In between

2019 - 2020

Oana Maria ENE and Maria NECHITA, 11th Grade

School: Colegiul Național "Bogdan Petriceicu Hasdeu", Buzău, Romania

Teachers: Mrs. Melania NICOLAE

THE SUBJECT:

You live in a street with houses only on one side. They are numbered starting from 1, 2, 3, … (no odd or even side, there's only one side). You notice that the numbers of the houses on your right add up to the same result as the numbers of the houses on your left.

What's your address?

EXAMPLES:

If I live on a street with 8 houses, numbered from 1 to 8, my address will be **number 6,** because

$$1+2+3+4+5=7+8$$

If I live on a street with 49 houses, numbered from 1 to 49, my address will be **number 35.**

If I live on a street with 288 houses, numbered from 1 to 288, my address will be **number 204**.

THE SOLUTIONS FOUND:

To solve this problem, we used two methods:

Method 1: the Gauss summation

Let's call:

- the number of houses: n, $(n \in \mathbb{N}^*)$
- the number of the house we are looking for: x, $x \in \mathbb{N}^*$

The string of numbers will look as follows:

$$1,\ 2, 3, 4, \dots, x-1, x, x+1, \dots, n-2, n-1, n$$

Our aim is to find out the value of the x which satisfies this condition:

$$\underbrace{1+2+3+\ \dots\ +(x-1)}_{S_1} = \underbrace{(x+1)+(x+2)+(x+3)+\ \dots\ +(n-2)+(n-1)+n}_{S_2}$$

It can be noted that the members of this equality are sums of consecutive numbers. As a result, for each of them, we can apply the Gauss summation:

1) $S_1 = 1 + 2 + 3 + \dots + (x-3) + (x-2) + (x-1)$
   $S_1 = (x-1) + (x-2) + (x-3) + \dots + 3 + 2 + 1$

After adding these two sums, we obtain the following relation:

$$2 \cdot S_1 = (1 + x - 1) + (2 + x - 2) + (3 + x - 3) + \dots + (x - 3 + 3) + (x - 2 + 2) + (x - 1 + 1)$$
$$= \underbrace{x + x + x + \dots + x}_{(x-1) \text{ times}}$$
$$= x(x-1) \Rightarrow S_1 = \frac{x(x-1)}{2}$$

2) $S_2 = (x+1) + (x+2) + (x+3) + \dots + (n-2) + (n-1) + n$

$S_2 = n + (n-1) + (n-2) + \dots + (x+3) + (x+2) + (x+1)$

After adding these two sums, we obtain the following relation:

$$2 \cdot S_2 = [(x+1) + n)] + [(x+2) + (n-1)] + [(x+3) + (n-2)] + \dots +$$
$$+[(n-2) + (x+3)] + [(n-1) + (x+2)] + [n + (x+1)]$$
$$= \underbrace{(x+n+1) + (x+n+1) + \dots + (x+n+1)}_{(n-x) \text{ times}}$$
$$= (x+n+1)(n-x) \Rightarrow S_2 = \frac{(x+n+1)(n-x)}{2}$$

After the sums are equaled, we will get the following relations:

$$\frac{x(x-1)}{2} = \frac{(x+n+1)(n-x)}{2}$$
$$2 \cdot x(x-1) = 2 \cdot (x+n+1)(n-x)$$
$$x^2 - x = n \cdot x - x^2 + n^2 - n \cdot x + n - x$$
$$x^2 - x = -x^2 + n^2 + n - x$$
$$2 \cdot x^2 = n^2 + n$$
$$x^2 = \frac{n^2+n}{2}$$
$$x = \pm\sqrt{\frac{n^2+n}{2}}$$

But $x \in \mathbb{N}^*$, so

$$x = \sqrt{\frac{n^2+n}{2}}, \ (x \in \mathbb{N}^*, \ n \in \mathbb{N}^*)$$

From here we can conclude that:

! x is a natural number if and only if $\frac{n^2+n}{2}$ is a perfect square.

DEFINITION: A **perfect square** is a number that can be expressed as the product of two equal integers (1).

The number of my house, which is a natural number, exists if and only if $\frac{n^2+n}{2}$ is a perfect square, where n is the total number of houses on my street.

Now it only remains to give n natural values, which must satisfy the conditions.

Let's take an example: let the number of houses be 8, so that using the algorithm below, we could calculate the address of the searched house.

The string of numbers will look as follows:

$$1, 2, 3, 4, 5, 6, 7, 8$$

Let the number of the house we are looking for be x, $x \in \mathbb{N}^*$.

Let the number of houses n be 8.

Our x must satisfy the following condition:

$$\underbrace{1 + 2 + 3 + \dots + (x - 1)}_{S_1} = \underbrace{(x + 1) + (x + 2) + (x + 3) + \dots + 7 + 8}_{S_2}$$

and it follows: $x = \sqrt{\frac{n^2 + n}{2}}$, $x \in \mathbb{N}^* \Rightarrow x = \sqrt{\frac{8^2 + 8}{2}} = 6$

Now we will replace the number n in the equation with 8. After calculating, x will be 6. This means the address of the searched house is 6.

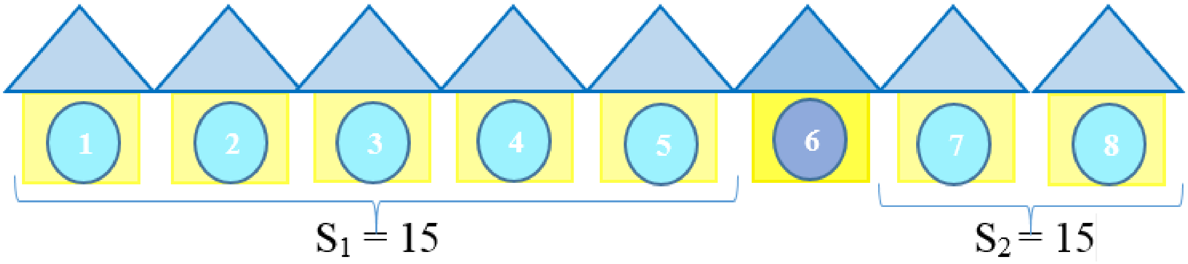

Figure 1: Illustration of the example above

Let's take another example, which is not valid: $n = 10$

$$\left.\begin{array}{l} x = \sqrt{\frac{10^2 + 10}{2}} \Rightarrow x = \sqrt{55} \\ \text{But } x \in \mathbb{N}^* \end{array}\right\} \Rightarrow x \in \emptyset$$

It is very difficult to find all the numbers n for which $\sqrt{\frac{n^2 + n}{2}}$ is a perfect square. Computer science helps us a lot in this case.

Method 2: PROGRAMMING

To make the values easier to find, we wrote some coding lines in C++ (programming language).

The program considers n streets, one by one (the street number i has i houses, numbered from 1 to i). Thanks to the fact that we go through them in ascending order (the street number i has i houses and the street number $i + 1$, $i + 1$ houses), we are allowed to use the information regarding the previous street in order to find answers for the current one.

In other words, when we are on the street number i, we store in the variable '`mijloc`', which means 'middle', the position of the house that satisfies the condition imposed by the problem, but for the previous street, $i - 1$ (2).

The next step is to add the house number i to the sum of the houses on the right side, then check if it is equal to the sum on the left side. In this case, we maintain the same position, as we have found the solution for the street number i.

If the sum on the left is smaller than the one on the right, we add the house number '`mijloc`' to the sum on the left and deduct it from the sum on the right. '`mijloc`' increases by 1, moving to the right, in the hope that the sums will eventually be equal.

Otherwise, if the sum on the right is smaller than the one on the left, we leave '`mijloc`' on the current position and try to increase the sum on the right as much as possible, in order to make it equal to the one on the left. Thus, we add the house number i to the sum on the right, obtained in the case of the previous street, $i - 1$.

```
#include <iostream>
using namespace std;
int main()
{
  long long n;
  cin >> n;
  long long dr = 1;
    long long sumSt = 0;
    long long sumDr = 0;
    long long mijloc = 1;
  while (dr <= n)
  {
    if (sumSt == sumDr)
    {
      cout << "The number of the house is " << mijloc << " for " << dr << " houses." << '\n';
      dr++;
      sumDr += dr;
    }
    else if (sumSt < sumDr)
    {
      sumSt += mijloc;
      mijloc++;
      sumDr -= mijloc;
    }
    else
    {
      dr++;
      sumDr += dr;
    }
  }
  return 0;
}
```

The program will display on the screen the address found and the total number of houses on the street:

```
The number of the house is 1 for 1 houses.
The number of the house is 6 for 8 houses.
The number of the house is 35 for 49 houses.
The number of the house is 204 for 288 houses.
The number of the house is 1189 for 1681 houses.
The number of the house is 6930 for 9800 houses.
The number of the house is 40391 for 57121 houses.
The number of the house is 235416 for 332928 houses.
The number of the house is 1.3721e+06 for 1940449 houses.
The number of the house is 7.99721e+06 for 11309768 houses.
The number of the house is 4.66112e+07 for 65918161 houses.
The number of the house is 8.52251e+07 for 120526554 houses.
The number of the house is 1.39834e+08 for 197754484 houses.
The number of the house is 1.62453e+08 for 229743340 houses.
The number of the house is 1.78448e+08 for 252362877 houses.
The number of the house is 1.94442e+08 for 274982414 houses.
The number of the house is 2.17061e+08 for 306971270 houses.
The number of the house is 2.33056e+08 for 329590807 houses.
The number of the house is 2.4905e+08 for 352210344 houses.
The number of the house is 2.7167e+08 for 384199200 houses.
The number of the house is 2.87664e+08 for 406818737 houses.
The number of the house is 2.94289e+08 for 416188056 houses.
The number of the house is 3.03659e+08 for 429438274 houses.
The number of the house is 3.10284e+08 for 438807593 houses.
The number of the house is 3.26278e+08 for 461427130 houses.
```

CONCLUSIONS

It can be noted that the number of houses on the street could be 1, 8, 49 or 288 in order to satisfy the condition mentioned above. The number of houses on a street is unlikely to be 1681 or 9800.

Mathematically speaking, there is an infinite number of possible addresses, depending on the number of houses on the street (3), but as a practical matter, there are only 4 possibilities: I can live at number 1, 6, 35 or 204.

EDITING NOTES

(1) Also, a number of the form $n(n+1)/2$ is called a *triangular number*, since it counts objects arranged in a triangle: dispose n dots on a row, $n-1$ on the row above, $n-2$..., and so on up to 1 dot on the n-th row, you get a triangle of dots and their total number is $1+2+\ldots+n$. Thus, the problem here is equivalent to looking for numbers that are both perfect squares and triangular numbers.

(2) More precisely, when i ('`dr`' in the program) takes a new value, '`mijloc`' is the position that satisfies the condition of the problem only if there exists such a position but in general it is the first position such that we have $S_1 \geq S_2$ in the $(i-1)$-th street. Then, comparing streets $i-1$ and i the values of S_1 are unchanged and that of S_2 are increased by i for all houses, so we are sure that $S_1 < S_2$ for all positions preceding '`mijloc`' and we can start testing at this position.

(3) This claim is not proven but it suggests that more could be done in this problem. A hint to go further: $n=8$ is twice a square and $n+1=9$ is a square; $n=49$ is a square and $n+1=50$ is twice a square; $n=288=2\times 12^2$ is twice a square and $n+1=289=17^2$ is a square, … When n is twice a square and $n+1$ is a square or vice versa, then $n(n+1)/2$ is a square (e.g. $n=a^2$, $n+1=2b^2$, $\frac{n(n+1)}{2}=a^2b^2$), so we have a solution. Is that the only possibility? Can we find infinitely many such integers n? …