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A dice game

[Year 2019 – 2020]

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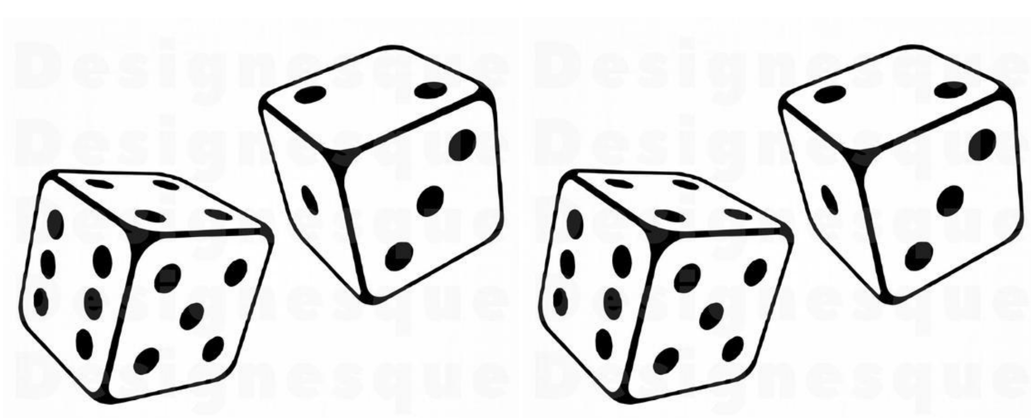
1. PRESENTATION OF THE RESEARCH TOPIC

The paper deals with the winning chances in a game of dice. We firstly consider the distribution of the sum of a set of four classical dice that are rolled together, then we consider other types of dice, which take the form of the remaining four Platonic solids.

2. BRIEF PRESENTATION OF THE CONJECTURES AND RESULTS OBTAINED

At a certain casino, four fair classical dice are thrown simultaneously. If a player bets on a certain sum, what are the chances of winning? Find the most probable sum of the points that appear on the top faces and the expected number of throws needed to obtain it for the first time?

You may also consider similar questions for the case of four identical dice that take the form of a general regular convex polyhedron.



THE SOLUTION

At a certain casino, four fair dice are thrown simultaneously. If a player bets on a certain sum, what are the chances of winning? What is the most probable sum and what is the expected number of throws needed to obtain it for the first time?

Method I.

Let's suppose that the numbers shown on the four dice after one throw are x_1, x_2, x_3, x_4, where $x_i \in \{1, 2, 3, 4, 5, 6\}$, $i = \overline{1,4}$.

Denote the sum of these values by $S_k = k = x_1 + x_2 + x_3 + x_4$ and the number of possible cases as P_k. We observe that $S_k \in \{4, 5, 6, ..., 23, 24\}$. Thus, it can be attributed 21 values. We will prove that the sum with the highest chance of appearance is $S_{14} = 14$ which means that P_{14} is the greatest out of the numbers P_k, $k = \overline{4, 24}$.

Because the equation:

$$x_1 + x_2 + x_3 + x_4 = k \text{, whilst } x_i \in \{1, 2, 3, 4, 5, 6\},\ i = \overline{1,4}$$

is equivalent to the equation:

$$(7 - x_1) + (7 - x_2) + (7 - x_3) + (7 - x_4) = 28 - k \text{, whilst } 7 - x_i \in \{1, 2, 3, 4, 5, 6\},\ i = \overline{1,4},$$

we have $P_k = P_{28-k}$, for any $k \in \{4, 5, ..., 13\}$.

Thus, it suffices to determine $P_4, P_5, ..., P_{13}$ and P_{14}. To do this, we shall prove the following lemma:

Lemma. The number of solutions of the equation:

(1) $$x_1 + x_2 + x_3 + x_4 = k,$$

where x_1, x_2, x_3, x_4, k are positive integers and $k \geq 4$ is equal to $\binom{k-1}{3}$.

Proof. Any solution of equation (1) can be obtained by placing the 3 *sticks* between the terms of a string of *k ones* (between two *terms* we can place no more than one *stick*). For example, the figure:

$$\underset{x_1=3}{1\,.\,1\,.\,1}\Big|\underbrace{1.1.1.1.1}_{x_2=5}\Big|\underbrace{1.1.1.1.1}_{x_3=5}\Big|\underset{x_4=1}{1}$$

represents the solution $x_1=3, x_2=5, x_3=5, x_4=1$ of the equation $x_1+x_2+x_3+x_4=14$.

The number of solutions of equation (1) is equal to the number of ways in which we can choose three *places* to place the three *sticks* out of $k-1$ *spaces* between the terms of the string. Thus, the result is $\binom{k-1}{3}$.

Let's return to the initial idea of finding the number of possibilities $P_4, P_5, ..., P_{13}$ and P_{14}.

According to the **Lemma**, equation (1) has $\binom{k-1}{3}$ solutions for any $k\in\{4,5,...,13,14\}$. But because x_1, x_2, x_3, x_4 are numbers written on dice, we have $x_i\in\{1,2,3,4,5,6\}$, $i=\overline{1,4}$, so some of the $\binom{k-1}{3}$ solutions will not work.

If (x_1, x_2, x_3, x_4) is a solution of equation (1) and at least 2 of its components are greater than 6, then:

$$x_1+x_2+x_3+x_4\geq 1+1+7+7=16$$

Because $k\leq 14$, we know that at most one of the components of the solution (x_1, x_2, x_3, x_4) could be greater than 6.

For $k\in\{4,5,6,7,8,9\}$, all the numbers x_1, x_2, x_3, x_4 are less than or equal to 6, therefore, in each one of the 6 cases, the number of solutions for equation (1) is $\binom{k-1}{3}$, so $P_4=\binom{4-1}{3}=1$, $P_5=\binom{5-1}{3}=4$, $P_6=\binom{6-1}{3}=10$, $P_7=\binom{7-1}{3}=20$, $P_8=\binom{8-1}{3}=35$, $P_9=\binom{9-1}{3}=56$.

For $k=10$, equation (1) has, except for the valid solutions, the solutions: (7, 1, 1, 1), (1,7,1,1), (1,1,7,1), (1,1,1,7). So, $P_{10}=\binom{10-1}{3}-4=80$.

For $k = 11$, equation (1) has, other than the valid solutions, the solutions: (8, 1, 1, 1) with its permutations (4 in total) and (7, 2, 1, 1) with its permutations (12 in total). So, $P_{11} = \binom{11-1}{3} - (4+12) = 104$.

For $k = 12$, equation (1) has the following non-valid solutions: (9, 1, 1, 1) and its permutations (4 in total), (8, 2, 1 ,1) and its permutations (12 in total), (7, 3, 1, 1) and its permutations (12 in total) and (7, 2, 2, 1) and its permutations (12 in total). Thus, $P_{12} = \binom{12-1}{3} - (4+12+12+12) = 125$.

For $k = 13$, equation (1) has, other than the valid solutions, the solutions: (10, 1, 1, 1) and its permutations (4 in total), (9, 2, 1 ,1) and its permutations (12 in total), (8, 3, 1, 1) and its permutations (12 in total), (8, 2, 2, 1) and its permutations (12 in total), (7, 4, 1, 1) and its permutations (12 in total), (7, 3, 2, 1), and its permutations (24 in total) and (7, 2, 2, 2) and its permutations (4 in total). Hence, $P_{13} = \binom{13-1}{3} - (4+12+12+12+12+24+4) = 140$.

For $k = 14$, equation (1) has the following non-valid solutions: (11, 1, 1, 1) and its permutations (4 in total), (10, 2, 1 ,1) and its permutations (12 in total), (9, 3, 1, 1) and its permutations (12 in total), (9, 2, 2, 1) and its permutations (12 in total), (8, 4, 1, 1) and its permutations (12 in total), (8, 3, 2, 1), and its permutations (24 in total), (8, 2, 2, 2) and its permutations (4 in total), (7, 5, 1, 1) and its permutations (12 in total), (7, 4, 2, 1) and its permutations (24 in total), (7, 3, 3, 1) and its permutations (12 in total), and (7, 3, 2, 2) and its permutations (12 in total), so $P_{14} = \binom{14-1}{3} - (4+12+12+12+12+24+4+12+24+12+12) = 146$.

We organise all the results in the following table:

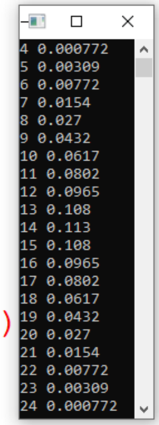
S_k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
P_k	1	4	10	20	35	56	80	104	125	140	146	140	125	104	80	56	35	20	10	4	1

Thus, the sum with the highest chance of appearing is $S_{14} = 14$.

Method II.

Due to the fact that $6^4 = 1296$ is a very small number (for a computer), we can take all the possibilities and count the sum at each step.

```
#include <bits/stdc++.h>
using namespace std;
double fr[40];
int main()
{
    for(int i=1;i<=6;i++)
        for(int j=1;j<=6;j++)
            for(int q=1;q<=6;q++)
                for(int l=1;l<=6;l++)
                    fr[i+j+q+l]++;
    for(int i=4;i<=24;i++)
        cout<<setprecision(3)<<i<<' '<<fr[i]/1296<<'\n';
    return 0;
}
```



We have written 21 lines, each line showing the sum and the probability of that sum to appear.

As we can see above, the sum $S_{14} = 14$ has the biggest chance of appearing. The probability of obtaining this sum is equal to:

$$p = \frac{number\ of\ \ favorable\ cases}{number\ of\ \ possbile\ cases} = \frac{146}{6^4} = \frac{146}{1296} = \frac{73}{648}$$

Let $q = 1 - p$.

Denote the number of necessary tries for obtaining sum 14 for the first time by the discrete random variable X and with p_k the probability of obtaining sum 14 for the first time after throw number k. If we don't succeed after $k-1$ tries and we do succeed after the next throw then, evidently, the first time we'll get the sum 14 is after throw number k. So, $p_k = q^{k-1} \cdot p$.

We have:

X	1	2	3	4	…	k	…
Prob.	$p_1 = p$	$p_2 = q \cdot p$	$p_3 = q^2 \cdot p$	$p_4 = q^3 \cdot p$	$\cdots$	$p_k = q^{k-1} \cdot p$	$\cdots$

The expected number of throws of the four dice until we observe sum 14 is:

$$E(X)=1\cdot p+2\cdot q\cdot p+3\cdot q^2\cdot p+...+k\cdot q^{k-1}\cdot p+...=\lim_{n\to\infty}\sum_{k=1}^{n}k\cdot q^{k-1}\cdot p\,.$$

Now, we need to compute the sum:

$$E_n=\sum_{k=1}^{n}k\cdot q^{k-1}\cdot p\,.$$

We have:

$$E_n-q\cdot E_n=p+2\cdot q\cdot p+3\cdot q^2\cdot p+...+n\cdot q^{n-1}\cdot p-q\cdot p-2\cdot q^2\cdot p-...-(n-1)\cdot q^{n-1}\cdot p-n\cdot q^n\cdot p$$

$$=p+q\cdot p+q^2\cdot p+...+q^{n-1}\cdot p-n\cdot q^n\cdot p$$
$$=p\cdot\frac{1-q^n}{1-q}-n\cdot q^n\cdot p=\frac{p\cdot(1-q^n)-n\cdot q^n\cdot p\cdot(1-q)}{1-q},$$

from which we deduce that:

$$E_n=\frac{p-p\cdot(1+n)\cdot q^n+n\cdot q^{n+1}}{(1-q)^2}\,.$$

Because $q\in(0,1)$, we have:

$$\lim_{n\to+\infty}(1+n)\cdot q^n=\lim_{n\to+\infty}n\cdot q^{n+1}=0\,,$$

$$E(X)=\lim_{n\to+\infty}\sum_{k=1}^{n}k\cdot q^{k-1}\cdot p=\lim_{n\to+\infty}E_n=\frac{p}{(1-q)^2}=\frac{p}{p^2}=\frac{1}{p}\,.$$

Thus, the expected number of throws of four dice until we first get sum 14 is:

$$\frac{1}{p}=\frac{648}{73}\approx 8{,}87\approx 9\,.$$

In order to find out the $E(X)$ we can also use this formula:

$$E(X)=E\left(X\middle|\text{for success in the first throw}\right)\cdot p+E\left(X\middle|\text{for failure in the first throw}\right)\cdot(1-p)$$
$$=1\cdot p+\left(E(X)+1\right)\cdot(1-p)=1+(1-p)E(X)\,,$$

so $E(X)-(1-p)E(X)=1$, therefore $E(X)=\dfrac{1}{p}$.

It is known for more than two thousands years that the only regular and convex polyhedra are the Platonic solids. These polyhedra are named after the ancient Greek philosopher Plato. Except for cube (regular hexahedron), they are:

1. regular tetrahedron
2. regular octahedron
3. regular dodecahedron
4. regular icosahedron

We now want to solve the same problem when the dice would have each of the above shapes.

1. If the dice are shaped as a regular tetrahedron, then the sum $S_k \in \{4, 5, 6, ..., 15, 16\}$. Using the same reasoning as with the regular cubic dice, we get the following table:

S_k	4	5	6	7	8	9	10	11	12	13	14	15	16
P_k	1	4	10	20	31	40	44	40	31	20	10	4	1

Thus, the sum with the highest chance of appearing is $S_{10} = 10$ and the expected number of throws of four dice until we first get sum 10 is:

$$\frac{1}{p} = \frac{64}{11} \approx 5,81 \approx 6 .$$

2. If the dice are shaped as a regular octahedron, then the sum $S_k \in \{4, 5, 6, ..., 31, 32\}$. Using the same reasoning as with the regular cubic dice, we get the following table:

S_k	4	5	6	7	8	9	10	11	12	13	14	15	16	17
P_k	1	4	10	20	35	56	84	120	161	204	246	284	315	336

S_k	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
P_k	344	336	315	284	246	204	161	120	84	56	35	20	10	4	1

Thus, the sum with the highest chance of appearing is $S_{18} = 18$ and- the expected number of throws of four dice until we first get sum 18 is:

$$\frac{1}{p}=\frac{8^4}{344}\approx 11,9\approx 12.$$

3. If the dice are shaped as a regular dodecahedron, then the sum $S_k \in \{4,5,6,...,47,48\}$. Using the same reasoning as with the regular cubic dice, we get the following table:

S_k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
P_k	1	4	10	20	35	56	84	120	165	220	286	364	451	544	640	736

S_k	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
P_k	829	916	994	1060	1111	1144	1156	1144	1111	1060	994	916	829	736	640

S_k	35	36	37	38	39	40	41	42	43	44	45	46	47	48
P_k	544	451	364	286	220	165	120	84	56	35	20	10	4	1

Thus, the sum with the highest chance of appearing is $S_{26}=26$ and the expected number of throws of four dice until we first get sum 26 is:

$$\frac{1}{p}=\frac{12^4}{1156}\approx 17,93\approx 18.$$

4. In the case of a regular icosahedron, we can see, by using a computer, that the sum with the highest chance of appearing is $S_{42}=42$ and the expected number of throws of four dice until we first get sum 42 is:

$$\frac{1}{p}=\frac{20^4}{5340}\approx 29,96\approx 30.$$

3. CONCLUSION

In this paper, we have found the distribution of the sum of points that appear on the top faces when four classical identical dice are rolled together. Also, we have considered a similar problem when the dice take the form of a: regular tetrahedron, regular octahedron, regular dodecahedron and regular icosahedron. Together with the regular hexahedron (the cube), these five polyhedra are the only regular convex polyhedra, also called Platonic solids.

In each case, we have calculated the expected number of throws until the most probable sum appears for the first time.