HYPERBOLIC TILINGS

Introduction

Throughout the school year 2019/2020 we were taking part in a project Erasmus+ called "Maths &Languages". We cooperated with our twin school Lycée des métiers Louis Vicat de Souillac.

Problem

A tiling of a flat surface is the covering of a plane using one or more geometric shapes with no overlaps and no gaps. Our task was to create tilings in Euclidean and hyperbolic geometry with only one polygon. First we were looking for polygons from which we could create tilings in Euclidean geometry and then in hyperbolic geometry. We were able to do it thanks to the Geogebra program and the tools we created in it.

Euclidean geometry - hyperbolic geometry

That is the geometry that you've been learning all this time. In his book "The Elements", Euclid of Alexandria focused on geometry as an axiomatic system, therefore he developed Euclidean geometry.

The essential difference between Euclidean geometry and hyperbolic geometry is the nature of parallel lines. In Euclidean geometry through a point not on a given line there is exactly one line parallel to the given line. In hyperbolic geometry, given a line and a point outside that line it is possible to trace at least two lines not intersecting the first one.

Isometry

Isometry is a transformation that keeps the same distance between the points: d(f(A),f(B))=d(A,B). Two figures that can be transformed into each other by an isometry are said to be congruent.
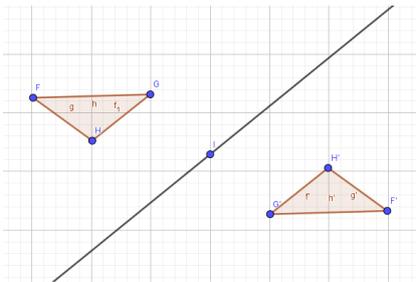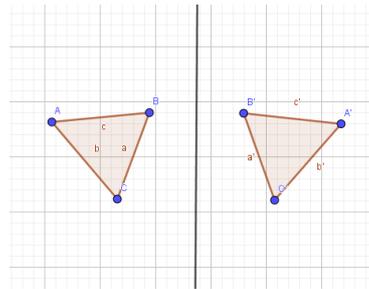
The examples of isometries:


Figure 1 - Central Symmetry


Figure 2 - Axial Symmetry

Euclidean tilings

We started from one polygon and we used central and axial symmetry to obtain a tilling. We could create tilings with:

- any triangle
- every convex quadrangle
- a pentagon which possess two right, subsequent angles
- a regular hexagon
- a hexagon which has one pair of parallel sides of the same length and two pairs nonparallel of the same length.
- a hexagon with three pairs of parallel sides
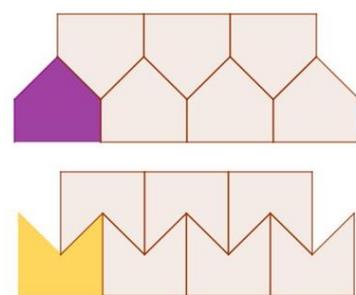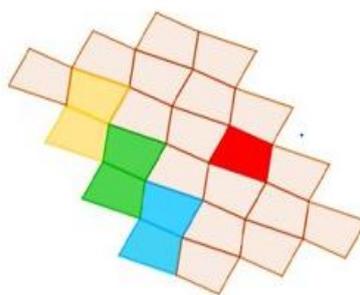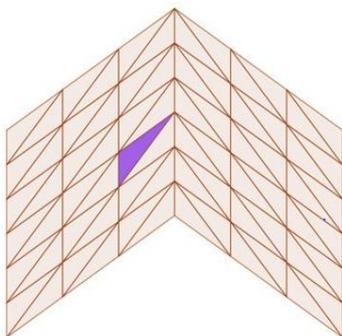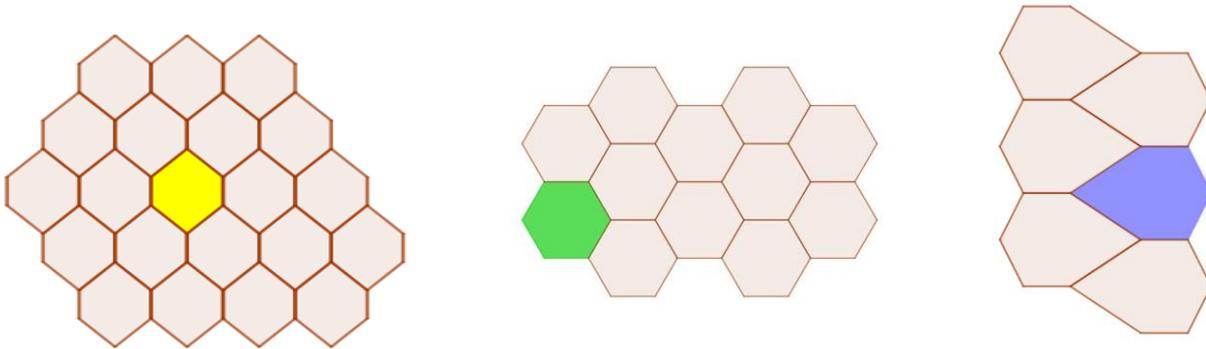
Poincaré disk model of hyperbolic geometry

The plane in the disk model is the interior of disk. Straight lines consist of all circular arcs contained within that disk that are orthogonal to the boundary of the disk and all diameters of the disk. We prepared our own tools in GeoGebra for drawing basic geometrical objects.
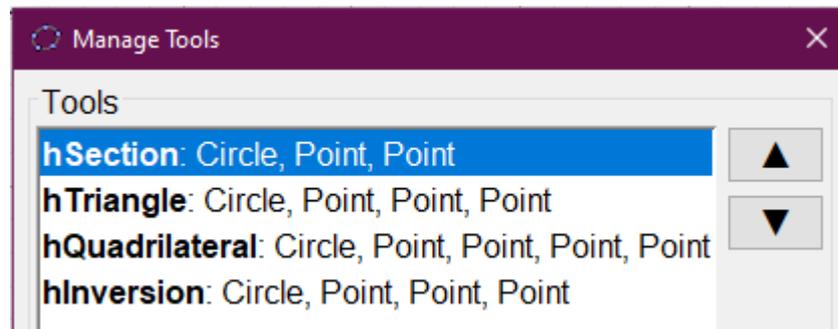

The most basic tool that we created is hSection which constructor takes three parameters:

• The circle which represents the Poincaré Disk

• Two points objects between which the **h**yperbolic **Section** is supposed to be created

Other tools were derived from hSection with the exception of hInversion, which was created from scratch.
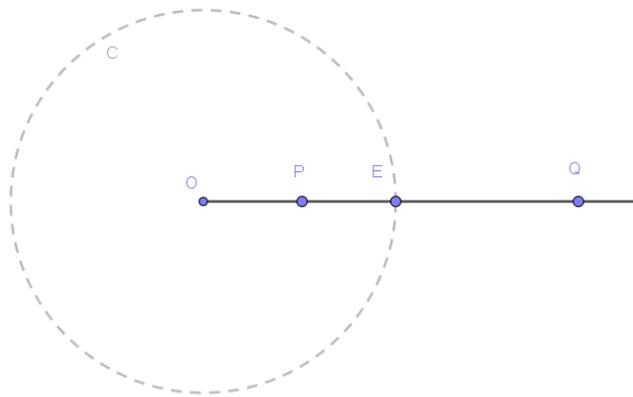
Given basic structures such as line, ray and section, we were able to construct basic figures known to us from Euclidean geometry and figures with infinite sides
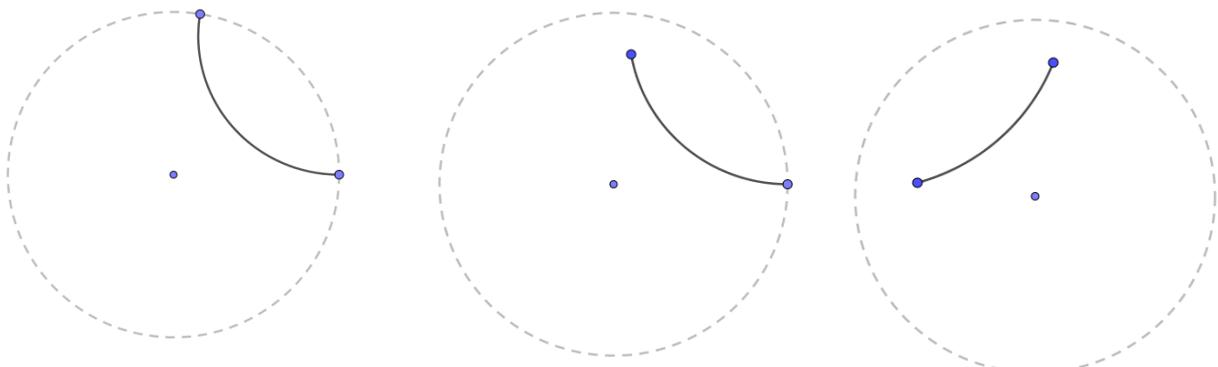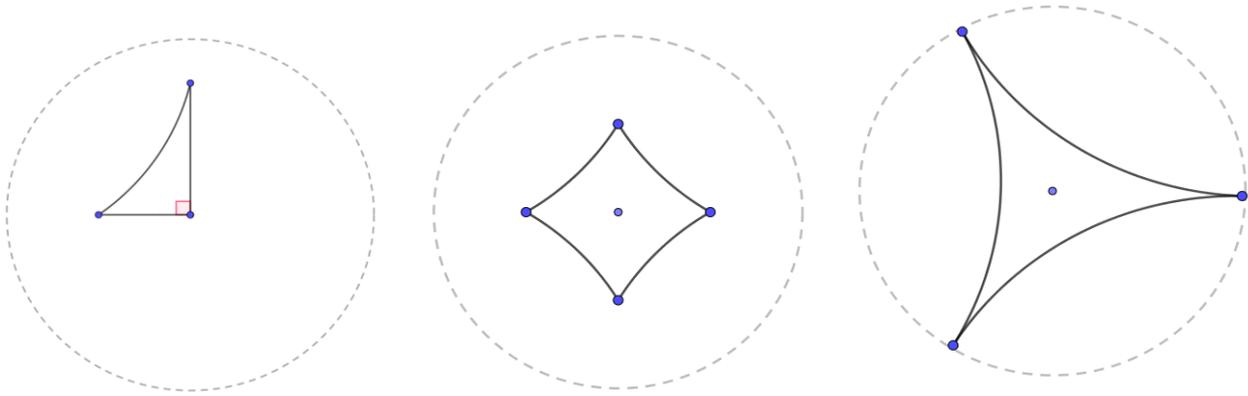
Inversion

Let P be a given point. Let c be a circle centred on O and radius r = [O, E]. The inverse of P with respect to c is a point Q on the line[O,P] such that distance[O,P] · distance[O,Q] = r^2.
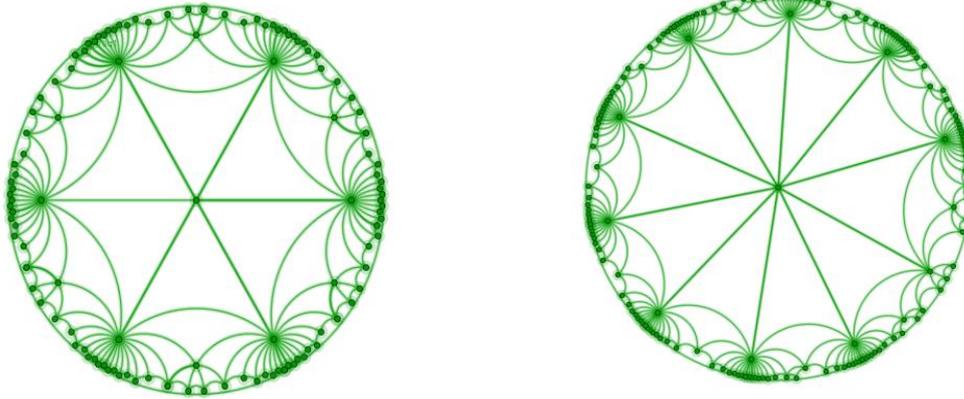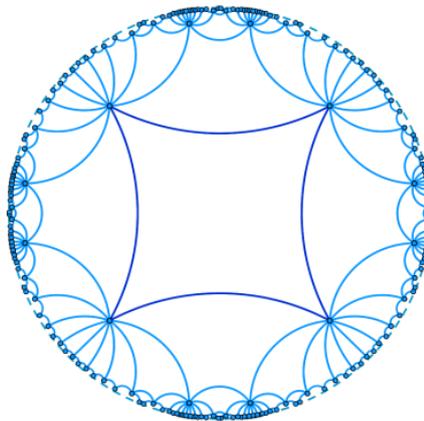

Hyperbolic tilings

We started from one polygon and we used inversions to obtain a tilling.

We created some tilings having one polygon and using only an inversion. Our first experiment was with 3, 4, 5, 6 and 10 triangles. All of them were possible.
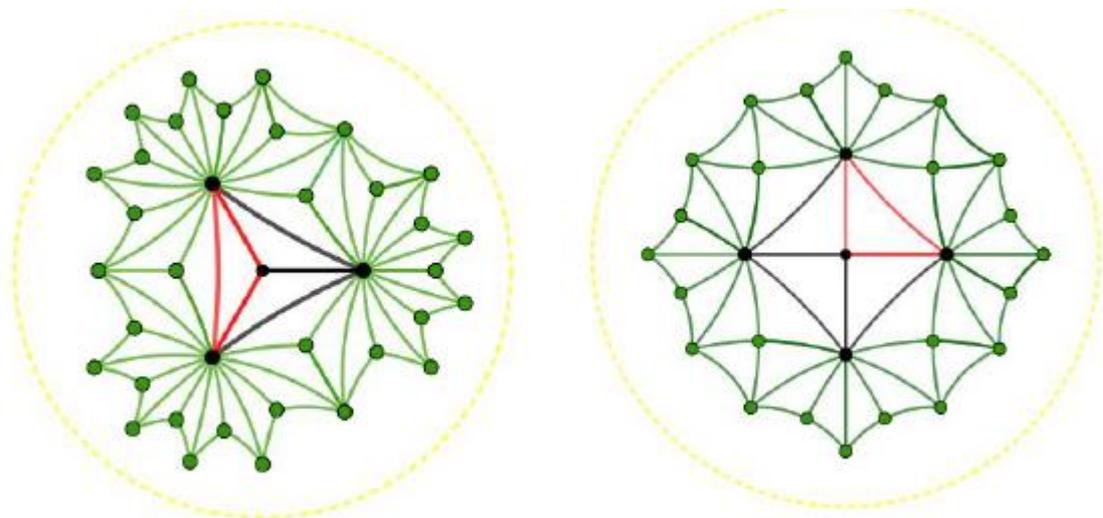
After, we tried with quadrangle. There, we also saw that our selected figures were correct to build a tilling.

Summary

We succeeded in building tilings with different figures. That being shown, we've concluded that it's possible to create tilings using a triangle and quadrangle. We also prepared animations for hyperbolic tilings with triangle and quadrangle. You will find them under the link https://www.geogebra.org/u/%C5%BCmichowska.

We encourage you to look for further possibilities.