No money without mathematics!

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1. Presentation of the research topic

This project aimed to initiate us into the field of economics and the strong relationship between mathematics and economics. The idea is to introduce the concept of risk and to try to generalize , which changed the face of today's microeconomics. There are also some important generalizations of the results. We will also discover a paradox that will make us think that maybe the classical way economists are analyzing risk might have its faults.

2. The Problem

1. Suppose that Alice wants to buy an insurance in order to avoid the costs of an accident, that happens with probability p that might produce a damage of x euro. An insurance firm wants to sell an insurance contract at price y. What will Alice take into account when choosing upon buying the contract?
2. What is the best insurance contract for Alice with an initial level of wealth w facing a risk of accident, represented by a utility function u, considering different cost structures for the insurance company? Take a numerical example?
3. What is the individual's risk premium and how do they choose coverage levels under different insurance premium structures? How does the willingness to pay for full coverage compare to partial coverage, and what insights does this provide about the individual's insurance decisions?
4. Is risk sharing between individuals always better? Are there any faults with the rational theory developed so far?

3. Elements from Probability Theory and Expected Utility Theory

3.1. Expectation and Variance

Definition 1. Let us consider a probabilistic experiment (for example, throwing a dice n times). A random variable is a function that assigns values to each possibility of the experiment.

Definition 2. For a random variable X defined on a probability space, the expected value $E[X]$ is given by:

$$\mathbb{E}[X] = \sum_{w \in \Omega} X(w) \cdot \mathbb{P}[w],$$

where Ω is the set of all possible outcomes, $X(w)$ is the value of X at outcome w, and $\mathbb{P}[w]$ is the probability of w.

Example 1. *Let us consider X to be the random variable that measures how many heads do we obtain when we toss twice a coin. By observing the probabilities, we have that:*

$$\mathbb{E}[X] = \mathbb{P}[X=0] \cdot 0 + \mathbb{P}[X=1] \cdot 1 + \mathbb{P}[X=2] \cdot 2 = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1.$$

This can be interpreted as the fact that, if we do the experiment that describes X many times, we expect to obtain, in average, 1 head.

Lemma 1 (Properties of Expectation)**.**

- *For two random variables X_1 and X_2, the expectation of their sum is the sum of their expectations:*

$$\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$$

- *If c is a constant, then:*

$$E[c \cdot X] = c \cdot \mathbb{E}[X]$$

- *If X is a random variable that takes non-negative values, then we have $\mathbb{E}[X] \geq 0$.*

Definition 3. The variance of a random variable X measures the spread of its values around the expected value. It is defined as:

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

Lemma 2. *If X is a real-valued random variable, then we have that:*

$$Var(X) = E[X^2] - (E[X])^2.$$

Proof. From the definition and Lemma 1, we have that:

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2 - 2X\mathbb{E}[X] + (\mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[2X\mathbb{E}[X]] + \mathbb{E}[(\mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + (\mathbb{E}[X])^2 = \mathbb{E}[X^2] - (E[X])^2.\end{aligned}$$

□

Remark 1. One of the first immediate consequences of the definition of variance is that, if $k \neq 0$ is a real number and X is a random variable, then $\mathrm{Var} kX = k^2 \mathrm{Var} X$.

Example 2. *Let us compute the variance for the random variable X representing the number of heads when flipping two coins. First of all, as we have previously seen, $\mathbb{E}[X] = 1$. Now, we calculate $E[X^2]$:*

$$\mathbb{E}[X^2] = 0^2 \cdot \mathbb{P}(X=0) + 1^2 \cdot \mathbb{P}(X=1) + 2^2 \cdot \mathbb{P}(X=2) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} = \frac{3}{2}.$$

Then, the variance is equal to: $Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{3}{2} - 1^2 = \frac{1}{2}$.

Definition 4. Two discrete random variables X and Y are independent if for all values x and y,

$$\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x]\mathbb{P}[Y = y].$$

Example 3. *We will not give an example of independent random variables, because it is very simple to construct them. Indeed, let us give a pair of non-independent random variables. Let X be the number of heads and Y the number of tails when tossing two fair coins. The possible outcomes and values of (X, Y) are:*

Outcome	(X, Y)
(H, H)	$(2, 0)$
$(H, T), (T, H)$	$(1, 1)$
(T, T)	$(0, 2)$

The probabilities are:

$$\mathbb{P}(X = 1) = \frac{2}{4}, \quad \mathbb{P}(Y = 1) = \frac{2}{4}, \quad \mathbb{P}(X = 1, Y = 1) = \frac{2}{4}.$$

We check independence:

$$\mathbb{P}(X = 1, Y = 1) = \frac{1}{2} \neq \mathbb{P}(X = 1)\mathbb{P}(Y = 1) = \frac{1}{4}.$$

*Thus, X and Y are **not independent**.*

Lemma 3. *If X and Y are independent random variables, then we have that:*

$$Var(X + Y) = Var(X) + Var(Y).$$

Proof. Using Lemma 2 for the random variable $X + Y$

$$\text{Var}(X + Y) = \mathbb{E}[(X + Y)^2] - (\mathbb{E}[X + Y])^2.$$

Now, by expanding both terms:

$$E[(X + Y)^2] = E[X^2] + 2E[XY] + E[Y^2]$$

Since X and Y are independent, we have that $E[XY] = E[X]E[Y]$. Hence:

$$E[(X + Y)^2] = E[X^2] + E[Y^2] + 2E[X]E[Y]$$

and also:

$$(E[X + Y])^2 = (E[X] + E[Y])^2 = E[X]^2 + E[Y]^2 + 2E[X]E[Y]$$

From the previous two relations, by subtracting, we have that:

$$\text{Var}(X + Y) = (E[X^2] - E[X]^2) + (E[Y^2] - E[Y]^2) = \text{Var}(X) + \text{Var}(Y)$$

□

3.2. The economic set-up

Definition 5. The utility function, usually denoted as u, represents an individual's preferences regarding different goods and services. It quantifies the satisfaction or happiness (utility) a consumer derives from consuming a combination of goods.

Definition 6. Marginal Utility is the additional satisfaction or benefit (utility) that a consumer gains from consuming one more unit of a good or service. It reflects how much extra enjoyment or value is derived from each incremental increase in consumption. Typically, it shows diminishing marginal utility, meaning that as a person consumes more of a good, the additional satisfaction gained from each extra unit decreases.

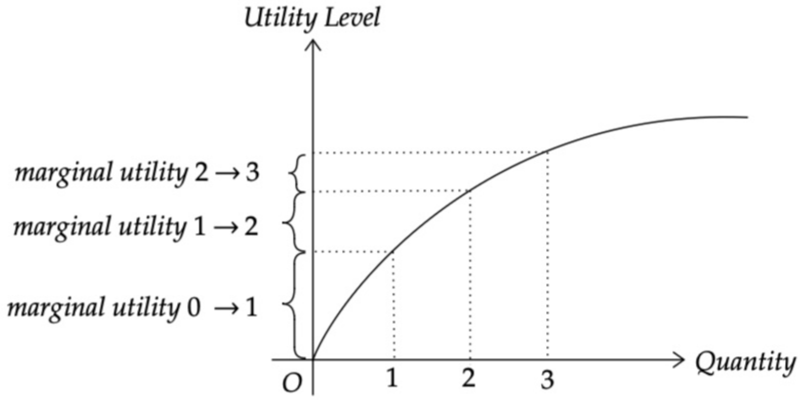


Figure 1: Utility function and Decreasing Marginal Utility

Remark 2. $u(x)$ isn't necessarily equal to $2u(x)$, where x is a real positive number. Indeed, we can think of it as the fact that the utility of consuming 2 apples does not equal 2 times the utility of consuming one apple since the first apple consumed produces more satisfaction than the second one.

Definition 7. Expected utility is a concept that represents the average or anticipated utility (satisfaction) an individual expects to obtain from a set of possible outcomes of a particular action, each outcome having a specific probability of occurring.

Let $u : \mathbb{R}^* \to \mathbb{R}$ be a utility function. For a lottery W (with the possible outcomes $W_1, \ldots, W_n$ and the corresponding associated probabilities $p_1, \ldots, p_n$), we have that the expected utility, $\mathbb{E}[u(W)]$, is equal to

$$\mathbb{E}[u(W)] = p_1 u(W_1) + p_2 u(W_2) + \cdots + p_n u(W_n),$$

while the expected payoff is

$$\mathbb{E}(W) = p_1 W_1 + p_2 W_2 + \cdots + p_n W_n.$$

The certainty equivalent is the guaranteed amount of money (or utility) that an individual would accept instead of taking on a risky prospect with an uncertain payoff. It is often used to measure an individual's risk tolerance.

Definition 8. For a lottery W, the certainty equivalent is a number $CE(W)$ so that we have

$$u(CE(W)) = \mathbb{E}[u(W)].$$

Remark 3. A relevant example is the following one: if there is a 50% chance of winning 2 coins and a 50% chance of winning nothing, and we have already decided that $u(x) = x$, x being an arbitrary number, we observe that

$$\mathbb{E}[u(L)] = \frac{1}{2}u(2) + \frac{1}{2}u(0) = 1.$$

Thus,

$$u(CE(L)) = \mathbb{E}[u(L)] \quad \text{so we obtain that} \quad CE(L) = 1.$$

There are three main types of risk preferences:

- **Risk-neutral:** Indifferent to risk; evaluate options purely based on their expected value, without concern for the associated risk. Ideal for business decision-making (companies). The certainty equivalent is equal to the expected value: $\mathbb{E}[u(W)] = u(\mathbb{E}[W])$. They would choose the option with the highest expected value, regardless of the risk.

- **Risk-lover (or risk-seeking):** Enjoys taking risks and is willing to accept uncertainty, even if it means potentially lower returns. For a risk lover, the certainty equivalent is higher than the expected value: $\mathbb{E}[u(W)] > u(\mathbb{E}[W])$. They prefer the excitement or potential of a larger payoff, even if the expected return is the same or lower.

- **Risk-averse:** Prefers certainty over uncertainty, avoids risky options, even if the potential reward is high (e.g., useful in insurance purchases). The certainty equivalent is lower than the expected value: $\mathbb{E}[u(W)] < u(\mathbb{E}[W])$. They value stability and would rather take a smaller, certain gain than a larger, uncertain one(1).

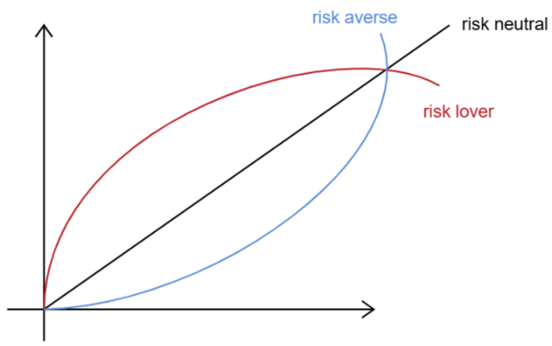


Figure 2: Qualitative graph of risk preferences

Let us now give an example of Certainty Equivalent and the different types of consumers.

Example 4. *Consider a lottery L that returns 2 Euros with probability $\frac{1}{2}$ and 0 Euros with probability $\frac{1}{2}$. Three consumers (0, 1, and 2) have respective utility functions:*

$$u_0(x) = x, \quad u_1(x) = x^2, \quad u_2(x) = \sqrt{x}$$

To compute the Certainty Equivalent (CE), follow these steps:
First of all, let us compute the expected utility:

$$\begin{aligned}
\mathbb{E}[u_0(L)] &= 0.5 \times u_0(0) + 0.5 \times u_0(2) = 0.5 \times 0 + 0.5 \times 2 = 1, \\
\mathbb{E}[u_1(L)] &= 0.5 \times u_1(0) + 0.5 \times u_1(2) = 0.5 \times 0 + 0.5 \times 2^2 = 2, \\
\mathbb{E}[u_2(L)] &= 0.5 \times u_2(0) + 0.5 \times u_2(2) = 0.5 \times 0 + 0.5 \times \sqrt{2} = 0.71.
\end{aligned}$$

We know invert the utility function, in order to obtain the certainty equivalent:

- *Consumer 0: Since* $\mathbb{E}[u_0(L)] = 1$, *solving* $u_0(CE) = 1$ *gives* $CE = 1$.
- *Consumer 1: Since* $\mathbb{E}[u_1(L)] = 2$, *solving* $u_1(CE) = 2$ *gives* $CE = \sqrt{2} = 1.41$.
- *Consumer 2: Since* $\mathbb{E}[u_2(L)] = 0.71$, *solving* $u_2(CE) = 0.71$ *gives* $CE = (0.71)^2 = 0.51$.

We summarize everything in the following table, from which we can deduce the likeliness towards risk:

Consumer	$\mathbb{E}[u_i(L)]$	CE	$\mathbb{E}(V)$
0	1	1	1
1	2	1.41	1
2	0.71	0.51	1

- u_0 *is risk-neutral since* $CE = \mathbb{E}(V) = 1$.
- u_1 *is risk-loving since* $CE = 1.41 > \mathbb{E}(V)$.

- u_2 *is risk-averse since* $CE \approx 0.51 < \mathbb{E}(V)$.

Definition 9. The *risk premium* $\pi(x)$ is the maximal amount that the individual is willing to pay to eliminate all risk associated with the lottery x, i.e.:

$$u(E(x) - \pi(x)) = \mathbb{E}[u(x)] \iff \pi(x) = \mathbb{E}[x] - CE(x).$$

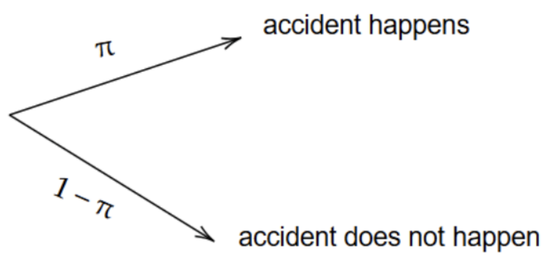
3.3. Choosing the best insurance

An insurance contract is a formal agreement between an insurer (insurance company) and the insured (policyholder) that defines the terms under which the insurer will provide financial protection or compensation for certain losses, damages, illness, or death in exchange for a premium. These contracts are crucial for managing financial risk, as they transfer potential economic burdens from the insured to the insurer.

The coverage q represents the specific protection provided by an insurance policy, detailing the risks, perils, or types of losses that the insurer agrees to cover. It specifies what kinds of incidents or damages the insurer will pay for if they occur.

The insurance premium p represents the amount of money an individual or business pays for an insurance policy. Insurance premiums are paid on policies that cover a variety of personal and commercial risks. If the policyholder fails to pay the premium, the insurance company may cancel the policy.

In order to choose a proper insurance agency, each individual should first analyze and learn the mechanism of it. In the case of an accident, the company earns the sum p, but has to pay the sum q, so the company ends up with the sum $p - q$. However, in the contrary case, the company earns the sum paid by the individual, without having to pay for any other damages.



As the company's main goal is to make profit, we have the following inequality:

$$\mathbb{E}[W] \geq 0 \quad \Rightarrow \quad \pi(p-q) + (1-\pi)p \geq 0 \quad \Rightarrow \quad p - \pi q \geq 0 \quad (1).$$

From this, we understand that the company actually has a utility function $u(x) = x$ for $x \geq 0$(2). Based on the anterior graphic, we can define that the firm is risk neutral. Let λ be defined as markup.

From the inequality (1),

$$p \geq \pi q \quad \Rightarrow \quad p \cdot \pi q \geq 1 \quad \Rightarrow \quad p \cdot \pi q = 1 + \lambda$$

$$p = (1+\lambda)(\pi q).$$

In this case, if $\lambda = 0$, the contract is called *actuarially fair*.

4. Solution

4.1. Risk Sharing is Always Better!

As mentioned in the statement, we will try to work on a numerical example:

Consider two individuals who face identically and independently[a] distributed risks. This risk corresponds to the following lottery: with probability 0.5, the individual loses 10 000, with probability 0.5 nothing happens. They then decide to form a pool, that is, to setup the following risk-sharing agreement: once the risk has been realized (that is, the outcome of the two lotteries is known), they share equally the total loss (this means that if only one of them lost, s/he is compensated 5 000 by the other).

1. Compute the mean and variance of the lottery.
2. Define the new lottery faced by each individual when they join the pool. Compute the mean and variance of this lottery. What are the benefits of joining the pool?
3. Suppose now that a third individual (facing the same risk) joins the pool (same risk-sharing rule as before). Define the new lottery faced by each individual when they join the pool. Compute the mean and variance of this lottery.
4. How do you think that mean and variance are going to evolve as more (similar) individuals join the pool?

[a]independent risks means that they don't depend one on other, give an example!

Using the expected value formula, we get that In this case (when 2 people join the lottery), $\mathbb{E}[X] = \frac{1}{2} \cdot 10.000\$ + \frac{1}{2} \cdot 0 = 5000\$$. In order to compute the variance, let us find $\mathbb{E}[X^2]$. We have that:

$$\mathbb{E}[X^2] = \frac{1}{2} \cdot 10.000^2 + \frac{1}{2} \cdot 0^2 = 50.000.000$$

Hence, by the formula of the variance,

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 25.000.000.$$

Let us model what happens when we form a pool with two participants. Because the total loss is shared equally between the two individuals, the possible outcomes are presented in 1.

Scenario	**Probability**	**Outcome for Each Individual**
Both individuals suffer no loss	$0.5 \times 0.5 = 0.25$	$\frac{0+0}{2} = 0$
One individual loses 10,000, the other loses nothing	$0.5 \times 0.5 + 0.5 \times 0.5 = 0.5$	$\frac{10{,}000+0}{2} = 5{,}000$
Both individuals lose 10,000	$0.5 \times 0.5 = 0.25$	$\frac{10{,}000+10{,}000}{2} = 10{,}000$

Table 1: Probabilities and Outcomes for Each Scenario when $n = 2$

Thus, the new lottery for each individual has the following possible outcomes and probabilities that we can see in Figure 3.

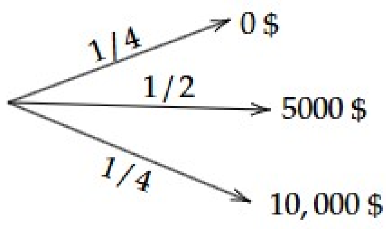


Figure 3: The new lottery when we have $n = 2$ participants

The expected value (mean) of this new lottery can be computed as:

$$\mathbb{E}[S] = \frac{1}{4} \times 0 + \frac{1}{2} \times 5{,}000 + \frac{1}{4} \times 10{,}000 = \mathbb{E}[S] = 0 + 2{,}500 + 2{,}500 = 5{,}000.$$

Let us now compute the variance. First, we calculate $\mathbb{E}[S^2]$:

$$\mathbb{E}[S^2] = \frac{1}{4} \times (0)^2 + \frac{1}{2} \times (5{,}000)^2 + \frac{1}{4} \times (10{,}000)^2.$$

$$\mathbb{E}[S^2] = 0 + \frac{1}{2} \times 25{,}000{,}000 + \frac{1}{4} \times 100{,}000{,}000 = 37{,}500{,}000.$$

Now, subtract the square of the mean:

$$\text{Var}(S) = 37{,}500{,}000 - (5{,}000)^2 = 37{,}500{,}000 - 25{,}000{,}000 = 12{,}500{,}000.$$

We can already observe that joining the pool makes the lottery less risky for each person as the variance has decreased. Let us continue the argument by seeing what happens when we do have $n = 3$ individuals joining the pools. When a third individual joins the pool, they share the total loss equally with the same risk-sharing rule. Each individual now faces a lottery where the total loss is divided among three participants.

Scenario	**Probability**	**Outcome for Each Individual**
None of the three lose anything	$\frac{1}{8}$	0
One person loses 10,000, the others lose nothing	$\frac{3}{8}$	$\frac{10{,}000}{3} = 3{,}333.333$
Two people lose 10,000, one loses nothing	$\frac{3}{8}$	$\frac{20{,}000}{3} = 6{,}666.666$
All three lose 10,000	$\frac{1}{8}$	10,000

Table 2: Probabilities and Outcomes for Each Scenario with Three Individuals

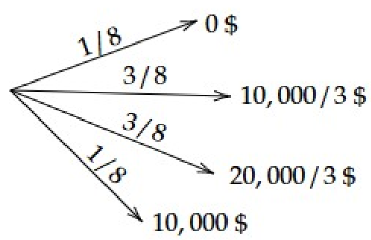


Figure 4: New lottery outcomes when we have $n = 3$ participants

The expected value of the new lottery remains the same:

$$\mathbb{E}[S] = \frac{1}{8} \times 0 + \frac{3}{8} \times \frac{10{,}000}{3} + \frac{3}{8} \times \frac{20{,}000}{3} + \frac{1}{8} \times 10{,}000 = 5{,}000.$$

Similarly, we can compute the variance:

$$\text{Var}(S) = \mathbb{E}[S^2] - (\mathbb{E}[S])^2$$

$$\mathbb{E}[S^2] = \frac{1}{8} \times (0)^2 + \frac{3}{8} \times \left(\frac{10{,}000}{3}\right)^2 + \frac{3}{8} \times \left(\frac{20{,}000}{3}\right)^2 + \frac{1}{8} \times (10{,}000)^2 = \frac{100{,}000{,}000{,}000}{3}$$

$$\text{Var}(S) = \frac{100{,}000{,}000{,}000}{3} - 25{,}000{,}000 = \frac{25{,}000{,}000{,}000}{3},$$

and hence, we observe that the variance decreased as well. We will now consider the general case when we have n participants in the pool. The mean of the lottery for each individual remains the same as more people join the pool. This is because the expected loss per person does not depend on how many individuals share the total loss.

$$E[X_i] = \frac{1}{n}\sum_{i=1}^{n} E[L_i]$$

where L_i is the individual loss in a pool with n people. The variance decreases as more individuals join the pool. In a pool of n individuals, the loss for each individual is the average of n independent and identically distributed losses $X_1, X_2, \ldots, X_n$ and we will now show that:

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{\mathrm{Var}(X)}{n}$$

We will show the result by induction. The base case ($n = 2$) is true by Lemma 3:

$$\mathrm{Var}\left(\frac{X_1 + X_2}{2}\right) = \frac{1}{4}\left(\mathrm{Var}(X_1) + \mathrm{Var}(X_2)\right) = \frac{\mathrm{Var}(X)}{2}$$

Using the inductive hypothesis (assuming that the formula works for $n = k$, where k is an arbitrary chosen natural number), we have to prove that the formula works for $n = k + 1$.

Now we know that:

$$\mathrm{Var}\left(\frac{1}{k}\sum_{i=1}^{k} X_i\right) = \frac{\mathrm{Var}(X)}{k}$$

and by adding X_{k+1}, we obtain:

$$\mathrm{Var}\left(\frac{1}{k+1}\sum_{i=1}^{k+1} X_i\right) = \frac{1}{(k+1)^2}\left(k \cdot \mathrm{Var}(X) + \mathrm{Var}(X)\right)$$

Simplifying:

$$\mathrm{Var}\left(\frac{1}{k+1}\sum_{i=1}^{k+1} X_i\right) = \frac{\mathrm{Var}(X)}{k+1}$$

Thus, the formula holds for $n = k + 1$.

Now, using the base case and the inductive step, the principle of induction tells us that:

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{\mathrm{Var}(X)}{n}$$

for any n independent variables, and our proof is finished.(3)

4.2. How do we establish insurance contracts?

Again, we will begin by analyzing a particular numerical case:

> An individual has an initial wealth equal to 40 and faces a risk of accident. An accident occurs with probability 0.5 and generates a loss of 20. The individual's preferences can be represented by the utility function $u(R) = \sqrt{R}$ or $\ln(R)$, where R is the individual's final wealth. An insurance company proposes a contract that offers a reimbursement T of accident against a fixed premium equal to p. What is the best insurance contract (from the individual's point of view) that guarantees a non-negative expected profit to the insurance company is a full coverage contract? How is the risk premium as a function of wealth ?

We will start by considering the case when the utility function is the natural logarithm, and then the square root(4). In our problem, there are two possible scenarios: accident ($P = 0.5$) and no accident ($P = 0.5$). The utility function is $u(W) = \sqrt{W}$. The final wealth W depends on whether an accident occurs, the premium p, and the reimbursement T.

The expected utility can be expressed as:

$$\mathbb{E}[U] = 0.5 \cdot \sqrt{40-p} + 0.5 \cdot \sqrt{20-p+T}.$$

To maximize expected utility, we first observe that expected utility is maximized when the premium p is minimized. However, there is a condition that must be satisfied: the premium p must be large enough to ensure the insurer can cover the reimbursement T. Therefore, $p \geq \frac{T}{2}$ is a necessary condition.

Let us substitute $T = 2p$ into the expected utility formula and then we get :

$$\mathbb{E}[U] = 0.5 \cdot \sqrt{40-p} + 0.5 \cdot \sqrt{20-p+2p}.$$

Simplifying the second term, we deduce that:

$$\mathbb{E}[U] = 0.5 \cdot \sqrt{40-p} + 0.5 \cdot \sqrt{20+p},$$

and we want to maximize it over p. In order to obtain this result, we will use the following lemma:

Lemma 4. *For all $a, b > 0$,*

$$\sqrt{a} + \sqrt{b} \leq \sqrt{2(a+b)}.$$

Equality holds when $a = b$.

Proof. To prove this, square both sides:

$$(\sqrt{a} + \sqrt{b})^2 \leq 2(a+b).$$

Expanding the left-hand side:

$$a + b + 2\sqrt{ab} \leq 2(a+b).$$

By simplifying, we get that:

$$2\sqrt{ab} \leq a + b,$$

and this inequality represents the geometric mean and arithmetic mean inequality, which holds true with equality when $\sqrt{a} = \sqrt{b}$, i.e., $a = b$.

□

Going back to our problem, using the inequality $\sqrt{a} + \sqrt{b} \leq \sqrt{2(a+b)}$, where $a = 40 - p$ and $b = 20 + p$, we have:

$$\mathbb{E}[U] \leq 0.5 \cdot \sqrt{2((40-p)+(20+p))}.$$

Simplify the terms, we obtain that:

$$\mathbb{E}[U] \leq 0.5 \cdot \sqrt{2 \cdot 60} = \sqrt{30}.$$

Thus, the maximum expected utility is $\sqrt{30}$.

Equality holds when $\sqrt{40-p} = \sqrt{20+p}$. Squaring both sides, we deduce that $40 - p = 20 + p$, so $p = 10$.

The maximum expected utility is achieved when $p = 10$, and its value is equal to $\mathbb{E}[U] = \sqrt{30}$.

Now, we are considering the case where the utility function is represented by the logarithmic function:

$$\mathbb{E}[U] = 0.5 \cdot \ln(40-p) + 0.5 \cdot \ln(20+p).$$

Combining the logarithms:

$$\mathbb{E}[U] = 0.5 \cdot \ln\big((40-p)(20+p)\big).$$

In order to maximize the expression, we simply observe that the term $(40-p)(20+p)$ expands to:

$$(40-p)(20+p) = 800 + 20p - 40p - p^2 = 800 - p^2 - 20p.$$

The expected utility is maximized when the function $800 - p^2 - 20p$ is maximized (5) and here we obtain that, since $p \geq \frac{T}{2} = 10$, the maximum is again achieved for $p = 10$.

This confirms that the premium of $p = 10$ ensures the optimal insurance decision under the given constraints.

We also want to find the risk premium in the case when the utility function is the natural logarithm (6).

Let us compute now the expected utility:

$$u\left(\mathbb{E}(\widetilde{W}) - \pi\right) = \mathbb{E}(u(\widetilde{W})) \iff$$
$$\ln(W - 5 - \pi) = \frac{1}{2} \cdot \ln(W - 10) + \frac{1}{2} \cdot \ln(W) \quad (\star)$$

Using the previous relation, we can quickly find the formula for the risk premium:

$$\mathbb{E}(\widetilde{W}) = \frac{1}{2}(W - 10) + \frac{1}{2}W = W - 5.$$
$$(\star) \iff \ln(W - 5 - \pi) = \frac{1}{2}\ln(W(W - 10)) \iff$$
$$\ln(W - 5 - \pi) = \ln\left(\sqrt{W(W - 10)}\right) \iff$$
$$W - 5 - \pi = \sqrt{W(W - 10)} \iff$$
$$\pi = W - 5 - \sqrt{W(W - 10)}.$$

We will try to figure out if the function is increasing or decreasing considering

$$W_2 > W_1:$$
$$\pi(W_2) - \pi(W_1) = W_2 - 5 - \sqrt{W_2(W_2 - 10)} - W_1 + 5 + \sqrt{W_1(W_1 - 10)}$$
$$= W_2 - W_1 + \sqrt{W_1(W_1 - 10)} - \sqrt{W_2(W_2 - 10)}$$
$$= W_2 - W_1 + \frac{W_1(W_1 - 10) - W_2(W_2 - 10)}{\sqrt{W_1(W_1 - 10)} + \sqrt{W_2(W_2 - 10)}}$$
$$= W_2 - W_1 + \frac{W_1^2 - W_2^2 + 10(W_2 - W_1)}{\sqrt{W_1(W_1 - 10)} + \sqrt{W_2(W_2 - 10)}}$$
$$= (W_2 - W_1)\left(1 + \frac{-W_1 - W_2 + 10}{\sqrt{W_1(W_1 - 10)} + \sqrt{W_2(W_2 - 10)}}\right),$$

and we cannot deduce the monotonicity by this analysis (7). One way to find the monotonicity of π is to use for instance a graph calculator. Hence, by looking at the Figure 5 for $W > 10$, we deduce that the function π is decreasing with respect to W, as we intuitively expected.

As wealth increases, the marginal utility of money decreases, making potential losses less painful. Wealthier individuals can better absorb financial risks, reducing their need for high risk premiums. Consequently, they demand lower extra returns to take on risky investments.

4.3. Allais Paradox

The Allais paradox demonstrates deviations from expected utility theory when participants make choices between two gambles in two experiments. This means that, even though we analyzed the way people make decisions, subconsciously, by asserting a utility function for each outcome and computing/comparing the expected utility, in reality, this theory might have its faults. For this paradox, we will consider 2 experiments: 1 and 2 (each containing 2 possible lotteries) and each participant to

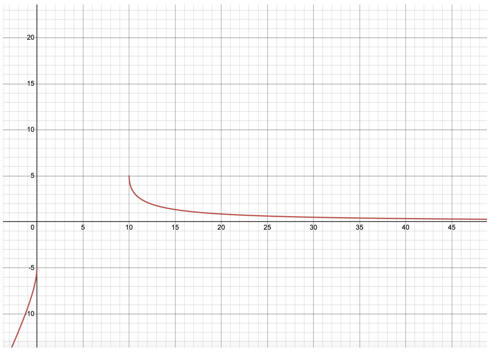


Figure 5: Graph of π as a function of W

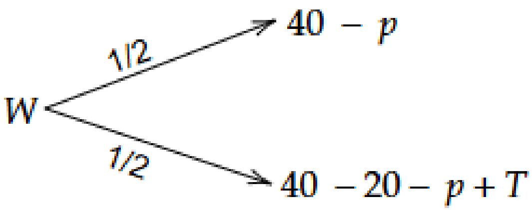


Figure 6: Lottery corresponding to the insurance contract

Experiment	Gamble	Winnings	Chance (%)
1	1A	\$1M	100
	1B	\$1M	89
		Nothing	1
		\$5M	10
2	2A	Nothing	89
		\$1M	11
	2B	Nothing	90
		\$5M	10

Table 3: Experimental setup for the Allais paradox.

the survey is asked to choose the lottery he prefers from the corresponding 2. There are variations of these experiments but we will consider the classical one[1].

Studies have consistently shown that most participants choose 1A over 1B and 2B over 2A, even though this violates the principles of expected utility theory. The expected average outcomes of the gambles (in millions) are:

$$1A : 1.00, \quad 1B : 1.39, \quad 2A : 0.11, \quad 2B : 0.50.$$

Allais observed that choosing 1A and 2B together was inconsistent with the developed theory, which predicts either:

- Choosing 1A and 2A, or
- Choosing 1B and 2B.

The expected payouts of the combinations (in millions) are:

$$1A + 2A : 1.11, \quad 1B + 2B : 1.89, \quad 1A + 2B : 1.50, \quad 1B + 2A : 1.50.$$

[1] the one that can be found at https://en.wikipedia.org/wiki/Allais_paradox

The independence axiom of the expected utility theory would state that equal outcomes in two choices should cancel out and have no bearing on the relative preference between the two gambles. However, this axiom does not align with human behavior.

By disregarding the common outcome (89% probability) shared across the gambles, the revised payoffs become:

Experiment	**Gamble**	**Winnings**	**Chance (%)**
1	1A	\$1M	11
	1B	Nothing	1
		\$5M	10
2	2A	\$1M	11
	2B	Nothing	1
		\$5M	10

Table 4: Revised payoffs after disregarding common outcomes.

To demonstrate the inconsistency observed in the Allais Paradox, let $U(W)$ represent the utility function of wealth W. The preferences are derived from Experiments 1 and 2 as follows:

The choice 1A is preferred over 1B, implying:

$$U(\$1M) > 0.89U(\$1M) + 0.01U(\$0M) + 0.1U(\$5M).$$

Simplifying, we obtain that:

$$1U(\$1M) - 0.89U(\$1M) > 0.01U(\$0M) + 0.1U(\$5M),$$

$$0.11U(\$1M) > 0.01U(\$0M) + 0.1U(\$5M). \tag{1}$$

The choice 2B is preferred over 2A, implying:

$$0.89U(\$0M) + 0.11U(\$1M) < 0.9U(\$0M) + 0.1U(\$5M).$$

Simplifying, we obtain that:

$$0.11U(\$1M) < 0.01U(\$0M) + 0.1U(\$5M). \tag{2}$$

The contradiction follows quickly by looking at the relations 1 and 2.

Remark 4. The choice between 1B and 2B becomes identical, as does the choice between 1A and 2A. The independence axiom would thus suggest consistency across experiments, which contradicts observed choices.

The Allais paradox highlights the need to reconsider the independence axiom and develop theories that better account for human behavior under uncertainty.

4.3.1 Conducting Allais Paradox in our school!

We also conducted the Allais experiment by doing a survey in our school and to other friends, all of them students in high school. We created Our sample was **137** people, and the results we accumulated have shown that:

- 77 of the participants have chosen 1A over 1B,
- 101 of the participants have chosen 2B over 2A.

Hence, we reached empirically the same conclusions as Allais did and the fact that the expected utility theory is inconsistent with reality.

Remark 5. It is true that the behavior of the people towards risk changes throughout the time and when you are not an adult, you tend to be riskier. However, this comes in favour of our experiment, showing that the results with an even sample would imply a higher contrast between expected utility theory and real-life events.

5. Conclusion

This project explored the strong connection between mathematics and economics, focusing on risk, decision-making, and insurance models. We analyzed concepts such as expected utility theory, risk preferences, and the benefits of risk-sharing, showing how mathematical principles help manage financial uncertainty.

Our study of the Allais paradox revealed that human decisions often deviate from classical economic models, a finding reinforced by our experiment. This highlights the need to complement mathematical models with behavioral insights. Ultimately, while mathematics is essential in economics, real-world decision-making can often be influenced by psychological factors.

Notes d'édition

(1) The figure shows the graphs of different possible utility functions u. Of course the blue curve should not go backwards after crossing the black line, otherwise it would not represent a function.

(2) Indeed, companies have such utility function because grouping the individual risks allows them to evaluate risks solely through their expected values (they are risk-neutral).

(3) The conclusion of all this study is that it is better to have a larger number of persons sharing the risks as it decreases the variance and hence the risk.

(4) It is the converse, first the square root and then the logarithm.

(5) It is so because the natural logarithm is an increasing function.

(6) Now the computation is given for any initial wealth W (instead of only $W = 40$).

(7) Yes we may conclude now! When W_1 and W_2 are bigger than 10 (in order to be able to pay the premium) with $W_2 > W_1$, then the first factor is positive, and the second one is negative. Indeed, using twice the arithmetic-geometric inequality again, we can state that

$$\sqrt{W_1(W_1-10)} + \sqrt{W_2(W_2-10)} < W_1 - 5 + W_2 - 5 = W_1 + W_2 - 10.$$

Dividing the right-hand side by the left-hand side proves the second factor negative.