

Paper Cups

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SURNAMES AND FIRST NAMES OF STUDENTS, GRADES:
CRISTEA RADU, 8th and ROBILA MATEI, 8th

SCHOOL: "COSTACHE NEGRUZZI" National College, Iasi, Romania

TEACHER: IONESEI SILVIANA

RESEARCHER: STOLERIU IULIAN

Abstract.

Our task to complete is about a waiting clinic in which patients can have water using disposable paper cups in the shape of a cylinder, a cone or a cone frustum. We have to discover a method so we can use as less material as possible, which means a spare of material resources and a minimal cost of production. We realized that we can solve the problem using inequalities and studying functions through derivatives. Thus, we discover minimal area of the cups in terms of a constant volume.

Keywords: Cylinder, cone, cone frustum, constant volume, minimal area, inequalities, derivatives.

1. PRESENTATION OF THE RESEARCH TOPIC

Our task to complete is:

In the waiting room of a clinic, the patients can have water using disposable paper cups, in the shape of a right circular cone. A glass may contain a maximum of 125 cm^3 of water. In order not to waste paper, it is desirable to use as little paper as possible when making these cups.

- Determine the minimal area of the paper used to make such a cup.
- Solve the same problem considering that the shape of the glasses is cylindrical or conical frustum.
- The production cost for these cups is $0,25 \text{ \$/m}^2$. Calculate the prices for the most economical cups of each type.

A plastic lid can be added on each type of cup. The production cost of such a lid is $1\text{\$/m}^2$.

- Keeping in mind this detail, which of the three cup shapes can be made with minimal costs?

2. BRIEF PRESENTATION OF THE CONJECTURES AND RESULTS OBTAINED

THE SOLUTION

Cylinder (without lid)

Using cylinder volume and area formulas, we get:

$$V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2}, \text{ thus the area becomes:}$$

$$A = 2\pi r h + \pi r^2 = \frac{V}{r} + \frac{V}{r} + \pi r^2.$$

Using the inequality between arithmetic mean and geometric mean, we get:

$$A \geq \sqrt[3]{\pi V^2}, \text{ where equality takes place } \Leftrightarrow \frac{V}{r} = \pi r^2 \Rightarrow h = r \text{ which means:}$$

$r = \sqrt[3]{\frac{V}{\pi}} = \frac{5}{\sqrt[3]{\pi}} = h \cong 3,41 \text{ cm.}$ Thus the minimal area of the cylinder-shaped cup without a lid is:

$$A_{\min} = 75\sqrt[3]{\pi} \cong 109,8 \text{ cm}^2, \text{ and the minimal production cost is } 0.0027 \text{ \$ per cup.}$$

Now we'd like to present a solution to the same problem with derivatives.

Cylinder (no lid, with derivatives)

Using cylinder volume and area formulas, we get:

$$V = 125 \text{ cm}^3 = \pi r^2 h \Rightarrow h = \frac{125}{\pi r^2} \text{ and } A = \frac{250 + \pi r^3}{r} \text{ cm}^2$$

Let the function be: **(1)**

$$f: [0, \infty) \rightarrow \mathbb{R}, f(x) = \frac{250 + \pi x^3}{x} \text{ with its derivative: } f'(x) = \frac{2\pi x^3 - 250}{x^2}$$

$$f'(x) = 0 \Rightarrow x^3 = \frac{125}{\pi} \Rightarrow x = \frac{5}{\sqrt[3]{\pi}}$$

We make a chart that defines the possible values of the function:

x	[0	$\frac{5}{\sqrt[3]{\pi}}$	$\infty)$
f'(x)	-	0	+
f(x)	↓	$f\left(\frac{5}{\sqrt[3]{\pi}}\right)$	↑

Thus, we get that $x = \frac{5}{\sqrt[3]{\pi}}$ is the minimum point of the function, and its minimal value is $f\left(\frac{5}{\sqrt[3]{\pi}}\right) = 75\sqrt[3]{\pi}$. That gives us the minimal area which is: $A \cong 109.84 \text{ cm}^2$, with the radius of the base $r = \frac{5}{\sqrt[3]{\pi}} \cong 3.41 \text{ cm}$ and the height of the cup $h = \frac{5}{\sqrt[3]{\pi}} = r$.

From this equation, we get the minimal cost of production for a cylinder-shaped cup: 0.0027 \$

Cylinder (with lid) (2)

Using cylinder volume and area formulas, we get:

$V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2}$. In this case, we increase the total using the inequality between means:

$$A_{\text{tot}} = \frac{V}{r} + \frac{V}{r} + 2\pi r^2 \geq \sqrt[3]{2\pi V^2}, \text{ where equality takes place } \Leftrightarrow \frac{V}{r} = 2\pi r^2 \Rightarrow h = 2r,$$

$$r = \frac{5}{\sqrt[3]{2\pi}} \Rightarrow r \cong 2.71 \text{ cm and } A_{\text{lid}} = \pi r^2 = 23.1 \text{ cm}^2.$$

Thus, $A_{\text{min}} = 138.37 \text{ cm}^2$, and the minimal cost for a cone-shaped cup with a lid is: $0,00288 \text{ \$ (cup)} + 0.00231 \text{ \$ (lid)} = 0.00519 \text{ \$ (total)}$.

Now we'd like to present a solution to the same problem with derivatives.

Cylinder (with lid, with derivatives)

Using cylinder volume and area formulas, we get:

$$V = 125 \text{ cm}^3 = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2} \text{ and } A = \frac{2\pi r^3 + 2V}{r}.$$

Let the function be:

$$f: [0, \infty) \rightarrow \mathbb{R}, f(x) = \frac{2\pi x^3 + 2V}{x}, \text{ with its derivative:}$$

$$f'(x) = \frac{(2\pi x^3 + 2V)'x - x'(2\pi x^3 + 2V)}{x^2} = \frac{4\pi x^3 - 2V}{x^2}$$

$$f'(x) = 0 \Rightarrow 2\pi x^3 - V = 0 \Rightarrow x = \frac{5}{\sqrt[3]{2\pi}}$$

We make a chart that defines the possible values of the function:

x	[0	$\frac{5}{\sqrt[3]{2\pi}}$	$\infty)$
f'(x)	-	0	+
f(x)	↓	$f\left(\frac{5}{\sqrt[3]{2\pi}}\right)$	↑

This way, the radius and the height of the cup are:

$$r = \frac{5}{\sqrt[3]{2\pi}} \cong 2,71 \text{ cm and } h = \frac{10}{\sqrt[3]{2\pi}} = 2r,$$

which gives us the minimal area of the cup and the lid:

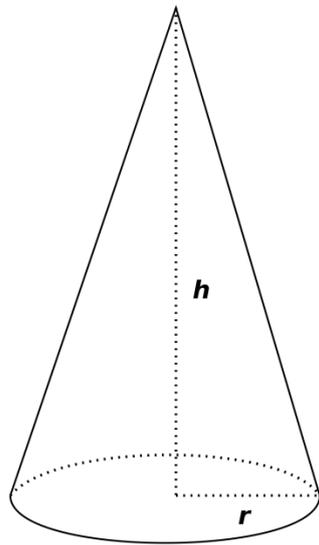
$$A = 75\sqrt[3]{2\pi} \cong 138,37 \text{ cm}^2$$

$$A_{\text{lid}} = \pi r^2 \cong 23,1 \text{ cm}^2$$

Last, but not least, the minimal production cost of the lid is $0.00231 \text{ \$}$, while the one for the cup is: $0.00288 \text{ \$} \Rightarrow \text{total cost} = 0.00519\text{\$}$

Cone (without lid)

Using cone volume and area formulas, we get:



$$V = \frac{h\pi r^2}{3} \Rightarrow h = \frac{3V}{\pi r^2}$$

$$A = \pi r g, \text{ where } g = \sqrt{\frac{9V^2}{r^4\pi^2} + r^2} \quad (3)$$

Using the inequality between arithmetic mean and geometric mean, we get:

$$A = r^4 + \frac{18V^2}{2\pi^2 r^2} \geq 3 \sqrt[3]{\frac{81V^4}{4\pi^4}}. \text{ Thus } A \geq 45 \sqrt[3]{\frac{3}{4\pi}}.$$

$$\text{This way, equality takes place } \Leftrightarrow r^4 = \frac{9V^2}{2\pi^2 r^2} \Leftrightarrow h = r\sqrt{2}.$$

Now we'd like to present a solution to the same problem with derivatives.

Cone (without lid, with derivatives)

Using cone volume and area formulas, we get:

$$V = 125 = \frac{\pi r^2 h}{3} \Rightarrow h = \frac{375}{\pi r^2}, A = \pi r g, \text{ where } g = \frac{\sqrt{375^2 + \pi^2 r^6}}{\pi r^2}.$$

$$\text{Thus, } A = \pi r \frac{\sqrt{375^2 + \pi^2 r^6}}{\pi r^2}.$$

Let the function be:

$$f : [0, \infty) \rightarrow \mathbb{R}, f(x) = \frac{\sqrt{375^2 + \pi^2 x^6}}{x} \text{ and}$$

$$f'(x) = \frac{\sqrt{375^2 + \pi^2 x^6}' \cdot x - \sqrt{375^2 + \pi^2 x^6} \cdot x'}{x^2}$$

$$f'(x) = 0 \Rightarrow x = 5 \sqrt[6]{\frac{9}{2\pi^2}}$$

A chart that defines the possible values of the function is:

x	[0	$5 \sqrt[6]{\frac{9}{2\pi^2}}$	$\infty)$
f'(x)	-	0	+
f(x)	↓	$f(5 \sqrt[6]{\frac{9}{2\pi^2}})$	↑

$$x_{\min} = 5 \sqrt[6]{\frac{9}{2\pi^2}} \cong 4,38 \text{ cm and } f_{\min} = 75 \sqrt[6]{\frac{3\pi^2}{4}}.$$

This way, the minimal area of the cone-shaped cup without a lid is:

$A \cong 104,7 \text{ cm}^2$, where the radius of the base is $r = 4,38 \text{ cm}$ and the height is $h = r\sqrt{2} \cong 6,2 \text{ cm}$.

From that, we get that the minimal production cost for a cone-shaped cup without a lid is: $0.0026 \text{ \$}$.

Cone (with lid, without derivatives) (4)

Using cone volume and area formulas, we get:

$$V = \frac{h\pi r^2}{3} \Rightarrow h = \frac{3V}{\pi r^2} \text{ which gives us the area:}$$

$A = \pi r^2 + \pi r g = \pi r^2 + \pi \sqrt{h^2 + r^2}$. Using the inequality between arithmetic mean and geometric mean, we get:

$$A^3 \geq 72\pi V^2, \text{ where equality takes place } \Leftrightarrow r^2 = \frac{A}{4\pi}, h = 2\sqrt{2} r \text{ and } g = 3r.$$

Thus, the minimal area of the lid is: $A_{\text{lid}} = \pi r^2 \cong 38.08 \text{ cm}^2$, while the minimal area for the cone-shaped cup is $A = 4\pi r^2 \cong 114.24 \text{ cm}^2$.

The production cost for the cone-shaped cup is $0.003808 \text{ \$}$, while the one of the lid is $0.009808 \text{ \$} \Rightarrow \text{total cost} = 0.007616 \text{ \$}$.

Now we'd like to present a solution to the same problem with derivatives.

Cone (with lid, with derivatives)

Using cone volume and area formulas, we get:

$$V = 125 \Rightarrow h = \frac{3V}{\pi r^2}$$

$$A = \pi r g + \pi r^2 = \frac{\pi r^3 + \sqrt{\pi^2 r^6 + 9V^2}}{r}$$

Let the function be:

$$f: [0, \infty) \rightarrow \mathbb{R}, f(x) = \frac{\pi x^3 + \sqrt{\pi^2 x^6 + 9V^2}}{x}, \text{ with its derivative:}$$

$$f'(x) = \frac{(\pi x^3 + \sqrt{\pi^2 x^6 + 9V^2})' \cdot x - x' \cdot (\pi x^3 + \sqrt{\pi^2 x^6 + 9V^2})}{x^2}$$

$$f'(x) = 0 \Rightarrow x^6 = \frac{9V^2}{8\pi^2}$$

We make a chart that defines the possible values of the function:

x	[0	$5\sqrt[6]{\frac{9}{2\pi^2}}$	$\infty)$
f'(x)	-	0	+
f(x)	↓	$f(5\sqrt[6]{\frac{9}{2\pi^2}})$	↑

It follows the radius of the base and the height:

$$x = r_{\min} = \sqrt[6]{\frac{9V^2}{8\pi^2}} = \sqrt[6]{\frac{9^2}{8} \cdot \frac{5^6}{\pi^2}}, \quad r_{\min} = 3,47 \text{ cm}, \quad h_{\min} = \frac{30}{\sqrt[3]{9\pi}} \Rightarrow h = 2\sqrt{2} r \text{ and}$$

$$g = 3r.$$

Thus, the minimal area of the cup-shaped cone with a lid is:

$$A = 4\pi r^2 = 50\sqrt[3]{9\pi} \text{ cm}^2 \text{ or } A \cong 152,32 \text{ cm}^2, \quad A_{\text{lid}} = \pi r^2 \cong 38.08 \text{ cm}^2.$$

That gives us the minimal production cost of the cup = 0.003808 \$ and the lid = 0.003808 \$ \Rightarrow total cost = 0.007616 \$

Conical frustum

Finding the minimum of a conical frustum's area was a real challenge for us. We decided the radius r of the small base to be 2,25 cm, because we've noticed that most of the cups we saw around were like this.

We define $k = \frac{R}{r}$, $k \geq 1$, where R is the radius of the other base.

Using the formula for calculating the volume:

$$V = \frac{\pi h}{3} (R^2 + Rr + r^2)$$

we get the height $h = \frac{2000}{27\pi \cdot (k^2 + k + 1)}$ and the side $g = \sqrt{\frac{2000^2}{729^2 \cdot \pi^2 \cdot (k^2 + k + 1)} + \frac{81}{16} \cdot (k - 1)^2}$, so the area to be investigated is: **(5)**

$$A = \frac{9\pi}{4} \cdot (k + 1) \cdot \sqrt{\frac{2000^2}{729^2 \cdot \pi^2 \cdot (k^2 + k + 1)} + \frac{81}{16} \cdot (k - 1)^2}$$

We couldn't find an algebraic solution, so we tried to solve the problem by means of derivatives.

Let $f: (1, \infty) \rightarrow \mathbb{R}$, $f(x) = \alpha \cdot \left(\frac{x+1}{x^2+x+1}\right)^2 + \beta \cdot (x^2 - 1)^2$,

$$\alpha = \frac{2000^2}{3^6 \cdot \pi^2} \cong 555,9238 \quad \text{and} \quad \beta = \frac{81}{16} = 5,0525. \quad \text{To determine the minimal point for } f,$$

we calculated the derivative $f'(x) = \frac{2\alpha \cdot (x+1) \cdot (-x^2-2x) + 4\beta x \cdot (x^2-1) \cdot (x^2+x+1)^3}{4\beta x \cdot (x^2-1) \cdot (x^2+x+1)^3}$

and finally we were supposed to solve the following equation :

$$2\beta x \cdot (x - 1) \cdot (x^2 + x + 1)^3 - \alpha \cdot (x + 2) = 0$$

We used a wolframalpha program to calculate the solution, which is $x \cong 1,85243$. Then $R \cong 4,16 \text{ cm}$, $h \cong 3,76 \text{ cm}$ and $g \cong 4,21 \text{ cm}$.

The min area of the cup without lid is 100,63 cm² and the cost is 0,002515\$ per cup. If the conical frustum cup has a lid, then the area is 154,96 cm² and its price is 0,007948\$.

3. CONCLUSION

Shape	With lid	Without lid
Cylinder	0.00519\$	0.0027\$
Cone	0.007616\$	0.0026\$
Cone frustum	0.0079\$	0.0025\$

From that chart we can gather that the conical frustum cup without a lid is the cheapest option for a clinic.

Looking at the price of the material, a plastic cup is more expensive and also, it cannot be recycled. Thus, a paper cup helps protecting the environment and the shape of a cone helps spare the amount of paper used during the making of these cups.

In this case, the use of paper cups is more beneficial due to the high usage of them. On the other hand, the cups are made to be used in such way that the person will drink the proper amount of water (for taking pills, medications etc).

The name of the cylinder was given by Democritus (the fifth century BC) and the formula for the volume of the right circular cylinder was known in ancient Egypt (2000 BC). The name of the cone is also thanks to Democritus. The formula for the volume of the right circular cone was discovered by Eudoxus of Cnidus. A thorough study of round shapes was done by Archimedes in his two-volume work on the sphere and cylinder (approximate 225 BC).

4. BIBLIOGRAPHY

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Notes d'édition

- (1) La fonction f devrait être définie sur $(0, +\infty)$ en excluant 0 du domaine de définition.
- (2) Il y a une erreur de raisonnement : il n'est pas tenu compte, au moment où on cherche le « meilleur » r du fait que le couvercle coûte quatre fois plus cher à produire que le papier à surface égale (il est calculé l'aire totale globale, papier et couvercle confondus, en fonction de r , et r est choisi de sorte à ce que cette quantité soit minimale).
- (3) La formule donnant A n'est pas justifiée.
- (4) L'erreur se répète : il n'est pas tenu compte, au moment où on cherche le « meilleur » r du fait que le couvercle coûte quatre fois plus cher à produire que le papier à surface égale.
- (5) La formule donnant A n'est pas justifiée.