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Lattice

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Abstract:

In mathematics, a combination is an arrangement in which order does not matter. Often contrasted with permutations, which are ordered arrangements, a combination defines how many ways you could choose a group from a larger group.

This is also how lotteries work. The numbers are drawn one at a time, and if we have the lucky numbers(no matter what order) we win!

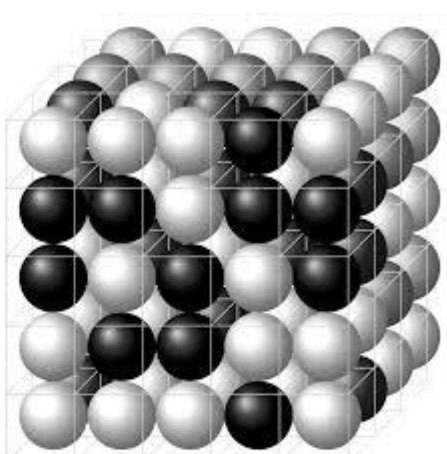


Figure 1. Lattice model (physics)

The problem:

a) Determine how many lines go through exactly two nodes of the network.
b) How many squares with peaks in network's nodes can be built? Generalize to a similar n×n type of network.
c) By randomly choosing 4 dots from the shown network, which is the probability that these dots are the peaks of a square?
d) By randomly choosing 3 dots from the network, which is the probability that these dots are the peaks of a triangle?
Random choices are made so that all dots have the same chance of being chosen.

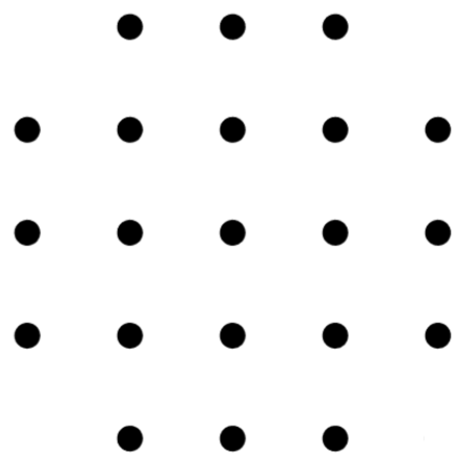


Figure 2. Problem's network

For the first subsection, we saw that we have a 5x5 lattice, but without the dots in the corners, meaning 21 dots. We used the combinatorial method. Decreasing from the initial number of lines that pass through exactly 2 dots on those that pass through 3, 4 and 5 points, finally getting the desired result. At subsection "b" we added the corners in the network and we noticed a pattern, after which we could obtain a formula for a square of the side k, with n points on the side, finally finding the solution with the principle of additivity. For the next subsection we used the formula of probability, thus calculating the probability of an event. We have learned the number of possible cases, with which we have reached the final result. The same formula I used for the subsection "d".

Solution of the problem

a) The technique used to solve this sub-item was **the combinatorial method**: (1)

$$C_n^m = \frac{n!}{(n-m)!m!}$$

- We calculated how many straight lines pass through any **2 points**, based on the above formula:

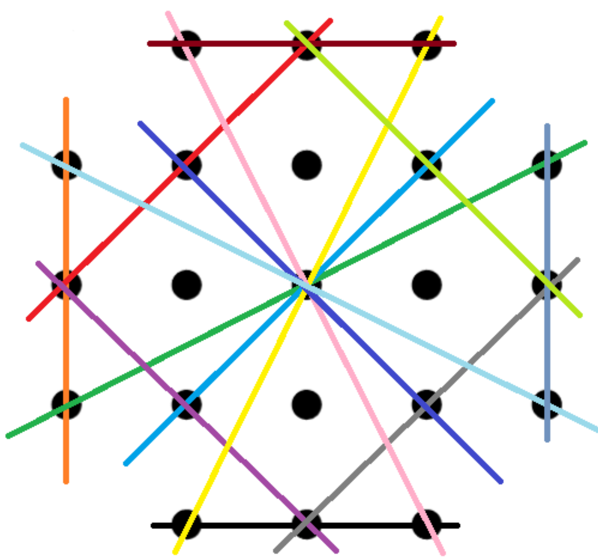
$$N_{\geq 2} = C_{21}^{2}$$

Total number of network's dots

$$C_{21}^{2} = \frac{21!}{2! \cdot 19!} = 210 \textbf{ straight lines}$$

- Next we counted how many lines we are going through exactly **3 points**, resulting in **14 lines** (**Figure 3**).

Figure 3:



But each line has already been counted twice:

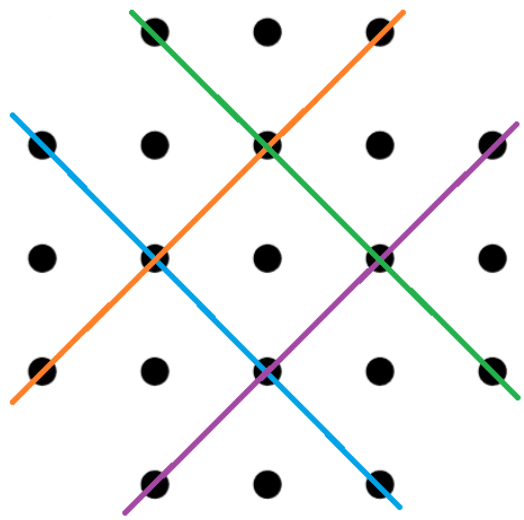
$$C_{3}^{2} = \frac{3!}{2! \cdot 1!} = 3$$

After these mathematical operations 14 lines · 3 lines= **52 lines** removed

The same thing we did for how many lines are crossing exactly 4 and exactly 5 dots.

- In the first case, we found counting **4 lines**, passing through fixed **4 points** (**Figure 4**).

Figure 4:



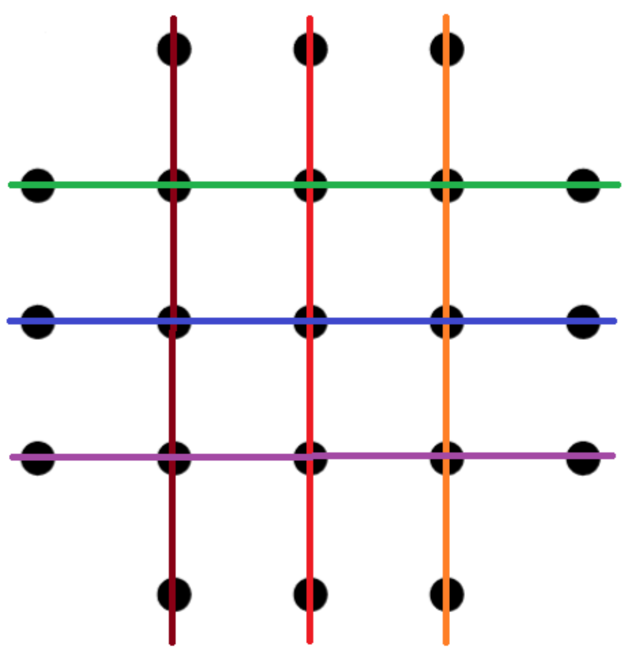
Each line was counted by:

$$C_4^2 = \frac{4!}{2!\cdot 2!} = 6\ \textbf{times}$$

Hence resulting that 4 lines · 6 lines=**24 lines** removed.

- Through exactly **5 dot,** we found **6 lines** by counting them (**Figure 5**).

Figure 5:



Each line was counted by:

$$C_5^2 = \frac{5!}{2!\cdot 3!} = 10 \textbf{ times}$$

Hence 6lines · 10 lines=60 **lines** removed.

- Finally, we have dropped from the initial number of lines that cross exactly 2 dots (210 lines), those eliminated during the problem:

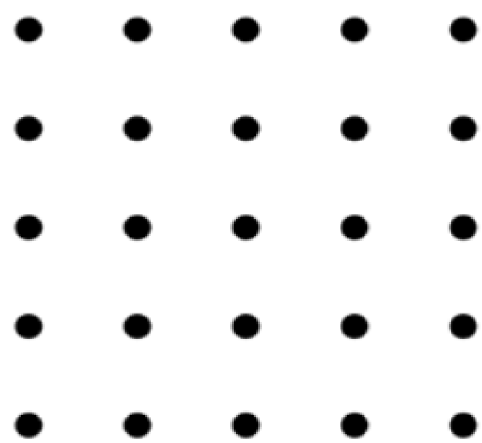
210-136=**74 lines which intersect exactly 2 dots**

b) In this subsection we had to figure out how many squares with the peak in the network's nodes can be built by generalizing to a similar network n·n.

First of all we took into account the 4 dots in the network so we can get a square with the 5-dot side and find out how many squares with the sides whole numbers and with the sides real numbers are in the figure.

From the Figure 6 we obtained squares with full number sides:

Figure 6:



- 16= $(5-1)^2$ *squares with the side of* 1
- 9= $(5-2)^2$ *squares with the side of* 2
- 4= $(5-3)^2$ *squares with side of* 3
- 1= $(5-4)^2$ *squares with side of* 4

By observing the pattern, we get the formula for a side square k, where n equals the number of points on one side: $(\boldsymbol{n}-\boldsymbol{k})^2=(5-\boldsymbol{k})^2$

And squares with sides which are measured in real numbers:

- $16 = (5-1)^2$ *squares with the side of* $\sqrt{2}$
- $18 = 2(5-2)^2$ *squares with the side of* $\sqrt{5}$
- $2 = 2 \cdot 1$ *squares with the side of* $\sqrt{10}$
- $2 = 2 \cdot 1$ *squares with the side of* $2\sqrt{2}$

We can write the general formula for **n** points:

$$N_n = (n-1)^2 + 2(n-2)^2 + 3(n-3)^2 + \dots\dots\dots + (n-1) \cdot 1^2$$

We check and get: $N_5 = 4^2 + 2 \cdot 3^2 + 3 \cdot 2^2 + 4 \cdot 1^2 =$ **50**

Then subtract all squares that pass through the 4 points added: **4+3+3+3= 13 squares**

And result: **50-13=37 squares**

c) We denote by: P=probability

The probability of an event is equal to:

$$P = \frac{number\ of\ favourable\ cases}{number\ of\ possible\ cases}$$

The number of favorable cases is equal to the number of squares that have peaks in our network's nodes(being also part of the result of the previous paragraph) because we have taken all the possibilities of obtaining a square(regardless of the length of the sides) with the peaks in the dots of the network.

Therefore, number of favorable cases is **37.**

The number of possible cases is determined by all the combinations in the formation of a square of the 21 available dots, resulting in **combinations of 21 taken 4 times.**

By a few mathematical calculations we have reached the exact number of possible cases:

$$C_{21}^{4}=\frac{21!}{4!\cdot(21-4)!}=\frac{18\cdot19\cdot20\cdot21}{2\cdot3\cdot4}=\mathbf{5985}$$

Consequently, the result of the probability is:

$$P=\frac{37}{5985}\cong 0{,}006$$

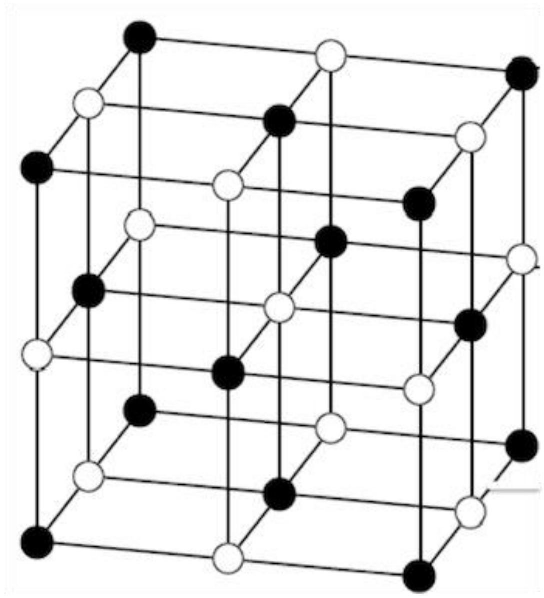


Figure 7: Crystal Lattice

d) As in the previous subsection we had to use the probability formula and applied the same solving strategy: we calculated all the possible combinations in which the dots are the vertices of a triangle (representing the number of possible cases), then we dropped all the lines that go through exactly 3.4 and 5 points and repeat. This value represents the number of favorable cases.

By not knowing this number (the number of favorable cases) we marked it with **X**.

For the beginning, we have calculated all possible dots that pass through **at least 3 dots** (combinations of 21 taken 3 times):

$C_{21}^{3}=\mathbf{1330}$ (the number of possible cases)

In the next part we counted and found out how many lines go through **exactly** 3.4 and 5 points and how many times they are counted:

- Through 3 dots = **14** lines counted **once**
- Through 4 dots = **4** lines counted $C_4^3 = \mathbf{4}$ **times**
- Through 5 dots = **6** lines counted $C_5^3 = \mathbf{10}$ **times**

There followed their decrease in the total number of lines and the finding of **X**:

$$1330 - 14 \cdot 1 - 4 \cdot 4 - 6 \cdot 10 = 1330 - 90 = \mathbf{1240}$$ (the number of favorable cases)

Returning to our probability and to the solution of this subsection:

$$P = \frac{1240}{1330} = \frac{124}{133} \cong \mathbf{0.93}$$

Conclusion

At first we thought it was easy to find the total number of lines, probabilities and combinations, but we realized that we needed an algorithm to help us solve the problem. After several searches we found the necessary formulas for the desired result. After a lot of work and hours spent in getting the inevitable answers, we have succeeded in finalizing the problem and enjoying our success together.

Edition note : (1) The notation used in this paper to denote binomial coefficients is an old one. Now one prefer use the notation, for n greater than m, $\binom{n}{m} = \frac{n!}{m!(n-m)!}$