The geometry of parallel parking

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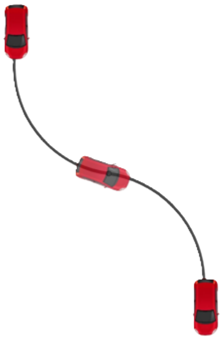
Abstract

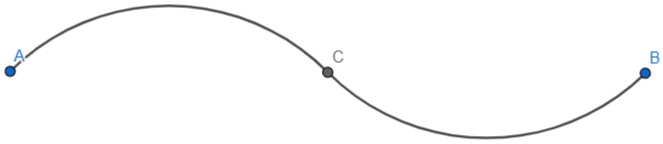
Vehicles while moving follow specific trajectories, more or less complex according to their purpose, which present particular and interesting aspects when studied with mathematical tools. The goal of this article is present the definition of a new geometry with a particular metric space, where distance is defined in terms of length of two quarters of circumference, which was inspired by the peculiar way that cars move while turning.

1 Arcs of circumference

From the analysis made previously we can assume that the trajectory that a car follows, during a parallel parking, is like this :



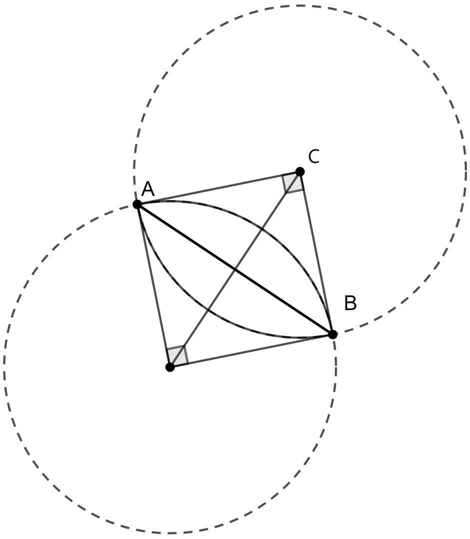
Therefore it can be described by two arcs of circumference, for example by two quarters of circumference with opposite concavity. From this idea we thought of building a plane geometry in which to go from a point A to a point B of the plane it is only possible to follow trajectories of this type :



First of all we asked ourselves if it is possible to move from any point to another following only one quarter of circumference and if this path is unique.

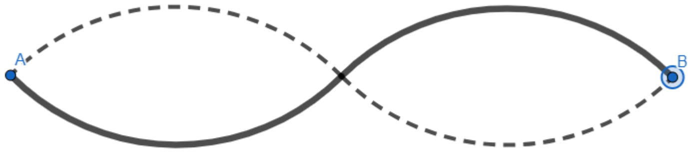
We answered this question with this theorem :

Theorem 1. *Given two distinct points of the plane A and B, there are only two distinct quarters of circumference symmetrical to the segment AB, with A and B as their endpoints and with a length of* $\frac{\sqrt{2}\,\pi}{4}\overline{AB}$.



Démonstration. We construct the axis of $\overline{AB}$ and on this axis we take the point C such that $A\widehat{C}B$ is a right angle. Therefore C is the center of the circumference $\mathcal{C}$ passing through A and B and with the arc $\widehat{AB}$ quarter of the circumference C with center C. This $\mathcal{C}$ is unique unless there is a symmetric construction, since the point C is the only one in the plane to satisfy these conditions. The length of the quarter $\widehat{AB}$ equals $\frac{r\pi}{2}$. $C\widehat{A}B \cong C\widehat{B}A$ because $\overset{\triangle}{ABC}$ is an isosceles triangle for the previous proof. $C\widehat{A}B = \frac{180° - 90°}{2} = 45$. So $\overset{\triangle}{ABC}$ is a 90°, 45°, 45° triangle, so $r = \frac{\overline{AB}}{\sqrt{2}}$. Therefore the length of the quarter $\widehat{AB}$ is $\frac{\sqrt{2}\pi}{4}\overline{AB}$. ☐

From the previous theorem we can deduce that there are only two possible trajectories that a car can follow to move from a point A to a point B of the plane made up by two consecutive congruent and opposite quarters of circumference.



We have then observed that the length of the trajectory composed by a single quarter and that composed of two quarters is the same. In fact :

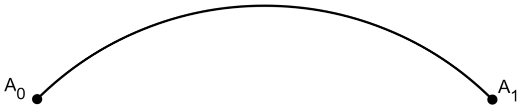
Theorem 2. *Given three aligned points A, B and C, the measure of the quarter* $\widehat{AC}$ *of circumference with A and C as endpoints is equal to the sum of the measures of the quarters of circumference* $\widehat{AB}$ *and* $\widehat{BC}$*.*

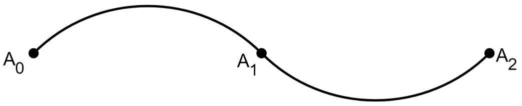
Démonstration. The length of $\widehat{AC}$ is $\frac{\sqrt{2}\pi}{4}\overline{AC}$. If we sum $\widehat{AB}$ and $\widehat{BC}$ we can factor out $\frac{\sqrt{2}\pi}{4}$ and obtain $\overline{AB}+\overline{BC}$ which equals $\overline{BC}$ so $\widehat{AC} \cong \widehat{AB} + \widehat{BC}$.

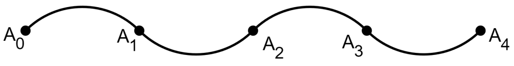
☐

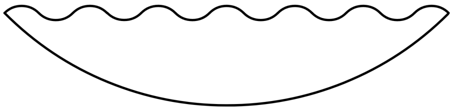
Generalizing :

Theorem 3. *Given* $n+1$ *aligned points* $A_0, A_1, \ldots, A_n$*, the measure of the quarter* $\widehat{A_0A_n}$ *of circumference is equal to the sum of the measures of the n quarters of circumference* $\widehat{A_0A_1}, \widehat{A_1A_2}, \ldots, \widehat{A_{n-1}A_n}$*.*









Démonstration. Considering the reasoning of the previous theorem we obtain that for n aligned quarters, the sum of all that is constant and equals $\frac{\sqrt{2}\pi}{4}\overline{A_0A_n}$ because we can collect $\frac{\sqrt{2}\pi}{4}$ which multiplies $\overline{A_0A_1}+\overline{A_1A_2}$... and the sum of all is $\overline{A_0A_n}$

□

So, even if the number of quarters increases the length of the trajectory does not change.

If n approaches infinity the sum of the infinite lengths of the quarters approaches the finite number $\frac{\sqrt{2}\,\pi}{4}\overline{AB}$.

Similarly we wondered what does the sum of the areas between the quarters of the circumference $\overset{\frown}{A_0A_1}$, $\overset{\frown}{A_1A_2}$, …, $\overset{\frown}{A_{n-1}A_n}$ and the segment A_0A_1, A_1A_2, …, $A_{n-1}A_n$ approach. In the following theorem the answer.

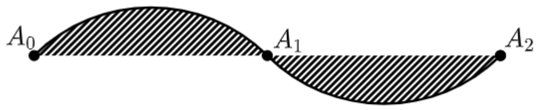
Theorem 4. *Given n+1 aligned points* A_0,A_1, …, A_n, *the sum of the areas between the quarters of the circumference* $\overset{\frown}{A_0A_1}$, $\overset{\frown}{A_1A_2}$, …, $\overset{\frown}{A_{n-1}A_n}$ *and the segment* A_0A_1, A_1A_2, …, $A_{n-1}A_n$ *is equal to* $\frac{\pi-2}{8n}$.

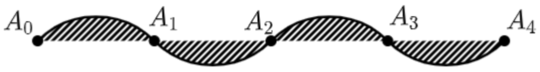
Démonstration. The area between the quarter of the circumference $\overset{\frown}{AB}$ and the segment AB is equal to the area of the circular sector minus the area of the right triangle $A\overset{\triangle}{O}B$ so $\frac{\overline{AB}^2(\pi-2)}{8}$.

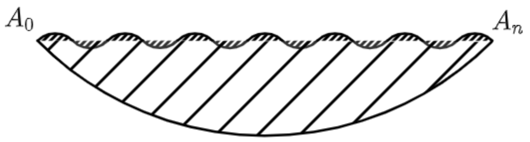
If we consider n equally spaced aligned points in the plane, the measure of the included area is $\frac{\overline{AB}^2(\pi-2)}{8n^2}$, therefore the sum of all the areas is $\frac{\overline{AB}^2(\pi-2)}{8n}$.

□

So, if n approaches infinity, the sum of the infinite areas between the quarters of the circumference $\overset{\frown}{A_0A_1}$, $\overset{\frown}{A_1A_2}$, …, $\overset{\frown}{A_{n-1}A_n}$ and the segment A_0A_1, A_1A_2, …, $A_{n-1}A_n$ approaches zero.



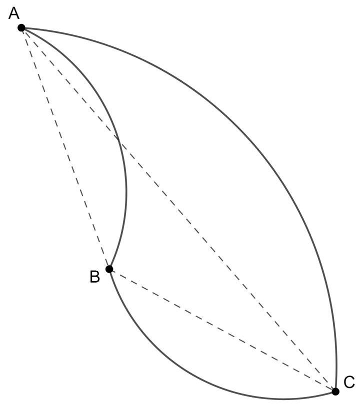




And what happens if the endpoints of the two quarters of circumference are not aligned ?

The following theorem answers :

Theorem 5. *Given three not aligned points A,B and C, the measure of the quarter $\widehat{AC}$ of the circumference having A and C as endpoints is smaller than the sum of the measures of the quarters $\widehat{AB}$ and $\widehat{BC}$.*



Démonstration. $\widehat{AB}+\widehat{BC}=\dfrac{\sqrt{2}\pi}{4}(\overline{AB}+\overline{BC})$ for previous proof. $\overline{AB}+\overline{BC}>\overline{AC}$ for the Triangle Inequality Theorem so $\widehat{AC}<\widehat{AB}+\widehat{BC}$.

□

2 A new metric space

Due to the previuos theorem, considering the metric

$$d(A,B) := \text{the length of the two consecutive quarters with opposite concavity}$$

it is possible to define a metric space on the set of points in the plane and the following properties :

- $d(A,B) > 0 \Leftrightarrow A \neq B$
- $d(A,B) = 0 \Leftrightarrow A = B$
- $d(A,B) = d(B,A)$
- $d(A,B) \leqslant d(A,C) + d(C,B)$ with C being a point of the segment AB

Definition 1. *The set of points P in the plane equidistant (using the distance d of the new metric) from a point C is a circumference with $\widehat{CP}$ as its radius.*

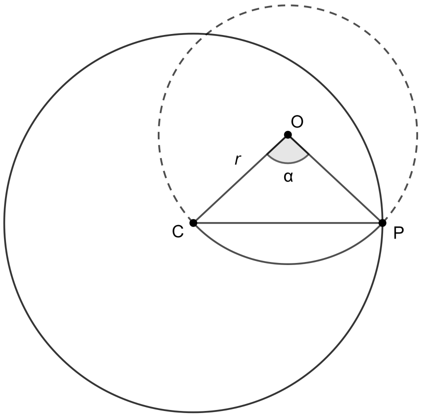
Theorem 6. *The lenght of the circumference is $4\sqrt{2} \cdot \widehat{CP}$. So, in this case, $\Pi = 2\sqrt{2} \approx 2,8284$.*

Démonstration. The measure of the traditional circumference is $2\pi r$. But now the radius is a quater of circumference. For the previous proof $r = \dfrac{4\widehat{CP}}{\pi\sqrt{2}}$.

So, substituting r in the first formula we have that the measure of the circumference of radius $\widehat{CP}$ is $C = 4\sqrt{2}\widehat{CP}$ ☐

What happens if the distance between two points is not a quarter but another arc of circumference ?

Theorem 7. *Given a circumference with C as its center and a central angle measuring $0 \leqslant \alpha \leqslant \pi$ radians, let P be a point of the circumference and $\widehat{CP}$ the arc that subtends such angle, the length of the circumference is $k \cdot \widehat{CP}$ where k is the solution of the equation $k \cdot \alpha = 4\pi \sin \dfrac{\alpha}{2}$.*
Hence we get the following value for pi : $\Pi = \dfrac{k}{2}$

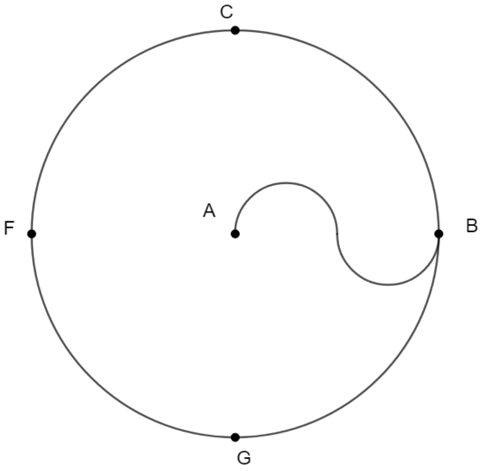


Démonstration. We wondered whether the circumference could be an integer number of times the arc $\widehat{CP}$ by changing the center arc of $\widehat{CP}$. So we matched $2\pi\overline{CP} = k\widehat{CP}$ and then we obtained $4\pi \sin \dfrac{\alpha}{2} = k\alpha$ thanks to the chord theorem.

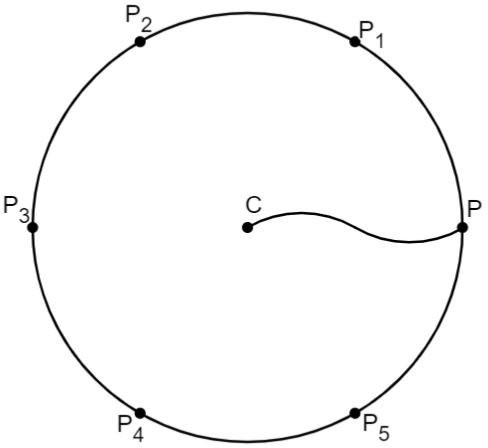
We know that $\Pi = \dfrac{C}{2\widehat{CP}}$. Then we substituited C with k $\widehat{CP}$ and by simplifying $\widehat{CP}$ we obtained that $\Pi = \dfrac{k}{2}$. With Π we mean the new value of pi, which is no longer 3,14..., but changes with the angle we choose for the arc. ☐

Here are some examples :

- $\alpha = \pi$ radians→ the length of the circumference is $4 \cdot \widehat{CP} \rightarrow \Pi = 2$.



— $\alpha = \dfrac{\pi}{3}$ radians→ the length of the circumference is $6 \cdot \widehat{CP} \rightarrow \Pi = 3$.



— $\alpha = \dfrac{\pi}{4}$ radians → the length of the circumference is $8\sqrt{2-\sqrt{2}} \cdot \widehat{CP} \rightarrow \Pi = 4\sqrt{2-\sqrt{2}}$.

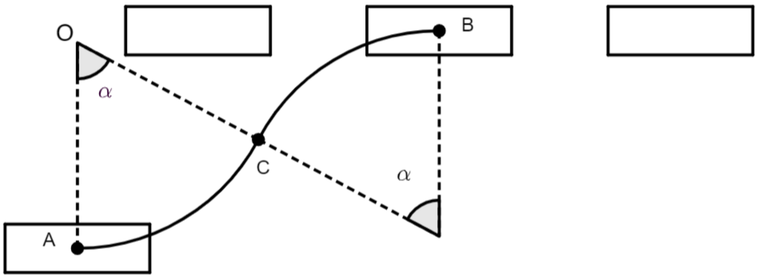
— $\alpha = \dfrac{2\pi}{3}$ radians → the length of the circumference is $3 \cdot \sqrt{3} \cdot \widehat{CP} \rightarrow \Pi = \dfrac{3\sqrt{3}}{2}$.

3 Parallel parking

Now we wanted to analyze the parallel parking. The aim is to reach a point B from a point A so that our car is parked between the two cars already present and it is also parallel to the sidewalk and aligned with the other two. To park the car parallel to the sidewalk, the car has to start moving already parallel to the sidewalk (to make sure you don't drive up onto the sidewalk). It is not always possible to follow a trajectory formed by quarters of a circumference from A to B. However we can find out the angle of the car's travels trajectories.

Theorem 8. *Starting from a generic point A and wanting to reach point B with the car parallel to the sidewalk, the car travels trajectories formed by two quarters of a circumference, with :*

$$\alpha = 2\arcsin\left(\frac{y_B - y_A}{\sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}}\right)$$



Démonstration. First of all we calcolate $\overline{AC}$ using the trigonometric chord theorem :

$$\overline{AC} = 2r\sin\left(\frac{\alpha}{2}\right) \qquad \text{so} \qquad \alpha = 2\arcsin\frac{\overline{AC}}{2r}$$

we know that $\overline{AC} = \sqrt{(x_A - x_C)^2 + (y_A - y_C)^2}$ and in order to find r we use the property

$$\overline{OA} = \overline{OC}$$

with $O(x_A, k)$ that is

$$(k - y_A)^2 = (x_A - x_C)^2 + (k - y_C)^2$$

so

$$k = \frac{(x_A - x_C)^2 + y_C^2 - y_A^2}{2(y_C - y_A)}$$

thus

$$r = k - y_A = \frac{(x_A - x_C)^2 + (y_A - y_C)^2}{2(y_C - y_A)}$$

so

$$\alpha = 2\arcsin\frac{\overline{AC}}{2r}$$

therefore

$$\alpha = 2\arcsin\left(\frac{y_B - y_A}{\sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}}\right)$$

□

4 Angles and triangles

In the new geometry the Plane and the Points are fundamental undefined terms analogously to those of Euclidean geometry. Let's now introduce new terms.

Definition 2. *The set of points of two quarters of a circumference, according to the Euclidean definition, consecutive and with opposite concavity, including the two endpoints, is called* ***segment****.*

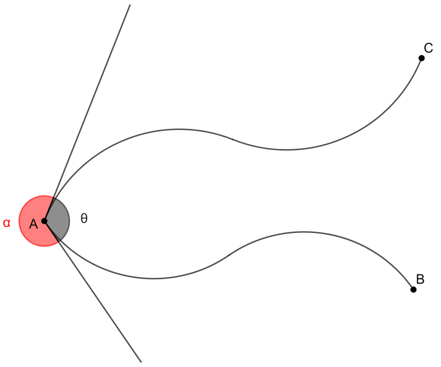
Definition 3. *A* ***polygonal chain*** *or* ***polygonal*** *is a set of segments such that each segment is consecutive but not adjacent to the next and each endpoint of the segments belongs to a maximum of two of them.*

Definition 4. *The segments of the polygonal are called* ***sides*** *of the polygonal.*

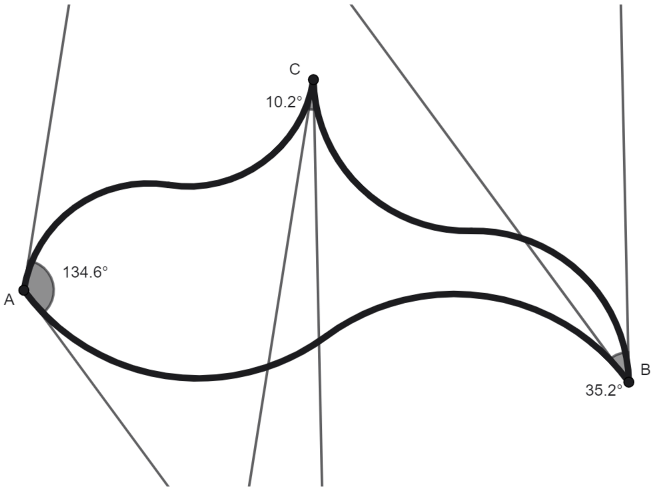
Definition 5. *The* ***triangle*** *is the set of the points of the plane inside a closed not self-intersecting polygonal of three sides and of the points of the polygonal itself. The* ***sides*** *of the polygonal are called sides of the triangle.*

Definition 6. *Given two segments AB and AC with the endpoint A in common, it is defined as* ***Angle*** *each of the two parts of the plane limited by the two tangent lines in the point A (vertex of the angle) to the first two quarters of circumference of the segments, AB and AC respectively.*

Definition 7. *The angles formed by the couples of consecutive sides, inside the triangle, are called the* ***angles of the triangle****.*



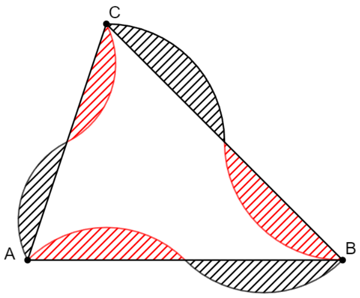
Theorem 9. *In the parallel parking geometry, the sum of the angles of a triangle equals a straight angle.*

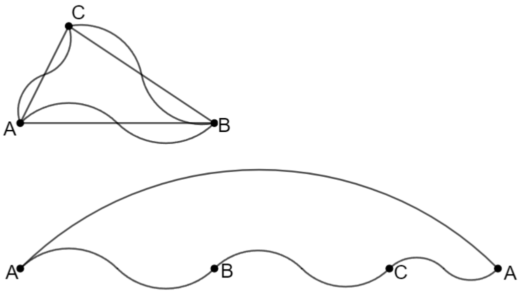


Démonstration. The sum of the interior angles of a triangle in Euclidean geometry and in parallel parking geometry is the same since each angle in the new geometry is obtained by adding two 45° angles or subtracting two 45° angles or adding a 45° angle and subtracting a 45° angle from the measure of the angle in Euclidean geometry. ☐

Theorem 10. *The area of the new triangle is equal to the Euclidean one and the perimeter is equal to* $P_\triangle \frac{\sqrt{2}\,\pi}{4}$

Démonstration. The area of the new triangle is equal to the respective Euclidean one because each side is formed by a convexity and a concavity. The area between the quarter of circumference and the side is equal in the convexity and in the concavity. So the convexity quarter adds the same amount of area that the concavity one reduces. This means that the area remains the same as the Euclidean triangle area. The length of each quarters of circumference is the respective segment multiplied by $\frac{\sqrt{2}\,\pi}{4}$. If we sum the lengths of all the quarters, we can factor out $\frac{\sqrt{2}\,\pi}{4}$ and obtain the sum of all the segments, which is the perimeter. So the perimeter of the triangle is $P_\triangle \frac{\sqrt{2}\,\pi}{4}$. ☐

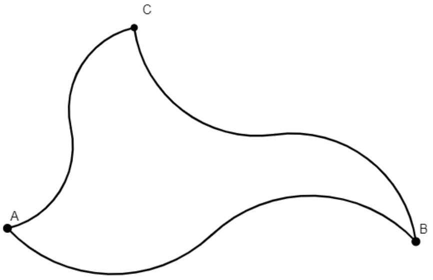




Theorem 11. *There are only two different types of triangles up to rotations in the plane.*

Démonstration. There exist only two type of triangle drawn with quaters of circumference as their sides instead of eight bacause seven of them are just rotation of the same figure.

The first type with all the angles formed by a concavity and convexity, we called it **homogeneous triangle**, and the second type formed by a double convexity, a concavity and a convexity and a doble convexity angles, we call it **non-homogeneous triangle**.



homogeneous

non-homogeneous

□

The question now is : given three non-aligned points in the plane, does there always exist a triangle that has these three points as vertices ?

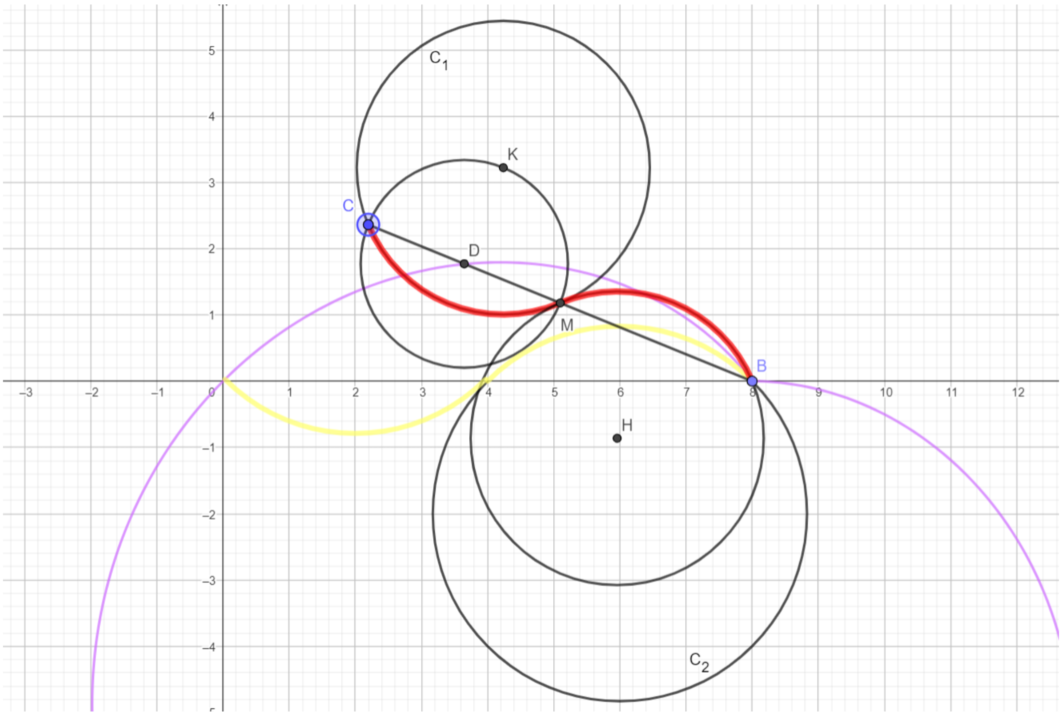
The answer is negative.

The next example describes the different cases that can occur.

Example.

Given three points $A(0,0)$, $B(0,8)$ and $C(x,y)$, the equation of the curve that establishes the existence of the triangle has the equation :

$$\left(\frac{(x-6)^2}{2}+\frac{(y+4)^2}{2}-10\right)^2=4(x-8)^2+4y^2$$



Given $B(x_B;0)$ and $C(x_C;y_C)$ so $M\left(\frac{x_B+x_C}{2};\frac{y_C}{2}\right)$ and $D\left(\frac{x_B+3x_C}{4};\frac{3y_C}{4}\right)$ and we can get K by a clockwise rotation around the point D of the point C by an angle of 90°. So $K\left(\frac{x_B+3x_C+y_C}{4};\frac{3y_C-x_C+x_B}{4}\right)$. The two Circumferences C_1 and C_2 intersect when $\overline{KH}=r_1+r_2$ with the radius $r_1=\frac{\sqrt{2}}{4}\overline{CB}$ and $r_2=\frac{\sqrt{2}}{4}x_B$

$$\left(\frac{x_B+3x_C+y_C}{4}-\frac{3}{4}x_B\right)^2+\left(\frac{3y_C-x_C+x_B}{4}+\frac{1}{4}x_B\right)^2=\left(\frac{\sqrt{2}}{4}x_B+\frac{\sqrt{(x_C-x_B)^2+y_C^2}}{2\sqrt{2}}\right)^2$$

if $x_C=x$, $y_C=y$ and for example $x_B=8$ we find the equation of the curve

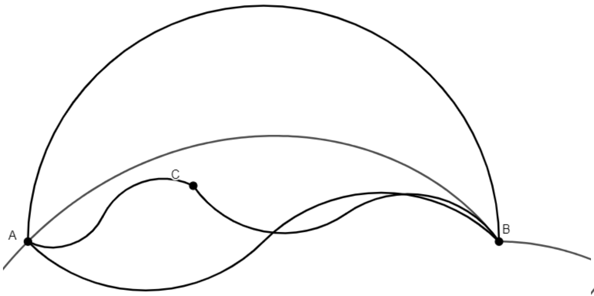
$$\frac{x^2}{2}+\frac{y^2}{2}-6x+4y+16=2\sqrt{(x-8)^2+y^2}$$

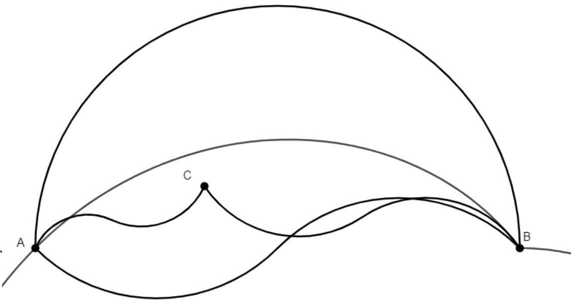
and we can write, simplifying,

$$\left(\frac{(x-6)^2}{2}+\frac{(y+4)^2}{2}-10\right)^2=4(x-8)^2+4y^2$$

not considering the points inside the circumference $\frac{x^2}{2}+\frac{y^2}{2}-6x+4y+16=0$

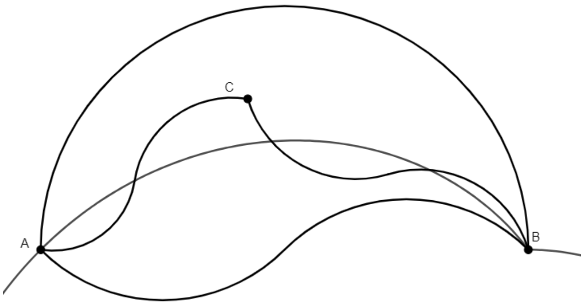
Let's now list the 6 possible cases :

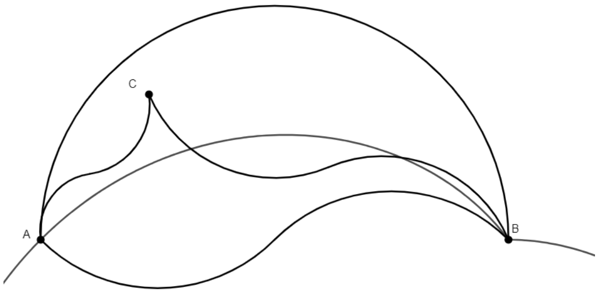




No homogeneous triangle exists

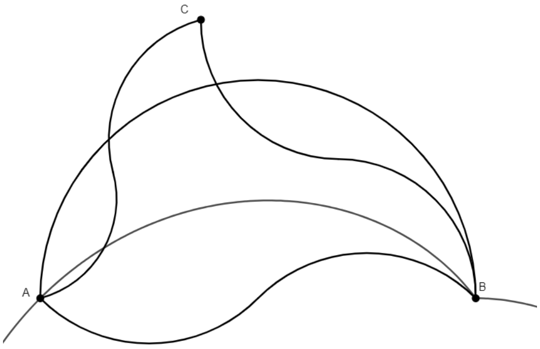
No non-homogeneous triangle exists

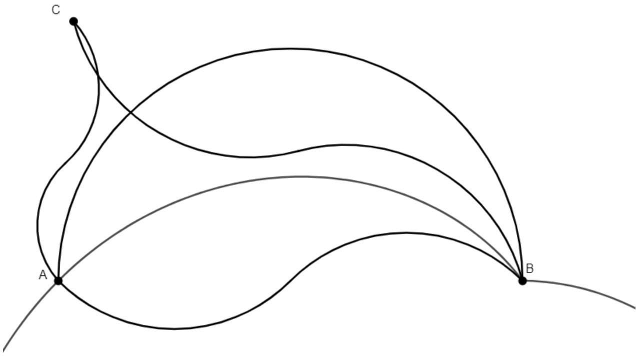




Homogeneous triangles exist

Non-homogeneous triangles exist





Homogeneous triangles exist

No non-homogeneous triangle exists