the remaining 3 letters of the words represent the words with 3 letters and an even number of α, so P_3words.

Using the equation $P_4 = P_{(=\alpha)} + P_{(\neq\alpha)}$ and replacing the unknown numbers with the values obtained in the above equations we get

$$P_4 = I_3 + 3P_3$$

In a similar way, we can assume a recurrence relation for I_n based on the principle used for P_4. We get that $I_{(=\alpha)} = P_3$ and $I_{(\neq\alpha)} = 3I_3$. From these 2 relations we get

$$I_4 = P_3 + 3I_3$$

By adding these two relations we get

$$P_4 + I_4 = 4^4$$

General case

To obtain the general case we will use the principle of multiplication. For a letter we have 4 possibilities ($\alpha, \beta, \gamma, \delta$) and we have 4^n words in total. From the particular case and from the principle of multiplication we deduce the general case:

$$P_n + I_n = 4^n$$

We have two cases for the first letter in the word. Using the equation $P_4 = P_{(=\alpha)} + P_{(\neq\alpha)}$ and replacing the unknown numbers with the values obtained in the above equations we get

$$P_n = I_{n-1} + 3P_{n-1}$$

For $a_1 = \alpha$, $P_{(=\alpha)} = I_{n-1}$, as the rest of the word contains an odd number of α and has length $n-1$. For $a_1 \neq \alpha$,$P_{(\neq\alpha)} = 3P_{n-1}$, as a_1 can have 3 values, and the remaining $n-1$ letters of the words represent the words with $n-1$ letters and an even number of α, so P_{n-1} words.

In a similar way, we can deduce a recurrence relation for I_n based on the principle used for P_n. We get that that $I_{(=\alpha)} = P_{n-1}$ and $I_{(\neq\alpha)} = 3I_{n-1}$. From these two relations we get

$$I_n = P_{n-1} + 3I_{n-1}$$

We get now these new recurrences, based on what we have noticed in the example above,

$$\begin{cases} P_n = I_{n-1} + 3P_{n-1} \\ I_n = P_{n-1} + 3I_{n-1} \end{cases}$$

$$P_n - I_n = 2P_{n-1} - 2I_{n-1} = 2(\, P_{n-1} - I_{n-1})$$

Let $x_n = P_n - I_n$. We have a recurrence: $x_n = 2x_{n-1} = 2^2x_{n-2} = \ 2^3x_{n-3} = \cdots = 2^{n-1}x_1$. We replace the unknowns from the recurrence created:

$$x_1 = P_1 - I_1 = 3 - 1 = 2;\ \ x_2 = 2 \times 2 = 4;\ \ x_n = P_n - I_n = 2 \times 2^{n-1} = 2^n \text{ for all } n \geq 1.$$

We know that $P_n + I_n = 4^n$:

$$\begin{cases} P_n + I_n = 4^n \\ P_n - I_n = 2^n \end{cases}$$

Adding these equations, we get $2P_n = 4^n + 2^n$ and so

$$P_n = 2^{n-1}(2^n + 1)$$

$$P_7 = 2^6 \times (2^7 + 1) = 64 \times 129 = \mathbf{8256} \textbf{ words.}$$

b) Let A_n be the number of words with n letters that don't contain a double α.

We are looking for a relation for A_n. We judge by the first letter of the word.

First case

If the first letter $a_1 \neq \alpha$, then we have $3A_{n-1}$ words as a_1 can have 3 values (β, γ, δ) and the rest of the words remaining from a_2 to a_n have the length $n-1$.

So the number of the words will be $3 \times A_{n-1}$.

Second case

If the first letter $a_1 = \alpha$ then there are $3A_{n-2}$ words, as $a_2 \neq \alpha$, so a_2 can have 3 values (β, γ, δ) and the rest of the words remaining from a_3 to a_n have the length $n-2$.

So the number of the words will be $3 \times A_{n-2}$.

In conclusion

$$A_n = 3A_{n-1} + 3A_{n-2} \text{ for all } n \geq 3$$

We can easily see that $A_1 = 4$ and $A_2 = 4 \times 4 - 1 = 15$.

For $n = 3$, we have $A_3 = 3A_2 + 3A_1 = 3 \times 15 + 3 \times 4 = 57$.

For $n = 4$, we have $A_4 = 3 \times (57 + 15) = 216$.

For $n = 5$, we have $A_5 = 3 \times (216 + 57) = 819$.

For $n = 6$, we have $A_6 = 3 \times (819 + 216) = 3105$.

For $n = 7$, we have $A_7 = 3 \times (3105 + 819) =$ **11772 words**.

c) Let us denote by C_n the number of n-letter words with the required property, $n \geq 1$. A n-letter word will be of the form $a_1 a_2 \dots a_n$.

In order to count how many such words we can write, we judge by the choice of the first letter in the word. We can distinguish three main cases:

First case

If $a_1 = \alpha$ or $a_1 = \beta$, the number of words will be $2C_{n-1}$ because we have 2 values for a_1 and the rest of the word is formed by the letters from a_2 to a_n, so is a word formed of $n-1$ letters which has the same conditions as mentioned and the number of such words is C_{n-1}.

Second case

If $a_1 = \delta$, the number of words will be the number of words with n letters for which the second letter is not γ and have length $n-1$, so C_{n-1} minus the number of $(n-1)$-letter words for which $a_2 = \gamma$.

Third case

If $a_1 = \gamma$, the number of words will be the number of words with n letters for which the second letter is not δ and have length $n-1$, so C_{n-1} minus the number of $(n-1)$-letter words for which $a_2 = \delta$.

Therefore, taking into account all these three cases, the number of words of n-words with the required property will be:

$$C_n = 2C_{n-1} + \{C_{n-1} \text{ minus the number of } (n-1)\text{-letter words for which } a_2 = \gamma\} + \{C_{n-1} \text{minus the number of } (n-1)\text{-letter words for which } a_2 = \delta\}$$

= $4C_{n-1} -$ {the number of $(n-1)$-letter words for which $a_2 = \gamma$ or $a_2 = \delta$}

= $4C_{n-1}$ $-\{C_{n-1}$minus the number of $(n-1)$-letter words for which $a_2 \neq \gamma$ and $a_2 \neq \delta\}$

= $4C_{n-1} - \{C_{n-1} - 2C_{n-2}\}$.

In conclusion, the recurrence relation is

$$C_n = 3C_{n-1} + 2C_{n-2} \text{ for all } n \geq 3$$

$$C_1 = 4$$
$$C_2 = 4 \times 4 - 2 = 14$$
$$C_3 = 3 \times 14 + 2 \times 4 = 50$$
$$C_4 = 3 \times 50 + 2 \times 14 = 178$$
$$C_5 = 3 \times 178 + 2 \times 50 = 634$$
$$C_6 = 3 \times 634 + 2 \times 178 = 2258$$
$$C_7 = 3 \times 2258 + 2 \times 434 = \mathbf{8042}$$

d) For any finite aggregate A we will note card (A) or $|A|$ the number of its elements.

$$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |B \cap C| + |C \cap A|) + |A \cap B \cap C|$$

We let this as a challenge for you! ☺ (3)

Editing Notes

(1) Actually the question of creating, coding and decoding passwords has very little to do with the problem dealt with in this article, which is combinatorial (see the text of the problem).

(2) The principle of multiplication in combinatorics is that if you have several *independent* choices, the total number of possibilities is the product of the numbers of possibilities for each choice. Here, for example, for a n-letter word you can choose independently the different letters among the 4 letters of the alphabet, so you have $4 \times 4 \times \cdots \times 4 = 4^n$ words of length n.

(3) The formula above is not difficult to prove, from the fact that each point in the union must be counted only once; for example we have to subtract $|A \cap B|$ since a point in $A \cap B$ but will be counted twice, in $|A|$ and in $|B|$; then, after subtracting the three intersections of two sets, the points in $A \cap B \cap C$ have to be added once.
Now, the real challenge is to find the number of elements in each of the intersections, which can be as difficult as the questions dealt with so far.