Sports tournament

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1. PRESENTATION OF THE RESEARCH TOPIC

In this paper, we count the number of ways in which two game tournaments between two parties, with a restriction, could unfold. Using this, we also calculate the chances that one player wins at the end of tournament or the game ends in a draw. Such questions may appear in practice, when calculating odds or chances for certain betting events.

2. BRIEF PRESENTATION OF THE CONJECTURES AND RESULTS OBTAINED

Tim and Bob play a tennis tournament against each other, consisting of seven consecutive games. For each player, a game could be won or lost, without possibility of a draw. It is known that Tim did not suffer more than three consecutive defeats.

(a) Find the number of ways in which the tournament could unfold for him (for example, WLLLWWL is a possible tournament run for Tim).

(b) Suppose that the players are equally good. What are the chances that Tim wins the tournament?

(c) Consider now the case of a seven round chess-tournament, with the possibility of a draw at each game. If Tim did not suffer more than three consecutive defeats, find the number of ways in which the tournament could unfold for him (for example, WDLLWDL is a possible tournament run for Tim).

(d) Suppose that the players are equally good, and the chance of a draw is 20%. What are the chances that at the end of the chess tournament the score will be even?

3.THE SOLUTION

In all the notations, we see the game in Tim's perspective. So, W means a win for Tim, while L means a loss.

(a) Find the number of ways in which the tournament could unfold for Tim.

We will begin by considering that the number of games the tournament has is *n*, a positive integer. Let w_n be the number of sequences of length *n* which end with W: $* * \ldots * \text{W}$.

Let l_n be the number of sequences of length *n* which end with L: $* * \ldots * \text{L}$.

We denote the total number of sequences of length *n* which satisfy our conditions with a_n.

So, $a_n = w_n + l_n$.

Obviously, $w_1 = l_1 = 1$ (W; L), $w_2 = l_2 = 2$ (LW; WW; WL; LL) and $w_3 = l_3 = 4$ (LWW; WWW; WLW; LLW,LWL; WWL; WLL; LLL)

We suppose that $n \geq 4$. At the end of any sequence of length $n-1$, we can add a W without any other extra condition. Because of that, we have the equation:

(1) $$w_n = w_{n-1} + l_{n-1}.$$

A sequence of length *n* having L in the final position can end in 3 ways:

$$* * \ldots * \text{W L}, \; * * \ldots * \text{W L L}, \text{ or } * * \ldots * \text{W L L L}.$$

So, we obtain the formula:

(2) $$l_n = w_{n-1} + w_{n-2} + w_{n-3},$$

for any integer *n* greater or equal than 4.

From relation (2), we obtain:

$$a_n = w_n + l_n = w_n + w_{n-1} + w_{n-2} + w_{n-3},$$

for any integer *n* greater or equal than 4.

Using the formulas (1) and (2), we will have:

$w_1 = l_1 = 1$; $w_2 = l_2 = 2$; $w_3 = l_3 = 4$; $w_4 = 8$, $l_4 = 7$; $w_5 = 15$, $l_5 = 14$; $w_6 = 29$, $l_6 = 27$; $w_7 = 56$, $l_7 = 52$.

Finally, $a_7 = w_7 + w_6 + w_5 + w_4 = 108$.

Before coming up with the mathematical solution, we elaborated a code that generated all the possible sequences, for checking the future results.

```
void bkt(int k) ///generates all the solutions of lenght n
{
    if(k<=n)
    {
        sol[k]='W'; ///we can put a W wherever we want
        bkt(k+1);   ///goes to the next index, with W on the current position
        if(k<=3||(k>3&&(sol[k-1]=='W'||sol[k-2]=='W'||sol[k-3]=='W'))) ///checks if we can add an L
        {
            sol[k]='L'; ///we put an L if it's possible
            bkt(k+1); ///goes to the next index, with L on the current position
        }
    }
    else ///if we get to the index n+1, we have a solution.
    {
        nrop++; ///number of solutions
        ///printing the found solution:
        for(int i=1;i<=n;i++)
            cout<<sol[i];
        cout<<'\n';
    }
}
```

After we found the recurrences, we came up with the following code:

```
w[0]=1;
w[1]=1,l[1]=1;
w[2]=2,l[2]=2; ///base cases
for(n=3;n<=7;n++) ///iterates with n from 3 to 7
{
    w[n]=w[n-1]+l[n-1];
    l[n]=w[n-1]+w[n-2]+w[n-3];  ///the formulas stated in the document
}
///we print w[n] and l[n], for any n, 0<n<8.
///the result is contained in w[7]+l[7].
for(n=1;n<=7;n++)
    cout<<"n="<<n<<": "<<"w[n]="<<w[n]<<", l[n]="<<l[n]<<", total="<<w[n]+l[n]<<'\n';
```

with the following output:

```
n=1: w[n]=1, l[n]=1, total=2
n=2: w[n]=2, l[n]=2, total=4
n=3: w[n]=4, l[n]=4, total=8
n=4: w[n]=8, l[n]=7, total=15
n=5: w[n]=15, l[n]=14, total=29
n=6: w[n]=29, l[n]=27, total=56
n=7: w[n]=56, l[n]=52, total=108

Process returned 0 (0x0)   execution time : 3.048 s
Press any key to continue.
```

We also created a code with the final formula of a_n, and we noticed the results were always the same.

(b) Suppose that the players are equally good. What are the chances that Tim wins the tournament?

We will begin by considering that the number of games the tournament has is *n*, a positive integer.

Let w_{ij} be the number of ways the tournament could unfold for Tim of length *i*, ending in W and containing *j* victories for Tim, where *i* and *j* are positive integers and $j \le i$.

Let l_{ij} be the number of ways the tournament could unfold for Tim of length *i*, ending in L and containing *j* victories for Tim, where *i* is a positive integer, *j* is a non-negative integer and $j \le i$. Obviously, $l_{ii} = 1$ for any *i*.

We have: $l_{10} = 1$ (L), $l_{20} = 1$ (LL), $l_{21} = 1$ (WL), $l_{30} = 1$ (LLL), $l_{31} = 2$ (WLL, LWL), $l_{32} = 1$ (WWL), $w_{10} = 1$ (L), $w_{11} = 1$ (W), $w_{20} = 1$ (LL), $w_{21} = 1$.(LW) and $w_{22} = 1$ (WW).

From now on, *w* and *l* follow the next recurrences:

$w_{ij} = w_{i-1,j-1} + l_{i-1,j-1}$, $i \ge 2$ and $1 \le j \le i$; (we can put a W after anything)

$l_{ij} = w_{i-1,j} + w_{i-2,j} + w_{i-3,j}$, $i \ge 4$ and $0 \le j \le i-3$;

The logic behind the formulas is similar to the one used in point (a), with the control of *j*.

Using the formulas above, we obtain $w_{71}=0$, $w_{72}=2$, $w_{73}=12$, $w_{74}=20$, $w_{75}=15$, $w_{76}=6$, $w_{77}=1$, $l_{70}=0$, $l_{71}=1$, $l_{72}=10$, $l_{73}=19$, $l_{74}=15$, $l_{75}=6$, $l_{76}=1$ and $l_{77}=0$.

Tim wins the tournament if he wins more games than he loses.

The probability that Tim wins the tournament is the ratio of the total number of favorable outcomes to the total number of possible outcomes: $p=\dfrac{w_{74}+w_{75}+w_{76}+w_{77}+l_{74}+l_{75}+l_{76}}{w_{71}+w_{72}+...+w_{77}+l_{71}+l_{72}+...+l_{76}}=\dfrac{64}{108}=\dfrac{16}{27}$ $\cong 0.5926$.

We can also build a solution the following way:

```
    n=7; ///we set n to 7 (we can choose, however, any number we want)
    W[0][0]=1;
    L[1][0]=1;
    L[2][0]=1,L[2][1]=1;
    ///we covered the base cases;
    for(i=1;i<=n;i++)
        for(j=0;j<=i;j++)
    {
        if(i>=1&&j>=1) ///the formula for W[i][j], only if i>=1 and j>=1 :
            W[i][j]=W[i-1][j-1]+L[i-1][j-1];
        if(i>=3&&j>=0) ///the formula for L[i][j], only if i>=3 and j>=0 :
            L[i][j]=W[i-1][j]+W[i-2][j]+W[i-3][j];
    }
    nrWL=0; ///the number of all the situations, winning or losing for Tim
    nrW=0; ///the number of winning situations for Tim
    for(i=0;i<=n;i++)
    {
        nrWL=nrWL+W[n][i]+L[n][i];
        if(i>n/2) ///equivalent to i>=n/2+n%2, the majority of the n symbols
            nrW=nrW+W[n][i]+L[n][i];
    }
    cout<<nrW<<"\"<<nrWL; ///prints the winning/(winning+losing) ratio
```

Second method

The total number of outcomes of the tournament without any restrictions is $2^7 = 128$ (for each game we can have two outcomes).

When we consider the restriction that Tim did not suffer more than three consecutive defeats, we must eliminate the following cases:

WWWLLLL	LLWLLLL	LLLLLWL
WWLLLLW	LWLLLLW	WLLLLLW
WLLLLWW	WLLLLWL	LLLLLWW
LLLLWWW	LLLLWWL	WLLLLLL
WWLLLLL	LLLLWLW	LLLLLLW
WLWLLLL	LLLLWLL	LLLLLLL
LWWLLLL	LWLLLLL	

So, the total number of ways in which the tournament could unfold is $128 - 20 = 108$.

For Tim to win the tournament he needs to win at least 4 games. Consequently, he loses at most 3 games, so he cannot lose more than 3 games in a row.

Thus, the number of games in which Tim wins is:

$$\binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7} = 64$$

The probability that Tim wins the tournament is: $p = \frac{64}{108} = \frac{16}{27} \cong 0.5926$.

Thus, Tim has almost 60% chances to win the tournament and, consequently, the odds that Tim wins the tournament are 3 *to* 2 *on* (three chances of winning out of a total of 5).

(c) Consider now the case of a seven round chess-tournament, with the possibility of a draw at each game. If Tim did not suffer more than three consecutive defeats, find the number of ways in which the tournament could unfold for him (for example, WDLLWDL is a possible tournament run for Tim).

We will begin by considering *n* a positive integer.

Let w_n be the number of sequences of length *n* which end with W, l_n be the number of sequences of length *n* which end with L and d_n be the number of sequences of length *n* which end with D.

We see that: $l_1 = 1$ (L), $w_1 = 1$ (W), $d_1 = 1$ (D); $l_2 = 3$ (DL; LL; WL); $l_3 = 9$ (WWL; WDL; WLL; DWL; DDL; DLL; LWL; LDL; LLL).

We have the following formulas:

$$d_n = w_{n-1} + l_{n-1} + d_{n-1},\ n \geq 2,$$

$$w_n = w_{n-1} + l_{n-1} + d_{n-1},\ n \geq 2,$$

$$l_n = (w_{n-1} + d_{n-1}) + (w_{n-2} + d_{n-2}) + (w_{n-3} + d_{n-3}),\ n \geq 4$$

Note. We observe that the formulas for d_n and w_n are equal, because we can add both a D or a W after the last character, without any conditions.

The way we calculate l_n is similar to the one used in point (a), but, this time, we can also have a D before a sequence of maximum 3 L's.

The number of ways in which the tournament could unfold for him is equal with $d_n + w_n + l_n$. So, with the following program, we obtain that $d_7 + w_7 + l_7$ is equal with 2106.

```
n=7; ///we set n with 7
W[0]=1;
L[1]=1;
L[2]=3; ///base-cases
for(i=1;i<=n;i++)
{
    D[i]=W[i-1]+D[i-1]+L[i-1]; ///formula for D[i], with i>=1
    W[i]=W[i-1]+D[i-1]+L[i-1]; ///formula for W[i], with i>=1
    if(i>=3)
    L[i]=(W[i-1]+D[i-1])+(W[i-2]+D[i-2])+(W[i-3]+D[i-3]);
}
cout<<W[n]+D[n]+L[n];///the number of sequences of lenght n which end in W, D and L
```

Second method

We will consider n a positive integer.

Let a_n be the number of ways to unfold a n-game tournament. In this case, we have 4 ways in which the tournament could start:

1) Tim starts with a W(win) or a D(draw): $\begin{matrix}W\\D\end{matrix} **\ldots*$.

This way we will have $2a_{n-1}$ ways to unfold the game.

2) Tim starts with a L(loss) and then he continues with a W or a D: $L\begin{matrix}W\\D\end{matrix} **\ldots*$.

In this case we will have $2a_{n-2}$ ways to unfold the game.

3) Tim starts with two L and then he continues with a W or a D: $LL\begin{matrix}W\\D\end{matrix} **\ldots*$.

This way we will have $2a_{n-3}$ ways to unfold the game.

4) Tim starts with three L and then he continues with a W or a D: $LLL\begin{matrix}W\\D\end{matrix} **\ldots*$.

In this case we will have $2a_{n-4}$ ways to unfold the game.

In the end, we obtain:

$$a_n = 2\cdot a_{n-1} + 2\cdot a_{n-2} + 2\cdot a_{n-3} + 2\cdot a_{n-4},\text{ for } n \geq 5.$$

It is easy to see that $a_1 = 3;\ a_2 = 9;\ a_3 = 27;\ a_4 = 80$ and afterwards, using the above formula, we get $a_5 = 238,\ a_6 = 708$ and $a_7 = 2106$. So, there are 2106 ways that the tournament could unfold.

(d) Suppose that the players are equally good, and the chance of a draw is 20%. What are the chances that at the end of the chess tournament the score will be even?

If the players are equally good, it means that the chance that Tim wins a game is equal to the chance that he loses it; more precisely, it is equal to $\frac{100-20}{2} = 40\%$.

We will begin by considering the number of games Tim wins *w*, a positive integer, the number of games Tim loses *l*, a positive integer and the number of draws *d*, a positive integer. Also, we will consider *n* (a positive integer) the number of games that Tim and Bob are playing. Obviously, $w + l \leq n$.

We compute the probability that Tim wins *w* games and loses *l*; the rest of $d = n - w - l$ games will be draws.

The *w* numbers corresponding to the games that Tim wins can be chosen in $\binom{n}{w}$ ways. For each configuration of the wins we can choose the *l* numbers of the games that Tim loses from the rest of $n - w$ variants in $\binom{n-w}{l}$ ways.

So, there are

$$\binom{n}{w}\cdot\binom{n-w}{l} = \frac{n!}{w!(n-w)!}\cdot\frac{(n-w)!}{l!(n-w-l)!} = \frac{n!}{w!\,l!\,(n-w-l)!}$$

ways to choose the *w* places for the wins and the *l* places for the losses.

Now for each game the probability that it is a win is $\frac{40}{100}=0.4$, the probability that it is a loss is also 0.4 and the probability that it is a draw is $\frac{20}{100}=0.2$. Moreover, the result of any game is not influenced by the results of the previous games and does not influence the results of the upcoming games. Consequently, denoting by e_{wl} the event that the tournament ends with *w* wins and *l* losses for Tim, we get that the probability that e_{wl} occurs is the product:

$$p(e_{wl})=\frac{n!}{w!l!(n-w-l)!}\cdot(0.4)^w\cdot(0.4)^l\cdot(0.2)^{n-w-l}=\frac{n!}{w!l!(n-w-l)!}\cdot(0.4)^{w+l}\cdot(0.2)^{n-w-l}.$$

Consider a 7 games tournament (in other words, $n=7$). For it to end with a even score we must have $w=l$, hence $d=7-w-l=7-2w$ must be odd. Thus,

$$(d,w,l)\in\{(1,3,3),(3,2,2),(5,1,1),(7,0,0)\}.$$

It is clear that the events $e_{33},\ e_{22},\ e_{11},\ e_{00}$ are incompatible two by two (any two of them cannot happen simultaneously). Hence, the probability that one of these events occurs is

$$p(e_{33}\cup e_{22}\cup e_{11}\cup e_{00})=p(e_{33})+p(e_{22})+p(e_{11})+p(e_{00}).$$

We compute:

$$p(e_{33})=\frac{7!}{3!3!1!}\cdot(0.4)^{3+3}\cdot(0.2)^1=\frac{4\cdot5\cdot6\cdot7}{6}\cdot8192\cdot10^{-7}=0.114688;$$

$$p(e_{22})=\frac{7!}{2!2!3!}\cdot(0.4)^{2+2}\cdot(0.2)^3=\frac{4\cdot5\cdot6\cdot7}{2\cdot2}\cdot2048\cdot10^{-7}=0.043008;$$

$$p(e_{11})=\frac{7!}{1!1!5!}\cdot(0.4)^{1+1}\cdot(0.2)^5=6\cdot7\cdot512\cdot10^{-7}=0.0021504;$$

$$p(e_{00})=\frac{7!}{0!0!7!}\cdot(0.4)^{0+0}\cdot(0.2)^7=0.0000128.$$

In conclusion,

$$p\left(e_{33} \cup e_{22} \cup e_{11} \cup e_{00}\right) = 0.114688 + 0.043008 + 0.0021504 + 0.0000128 = 0.1598592.$$

We can see that there are almost 16% chances that the tournament ends in a draw, which means the approximative odds are 1 *to* 5 *on* for a draw (1 chance out of a total of 6).

4. CONCLUSION

We have employed at least two methods to calculate the number of ways in which both tournaments could unfold for Tim. For each question, we have also attached a C++ code to verify our results. The methods that we have used here could also be adapted for other game tournaments in which the restriction is either removed or changed.