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From the swept path of a bicycle to the optimal trajectory of an F1 car

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Abstract

Vehicles while moving follow specific trajectories, more or less complex according to their purpose, which present particular and interesting aspects when studied with mathematical tools. The goal of this article is to analyse the trajectories of two different vehicles. First we will analyze the motion of a bicycle, focusing on its footprint, and then we will move to the development of a simple model to compute the optimal racing line of an F1 car on track.

Contents

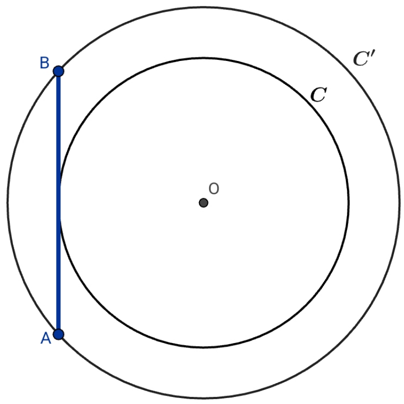
Chapter 1

Swept Path of a Bicycle

1.1 Circular motion of the bicycle

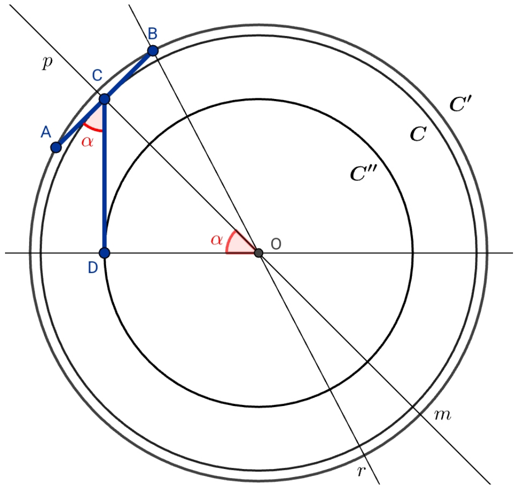
1.1.1 Bicycle as a single segment

A first modelling is made possible studying the circular motion of the segment AB tangent to the circumference C in its midpoint. During this motion its extremes describe the circumference C'. The swept path of the bicycle corresponds to the circular ring.



1.1.2 Bicycle as two segments: body and front wheel

We analyzed the motion of a bicycle whose front wheel is represented by the segment AB of length s and whose frame is represented by the segment CD of length l.
A first modelling is made possible by studying its motion keeping the angle α between the wheel and the bicycle constant. The extremes C, $A(B)$ and D describe respectively the circumferences C, C' and C''.

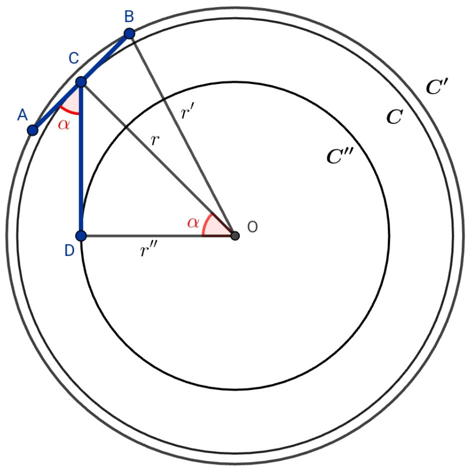


We have written the radii in function of l, s and α:

$$r'' = \frac{l}{\tan\alpha}$$

$$r = \sqrt{(r'')^2 + l^2} = \sqrt{\frac{l^2}{\tan^2\alpha} + l^2}$$

$$r' = \sqrt{r^2 + \left(\frac{s}{2}\right)^2} = \sqrt{\frac{l^2}{\tan^2\alpha} + l^2 + \frac{s^2}{4}}$$



The corresponding areas are:

$$A'' = \pi\frac{l^2}{\tan^2\alpha}$$

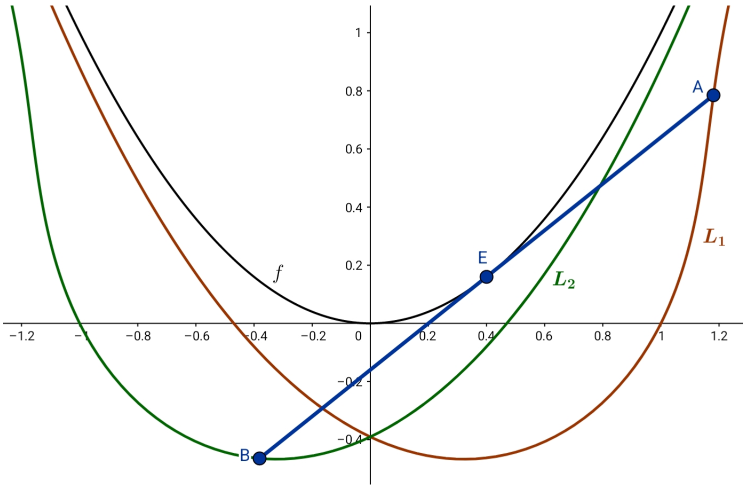
$$A = \pi\left(\frac{l^2}{\tan^2\alpha} + l^2\right)$$

$$A' = \pi\left(\frac{l^2}{\tan^2\alpha} + l^2 + \frac{s^2}{4}\right)$$

The swept path of the bicycle frame has area $A - A'' = \pi l^2$, the swept path of the bicycle wheel has area $A' - A = \pi \frac{s^2}{4}$, so the swept path of the entire bicycle has area $A_{tot} = A' - A'' = \pi \left(l^2 + \frac{s^2}{4}\right)$, which does not depend on the angle α.

1.2 Motion along a power function $y = x^n$

Before analyzing the motion of the bicycle along a generic function, we studied a simpler situation, that is moving a segment AB tangent to a power function $y = x^n$ in its midpoint E.



Our goal was to describe the trajectory of endpoints A and B with parametric equations L_1 and L_2.
Let $a \in \mathbb{R}$ be a real number such that the midpoint E has the following coordinates $(a; a^n)$. [1]
We considered the line tangent to the function in E where A and B lie at distance of $\frac{l}{2}$ from E. We know that the slope of this line is equivalent to the derivative of $y = x^n$ calculated in E, so in this case $\alpha = \arctan(na^{n-1})$. We obtained the x-coordinates of A and B by respectively adding and subtracting $\cos\alpha$ from x_E and, in the same way, the y-coordinates by respectively adding an subtracting $\sin\alpha$ from y_E.

$$x = a \pm \frac{1}{2}\cos\left[\arctan\left(na^{n-1}\right)\right]$$

$$y = a^n \pm \frac{1}{2}\sin\left[\arctan\left(na^{n-1}\right)\right]$$

We simplified these equations until we obtained the following parametric equations for L_1 and L_2:

$$L_1 : \begin{cases} x = a + \dfrac{l}{2\sqrt{n^2 a^{2(n-1)} + 1}} \\[2ex] y = a^n + \dfrac{lna^{n-1}}{2\sqrt{n^2 a^{2(n-1)} + 1}} \end{cases}$$

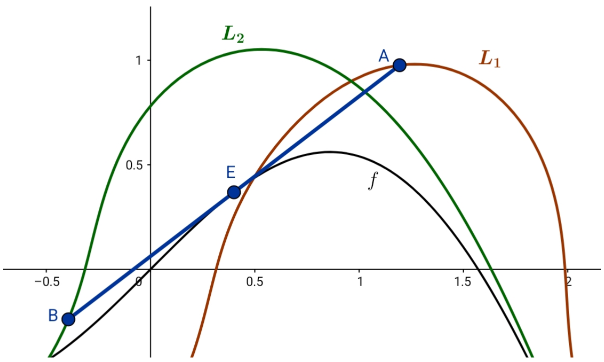
$$L_2: \begin{cases} x = a - \dfrac{l}{2\sqrt{n^2a^{2(n-1)}+1}} \\ y = a^n - \dfrac{lna^{n-1}}{2\sqrt{n^2a^{2(n-1)}+1}} \end{cases}$$

1.3 Motion along a generic function $y = f(x)$

After studying the motion of a segment along a power function, we extended the problem to a generic function $y = f(x)$, at first with a segment AB that simulates the body of the bicycle, then adding a second segment CD, which corresponds to the front wheel with midpoint A.

1.3.1 Bicycle as a single segment

We considered the segment AB with length l and midpoint $E(a; f(a))$ tangent to the graph of f.



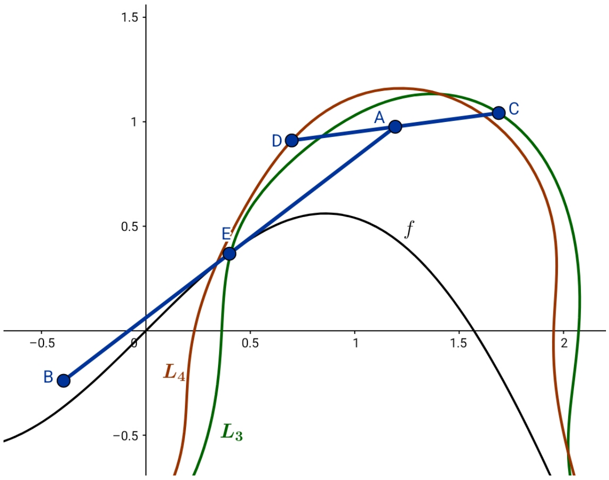
Similarly to what done before, we found the following equations for L_1 and L_2:

$$L_1: \begin{cases} x = a + \dfrac{l}{2\sqrt{[f'(a)]^2+1}} \\ y = f(a) + \dfrac{lf'(a)}{2\sqrt{[f'(a)]^2+1}} \end{cases}$$

$$L_2: \begin{cases} x = a - \dfrac{l}{2\sqrt{[f'(a)]^2+1}} \\ y = f(a) - \dfrac{lf'(a)}{2\sqrt{[f'(a)]^2+1}} \end{cases}$$

1.3.2 Bicycle as two segments: body and front wheel

We proceeded to add the segment CD of length s with midpoint A which corresponds to the front wheel.



We applied the previous method considering the line tangent to L_1 in A, C and D lie on this line at distance $\frac{s}{2}$ from A and their motion describes respectively L_3 and L_4 with the following parametric equations: [2]

$$L_3 : \begin{cases} x = a + \dfrac{l}{2\sqrt{[f'(a)]^2+1}} + \dfrac{s}{2}\dfrac{h(a)}{|h(a)|}\cos\left[\arctan\left(\dfrac{f'(a)}{h(a)} + \dfrac{f''(a)}{h(a)\sqrt{[1+(f'(a))^2]^3}}\right)\right] \\ \\ y = f(a) + \dfrac{lf'(a)}{2\sqrt{[f'(a)]^2+1}} + \dfrac{s}{2}\sin\left[\arctan\left(\dfrac{f'(a)}{|h(a)|} + \dfrac{f''(a)}{|h(a)|\sqrt{[1+(f'(a))^2]^3}}\right)\right] \\ \end{cases}$$

$$L_4 : \begin{cases} x = a + \dfrac{l}{2\sqrt{[f'(a)]^2+1}} - \dfrac{s}{2}\dfrac{h(a)}{|h(a)|}\cos\left[\arctan\left(\dfrac{f'(a)}{h(a)} + \dfrac{f''(a)}{h(a)\sqrt{[1+(f'(a))^2]^3}}\right)\right] \\ \\ y = f(a) + \dfrac{lf'(a)}{2\sqrt{[f'(a)]^2+1}} - \dfrac{s}{2}\sin\left[\arctan\left(\dfrac{f'(a)}{|h(a)|} + \dfrac{f''(a)}{|h(a)|\sqrt{[1+(f'(a))^2]^3}}\right)\right] \\ \end{cases}$$

where

$$h(a) = 1 - \frac{f'(a)f''(a)}{\sqrt{(1+(f'(a))^2)^3}}$$

We noticed that for x-coordinates, we multiply cosine for the sign function of $h(a)$ so that we add or subtract this quantity when the variation of x-coordinates is respectively positive or negative. We also noticed that for y-coordinates, we get the absolute value of $h(a)$ to do that the sign of the sine is equal to the sign of the variation of y-coordinates.

Chapter 2

Finding an optimal racing line

In this section we are going to develop a model to find an optimal racing line of an F1 car during a qualifying session.
We will proceed along the following steps:

- Discretize the space of the track into finite states (nodes);
- Generate links between the nodes (possible paths);
- Compute a cost τ_{ij} for each link;
- Use Dijkstra's algorithm to find the path with lower cost.

2.1 Definitions

Let's begin by stating some definitions.

Definition 1. *We define the* ***state*** *(or node) N as a couple of a cartesian point and a velocity vector:*

$$N := (P, \vec{v}) \quad \text{with} \quad P, \vec{v} \in \mathbb{R}^2$$

Definition 2. *We define the* ***set of edges*** *(or links) E as the set of all the possible couples of nodes:*

$$\mathbb{E} \subseteq \mathcal{N} \times \mathcal{N}$$

[3]

Definition 3. *We define the* ***track*** *$\mathbb{T}$ as the couple of a set of nodes and a set of edges. Therefore the track is a graph:*

$$\mathbb{T} := (\mathcal{N}, \mathbb{E})$$

Definition 4. *We define the* ***cost function*** *τ as the measure:*

$$\tau : \mathbb{E} \to [0, +\infty]$$

Definition 5. *We define a* ***path*** *as a subset of $\mathbb{E}$ such that every item of it has at least one state in common with another.*

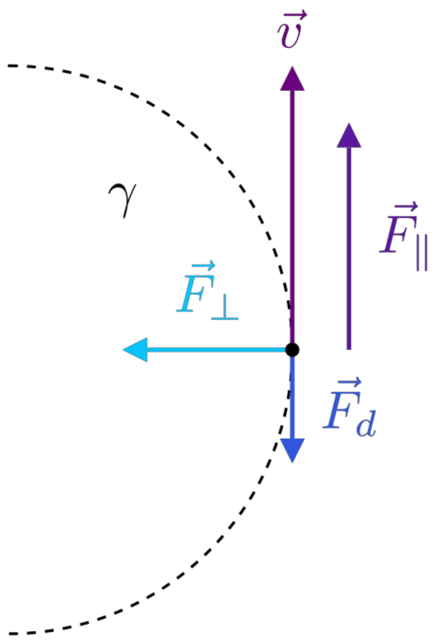
Definition 6. *We define the* ***optimal racing line*** *going from a point A to a point B as the path containing A and B whose sum of costs over the edges is minimum.*

2.2 Physical boundaries

Our next goal is to find a way to compute a cost for each link. In order do that, we should first analyze the physical boundaries of the vehicle and the forces acting on it, to see if the link is actually feasible.
We can split the forces the vehicle is subject to into:

- Forces caused by the vehicle control systems (throttle, brake and steering);
- Dissipative forces (drag).

We can also decompose the forces into their components perpendicular and parallel to the trajectory. The free body diagram of the car will therefore be:



The total force acting on the car can be written as:

$$\sum_i \vec{F}_i = \vec{F}_{\parallel} + \vec{F}_{\perp} + \vec{F}_d = m\frac{d\vec{v}}{dt}$$

While the norm of the drag can be calculated as:

$$F_d = \frac{1}{2} \cdot c_d \cdot \rho \cdot A \cdot v^2$$

If the total force is greater or lower of the chosen boundaries, the link won't be drivable.

2.3 Computing the costs

After analyzing the physical boundaries of the car, we can proceed by computing the cost of each link, respectively as the time needed to drive the link if it is feasible, or as plus infinity otherwise.
Therefore, we can compute each cost τ_{ij} as follows:

$$\tau_{ij} := \begin{cases} \dfrac{2\Delta S}{v_i + v_j} \text{ if } F \leqslant F_{max} \\ +\infty \text{ if } F > F_{max} \end{cases}$$

[4]

2.4 Dijkstra's algorithm

Finally we can use Dijkstra's algorithm to find the path of least cost in the graph we have just constructed.
The algorithm consists of the following steps: [5]

1. Create a set of all unvisited nodes;
2. Assign to each node a distance from the starting one: for the starting node we define it as being zero, while for all the other ones, we set it to be plus infinity;
3. From the unvisited set, select the current node to be the one with the smallest distance. If the unvisited set is empty, or contains only nodes with infinite distance or the target node is the current one, the algorithm terminates;
4. For the current node, consider all of its unvisited neighbors and update their distances to the current node. Then compare the newly calculated distance to the one currently assigned to the neighbor and assign the smaller one to it;
5. After considering all of the current node's unvisited neighbors, the current node is removed from the unvisited set and the algorithm proceeds by repeating from step 3;
6. Once the loop exits, every visited node contains its shortest distance from the starting node.

2.5 Data

The data we chose are meant to be an approximation of the ones of a 2024 f1 car:

- $m = 840\ kg$
- $v_{min} = 9\ \dfrac{m}{s}$
- $v_{max} = 90\ \dfrac{m}{s}$
- $A = 1\ m^2$
- $l = 9\ m$
- $c_d = 1$
- $\rho = 1.293\ \dfrac{kg}{m^3}$
- $F_{\perp max} = m \cdot 3.5g$
- $F_{\parallel max} = m \cdot 1.09g$
- $F_{\parallel min} = m \cdot 7g$

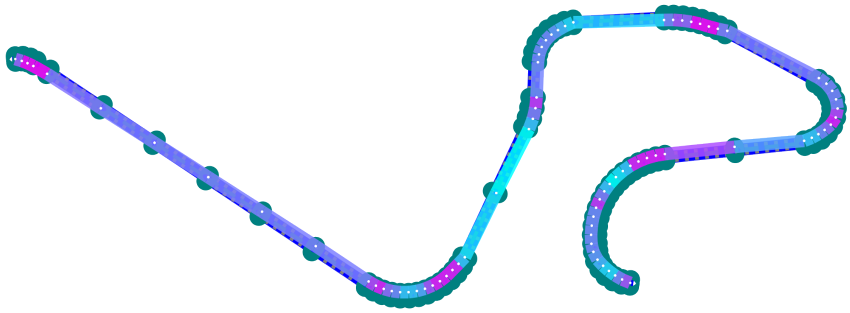
The circuit we chose to test the model on is the Barcelona circuit having the layout of the 2024 Grand Prix and the algorithm was implemented using Python and Numpy.

2.6 Results of the simulation

2.6.1 First sector

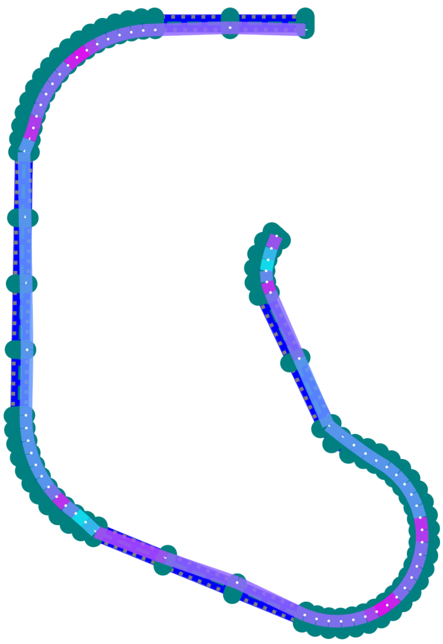
Computed time: $\Delta t_1 = 21.276s$

2.6.2 Second sector



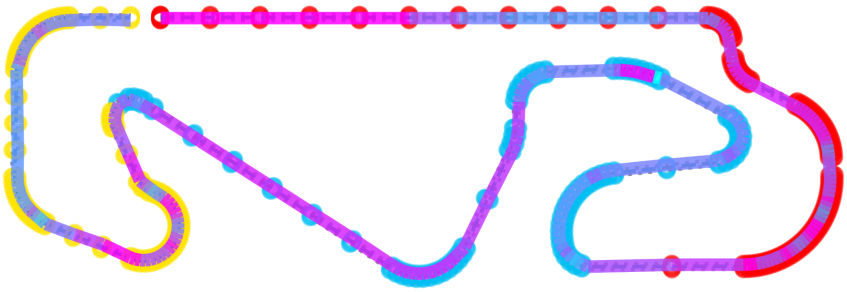
Computed time: $\Delta t_2 = 26.318s$

2.6.3 Third sector



Computed time: $\Delta t_3 = 21.261s$

2.6.4 Global computation



Overall times		
Sector	Computed time	Best 2024 qualifying time
I II III	0:21.276 0:26.318 0:21.261	0:21.383 0:28.402 0:21.598
All	**01:08.855**	**01:11:383**

We can assume that the difference between the simulated lap time and the best real qualifying time is caused by the ideal and discrete nature of the model and the approximate choice of the parameters.

EDITION NOTES

[1] This expression is somewhat convoluted. A clearer formulation would be: *We denote the coordinate of the midpoint E by* $(a, a^n) \in \mathbb{R}^2$.

[2] The definition of L_3 and L_4 should be explained in more detail.

[3] This definition needs to be fixed. The E in the first line is actually $\mathbb{E}$ and it should be stated that $\mathcal{N}$ is the set of all N.
More importantly, the expression "the set of all the possible couples of nodes" seems to conflict with the inclusion sign $\subseteq$ in $\mathbb{E} \subseteq \mathcal{N} \times \mathcal{N}$. Is $\mathbb{E}$ the set of all pairs of nodes? or a set of pairs of nodes?
As minor observations, 'pair' is preferable to 'couple' in this context and the adjective 'possible' is superfluous.

[4] It should be stated explicitly what ΔS denotes. It would also be useful to include some, possibly informal, explanation of why the cost becomes small for high velocities v_i and v_j.

[5] The following is an informal description of Dijkstra's algorithm. However, its connection with the previous mathematical constructions is not clear, and some explanation would be helpful.
A similar lack of correlation is observed between the previously developed mathematical framework and the data and images of a Formula 1 circuit.