Fixed points

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Abstract

Let f be a steadly increasing function from $\mathcal{D}_f = \{1, 2, \ldots, n\}$ to itself. We study this function asking if it can have no fixed points, even with different generalizations.

1 The problem

Let f be a function from $\mathcal{D}_f = \{1, 2, \ldots, n\}$ to itself, where n is a natural number. Let f be an increasing function, so if $x_1 < x_2$ then $f(x_1) \leqslant f(x_2)$ for all $x_1, x_2 \in \mathcal{D}_f$ [1]. Is there a natural number k such that $f(k) = k$? If so, it is called a fixed point. Analyse some other generalizations of f, such as from $[0, 1]$ to $[0, 1]$, limited to decimals, rationals, reals, ...

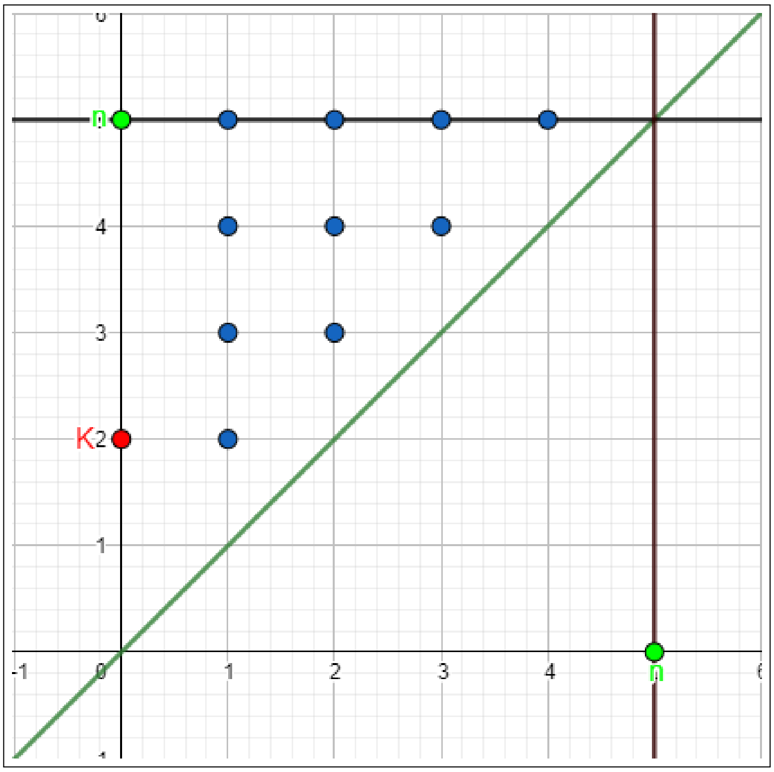
2 Natural numbers

Proposition 1 *Let $\mathcal{D}_f = \{1, 2, \ldots, n\}$, for $n \in \mathbb{N}$, $n > 0$. Let f be an increasing function from $\mathcal{D}_f$ to itself. Then f always has a fixed point, namely there exists $x \in \mathcal{D}_f$ such that $f(x) = x$.*

Proof. Assume for reductio ad absurdum that there exists an increasing function f from $\mathcal{D}_f$ to $\mathcal{D}_f$ without a fixed point, that is to say that $f(x) \neq (x)$ $\forall x \in \mathcal{D}_f$. Then, $f(1) = k$, with $k \in \mathcal{D}_f$. $k > 1$ otherwise f has a fixed point. If $k = 2$, 2 has at most $n - 2$ possible images, 3 has at most $n - 3$ possible images [2]. Iterating, n has at most $n - n = 0$ possible images, which is impossible because n must have a image. So in conclusion f must have at least one fixed point.

* *For French readers:* La classe IV correspond à la première et la classe V à la terminale.

If $k = n$, n is the only possible image of every $x \in D_f$, except for $x = n$, which does not have any possible image. All the intermediate cases are referable to the previous ones.



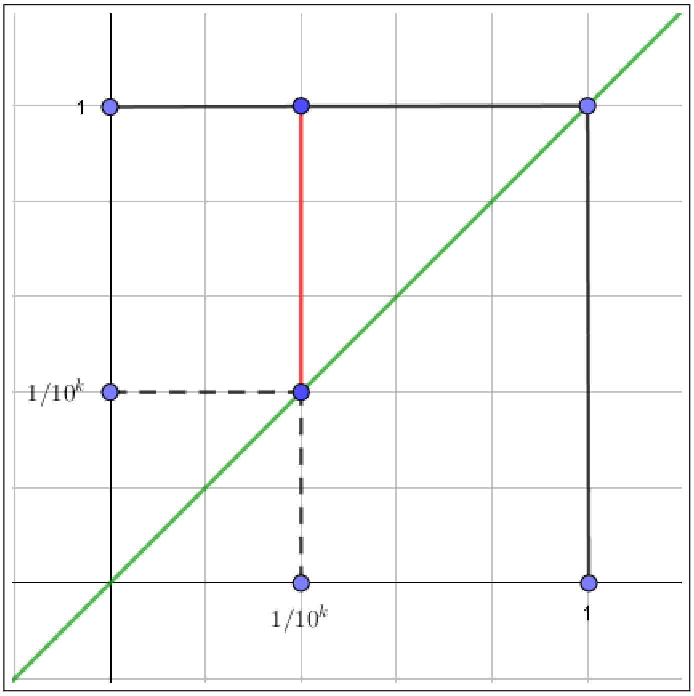
More generally, we proved that this type of functions (same ordered domain and codomain with finitely many elements) necessarily has a fixed point. That is because there is a one-to-one correspondence between a finite ordered set of cardinality n and the set of the first n natural numbers.

3 Decimal numbers

[3]

Proposition 2 *Let f be an increasing function from $\mathcal{D}_f$ to itself, with $\mathcal{D}_f = \{x | x \in \mathbb{R}, x = 1/10^n,\ n \in \mathbb{N}\} \cup \{0\}$. Then f always has a fixed point.*

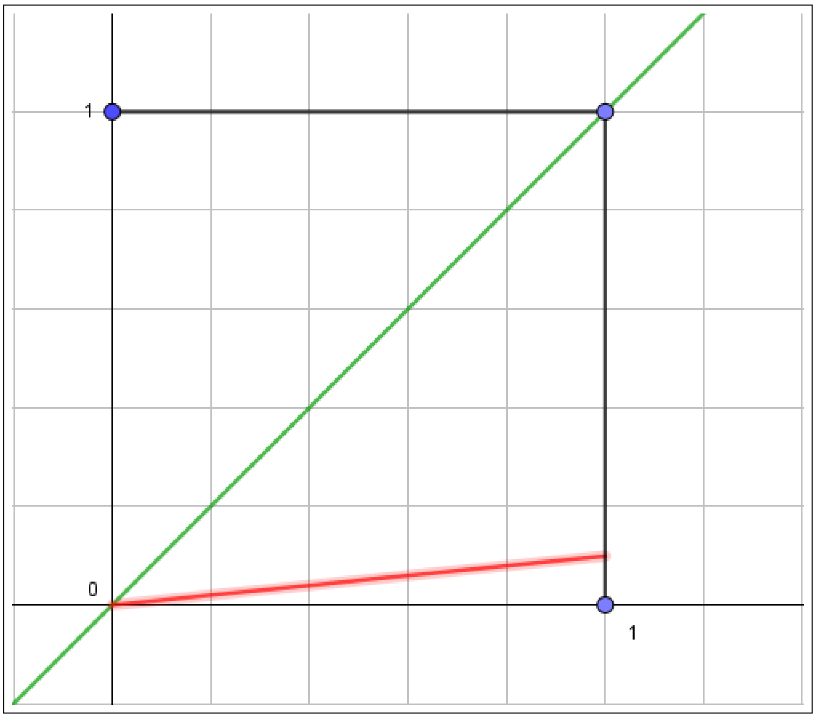
Proof. Assume for reductio ad absurdum that there exists f increasing, whose domain and codomain are $\mathcal{D}_f = \{x | x \in \mathbb{R}, x = 1/10^n, n \in \mathbb{N}\} \cup \{0\}$ and such that $f(x) \neq x$ for all $x \in \mathcal{D}_f$. Then $f(0) = 1/10^k$, for some k. $f(1) < 1$ because otherwise 1 would be a fixed point. If $n > k$, $f(1/10^n) \geqslant 1/10^k$ because the function is increasing. Thus, $f(1/10^n) > 1/10^n$; so, $f(x) > x$. If $n \leqslant k$, we can refer to the previous proof because we have a function which goes from a set with finitely many elements to the same set [4]. This leads to an absurdity.



3.1 Decimal numbers (not including 0)

There exists at least one increasing function f from $\mathcal{D}_f = \{x | x \in \mathbb{R}, x = 1/10^n,\ n \in \mathbb{N}\}$ to $\mathcal{D}_f = \{x | x \in \mathbb{R}, x = 1/10^n,\ n \in \mathbb{N}\}$ without a fixed point.

For example: $y = f(x) = x/10$.



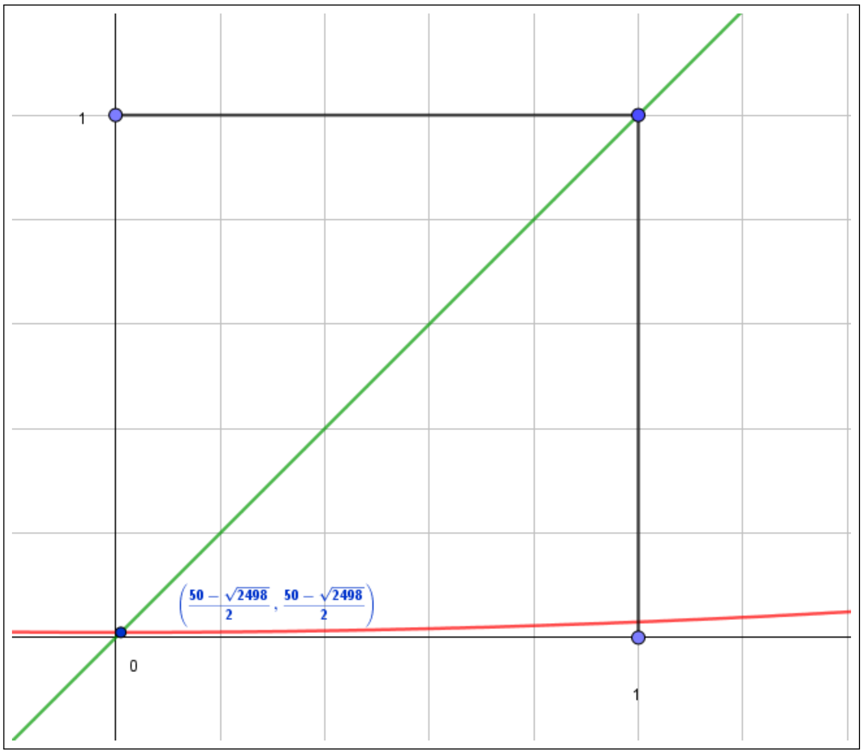
We have studied different functions of this type and we have proved that:

- there exists an infinite number of functions without fixed points;
- we can build infinitely many functions with one fixed point and one with an arbitrary number of fixed points;
- for every $x \in \mathcal{D}_f$ we can build a function having x as a fixed point.

4 Rational numbers

There exists at least one increasing function f from $\mathcal{D}_f = \mathbb{Q} \cap [0,1]$ without any fixed point.

For example: $f(x) = y = \frac{x^2}{50} + \frac{1}{100}$.



For a generic function $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{Q}$ and $a \neq 0$, the conditions to be increasing and without any fixed point are:

- the function must be increasing between 0 and 1, and there must not be a fixed point in $x = 0$, so $a > 0$ and $c \neq 0$;
- $0 < a + b + c < 1$ because the image of the function must be a subset of $[0; 1]$ [5];
- the intersection between $y = f(x)$ and $y = x$ must occur at an irrational number, so $\sqrt{(b-1)^2 - 4ac} \notin \mathbb{Q}$.

5 Real numbers

Proposition 3 *Let f be a steadly increasing function from $\mathcal{D}_f = \mathbb{R} \cap [0,1]$ to itself, with finitely many points of discontinuity. Then f always has a fixed point.*

Proof. Let us consider an interval $I = [a, b] \subseteq D_f$, with the limit points a and b included or not and where f is continuous; thus, a and b are discontinuous points [6]. Let us restrict the domain to I. Using the definition of continuous and steadly increasing function we can say that: since $a < x\ \forall x \in I$, $f(a) < f(x)$, so $f(a)$ is the limit inferior; then, since $x < b\ \forall x \in I$, $f(x) < f(b)$, so $f(b)$ is the limit superior. Now let consider the function $f(x) - x$ if a and b are included in the interval, its continuous extension otherwise. Using the intermediate zero theorem on this function and we can conclude that there must be a intersection between $y = f(x)$ and $y = x$, because $\lim_{x\to a^+} f(x) > a$ and $\lim_{x\to b^-} f(x) < b$. In conclusion, this type of function must have a fixed point.

Proposition 4 *Let f be a continuous steadly increasing function from $\mathcal{D}_f = \mathbb{R} \cap [0,1]$ to itself. Then f always has a fixed point, namely there exists $x \in \mathcal{D}_f$ such that $f(x) = x$* [7].

Proof. Using the definition of continuous function we can say that: since $0 < x$, $f(0) \leqslant f(x)$, so $f(0)$ is the global minimum; then, since $x \leqslant 1$, $f(x) \leqslant f(1)$, so $f(1)$ is the global maximum. Therefore, using the intermediate zero theorem to the function $f(x) - x$ there must be an intersection between $y = f(x)$ and the function $y = x$, because $f(0) \geqslant 0$ and $f(1) \leqslant 1$, so the function must have a fixed point.

6 Notes d'édition

[1] Throughout the article *increasing function* is to be understood as *non-decreasing function.*

[2] Since f is non-decreasing $f(2) \geqslant f(1) \geqslant 2$, and since it has no fixed point $f(2) \neq 2$; so, 2 has at most $n-2$ possible images $f(2)$. Similarly $f(3) > 3$ and 3 has at most $n-3$ possible images, and so on until we get that n has no possible image.

[3] This title "Decimal numbers" is not very appropriate since we are dealing with only one very particular example.

[4] Namely $f(1/10^n) \geqslant 1/10^k$ still holds for $n \leqslant k$ and the function f restricted to $\{1/10^n, 0 \leqslant n \leqslant k\}$ is then an increasing function from this finite set to itself without fixed point.

[5] There are some errors in these conditions. The conditions for f to be increasing on $[0,1]$ are $b \geqslant 0$ and $2a + b \geqslant 0$, and then for it to send [0,1] into itself, we need $c \geqslant 0$ and $0 \leqslant a + b + c \leqslant 1$.

[6] To apply the intermediate value theorem (below, at the end of the proof), we lack precisions. The function f is assumed to be increasing from $[0,1]$ to itself with finitely many points of discontinuity; so, if we denote by $a_1 \leqslant \ldots \leqslant a_{n-1}$ the discontinuity points and let $a_0 = 0$ and $a_n = 1$, since $f(0) \geqslant 0$, $f(1) \leqslant 1$ we necessarily have some index i with $f(a_{i-1}) \geqslant a_{i-1}$ and $f(a_i) \leqslant a_i$. We may then apply the authors' argument with $a = a_{i-1}$ and $b = a_i$, adding that $\lim_{x\to a^+} f(x) \geqslant f(a) \geqslant a$ and $\lim_{x\to b^-} f(x) \leqslant f(b) \leqslant b$.

[7] *Remark:* this proposition is contained in the previous one.