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Table arrangements

Year 2020- 2021

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Keywords:

Abstract

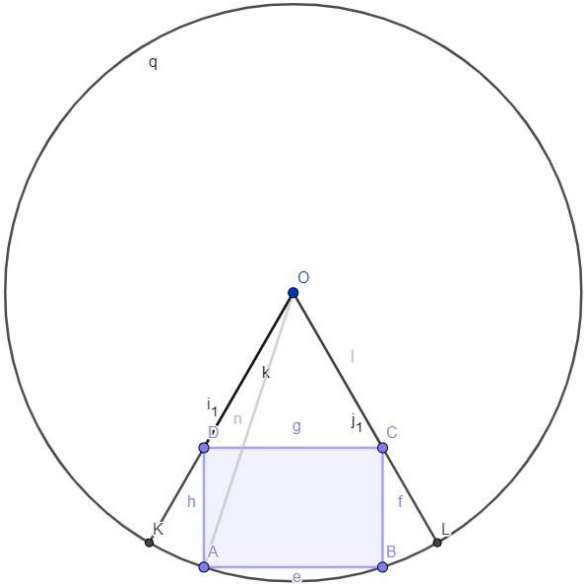
Our research deals with arranging trays on round tables in a high school canteen, so that the used tables are as small as possible. Given the number of trays, we must find the smallest radius of a table on which the trays, that are placed with the adjacent corners, fit.

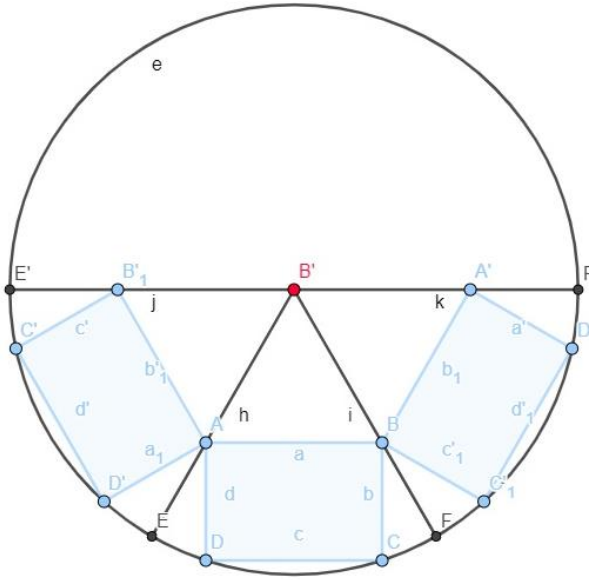
The problem

In the canteen of a certain high-school, all eating tables are round and of various sizes. The serving trays are of rectangular shape, x × y (x > y), all of the same size. Students place their trays with food on the table such that two adjacent corners of each tray are on the edge of the table. The trays are not overlapping and the students align them in the same way.

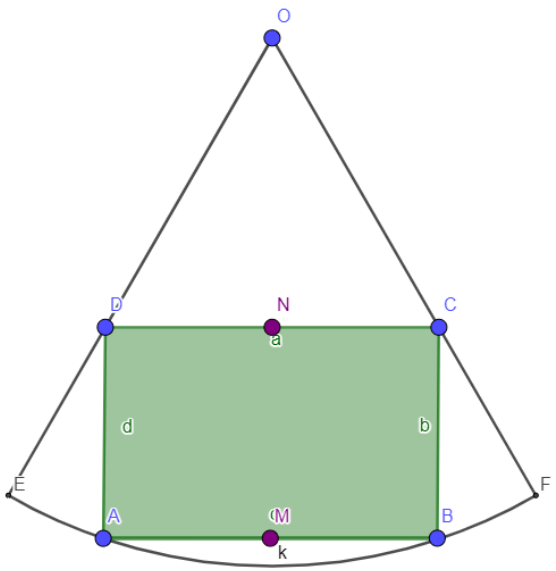
a) How should they align their trays so that the table needed is as small as possible?
b) What must be the radius of the smallest table that can accommodate n non-overlapping trays if were move the condition that two adjacent corners of each tray must sit on the edge of the table?
c) What about trays of a square shape?

We considered the first case, in which *R* is the radius of the table, and the trays are placed lengthwise. Their arrangement can be seen in the following drawings:





We note by n the number of trays, which is a constant.



$\measuredangle DOC = \frac{2\pi}{n}$ (1)

$\measuredangle DOC = 2\measuredangle MOC$ (2)

From relations (1) and (2) we deduced that $\measuredangle MOC = \frac{\pi}{n} \Rightarrow \alpha = \frac{\pi}{n}$

ΔAOM is a right triangle at M. If we use Pythagoras' Theorem, we will get that:

$AO^2 = AM^2 + MO^2$

Since the point M is the middle of the segment [AB], we deduce that $AM = \frac{AB}{2}$. Since AB represents the length of the tray, it means that it is equal to x. Thus, $AM = \frac{x}{2}$.

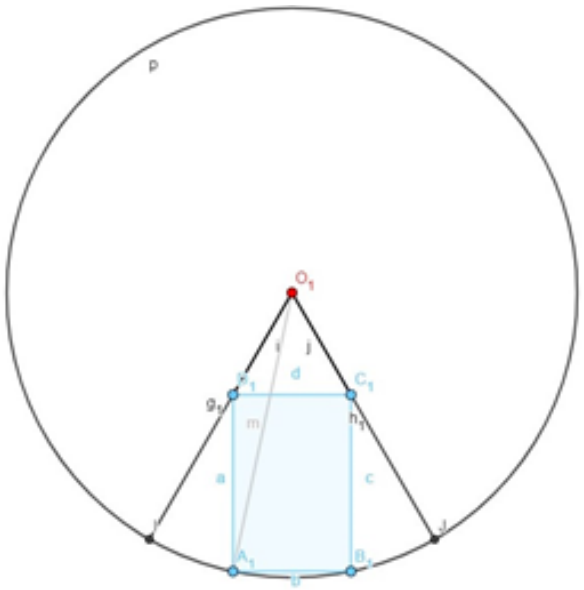
In the right triangle $\measuredangle ONC$ we can deduce that $ctg\alpha = \frac{ON}{NC}$, from which results that $ON = NC \times ctg\alpha$. Since the tray is a rectangle, it means that AB is equal to CD. Since M and N are their middle points, it follows that NC will be equal to AM, respectively to $\frac{x}{2}$. Thus, $ON = \frac{x}{2} \times ctg\ \alpha$ (3).

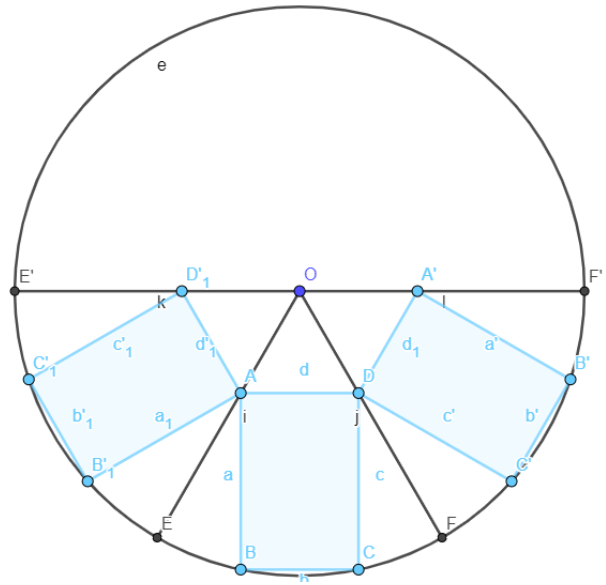
OM=ON+NM (4)

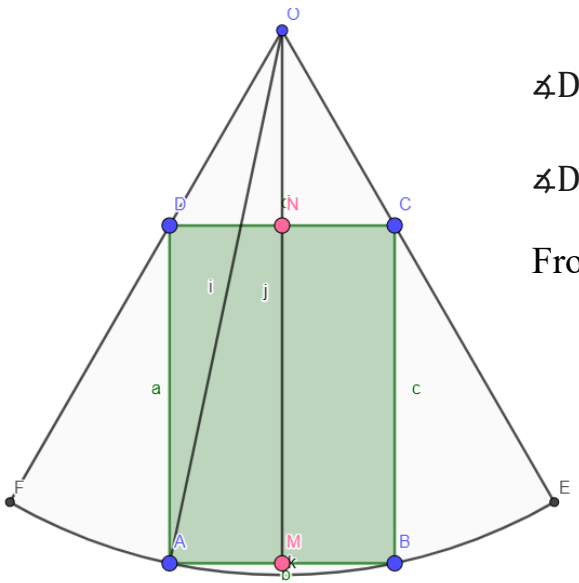
From relations (3) and (4) it results that OM = x / 2 × ctgα + y. Since OM is the radius of the table (R) and the triangle ΔOMN is rectangular at point M, we can deduce that:

$$R^2 = \left(\frac{x}{2}\right)^2 \times \left(\frac{x}{2} \times ctg\alpha + y\right)^2 = \frac{x^2}{4} + \left(\frac{x}{2} \times ctg\alpha + y\right)^2$$

In the second case *r* will be the radius of the table, and the trays will be placed on the width. Their arrangement can be seen in the following images:







$\measuredangle DOC = \frac{2\pi}{n}$ (1)

$\measuredangle DOC = 2\measuredangle MOC$ (2)

From relations (1) and (2) we deduced that $\measuredangle MOC = \frac{\pi}{n} \Rightarrow \alpha = \frac{\pi}{n}$

ΔAOM is a right triangle in M. Thus, using Pythagoras' Theorem, we obtain that:

$AO^2 = AM^2 + MO^2$

Since the point M is the middle of the segment [AB], we deduce that $AM = \frac{AB}{2}$. Since AB represents the width of the tray, it means that it is equal to y. Thus, $AM = \frac{x}{2}$.

In the right triangle $\measuredangle ONC$ we can deduce that $ctg\alpha = \frac{ON}{NC}$, from which it results that $ON = NC \times ctg\alpha$. Since the tray is a rectangle, it means that AB is equal to CD. Since M and N are their middle points, it follows that NC will be equal to AM, respectively to $\frac{y}{2}$. Thus, $ON = \frac{x}{2} \times ctg\ \alpha$ (3).

OM=ON+NM (4)

From relations (5) and (6) it results that $OM = \frac{y}{2} \times ctg\alpha + x$. Since OM is the radius of the table (r) and the triangle

ΔOMN is rectangular at point M, we can deduce that:

$$\boxed{r^2 = \left(\frac{y}{2}\right)^2 \times \left(\frac{y}{2} \times ctg\alpha + x\right)^2 = \frac{y^2}{4} + \left(\frac{y}{2} \times ctg\alpha + x\right)^2}$$

We compare r^2 with R^2 (because r> 0, R> 0):

$$R^2 - r^2 = \frac{x^2}{4} + \left(\frac{x}{2} \times ctg\alpha + y\right)^2 - \frac{y^2}{4} + \left(\frac{y}{2} \times ctg\alpha + x\right)^2$$

$$R^2 - r^2 = \frac{(x^2 - y^2)}{4} + \frac{x^2}{4} \times ctg^2\alpha + 2 \times \frac{xy}{2} \times ctg\alpha + y^2 - \frac{y^2}{4} \times ctg^2\alpha - 2 \times \frac{xy}{2} \times ctg\alpha - x^2$$

$$R^2 - r^2 = \frac{(x^2 - y^2)}{4} + y^2 - x^{2+} ctg^2\alpha \times \frac{(x^2 - y^2)}{4}$$

$$R^2 - r^2 = \frac{(x^2 - y^2)}{4} \times \left(\frac{1}{4} - 1 + \frac{ctg^2\alpha}{4}\right)$$

$$R^2 - r^2 = \frac{1}{4} \times [(x^2 - y^2) \times (ctg^2\alpha - 3)] \ (7)$$

$$x^2 - y^2 = \text{ constant } (8)$$

$$\frac{1}{4} = \text{constant } (9)$$

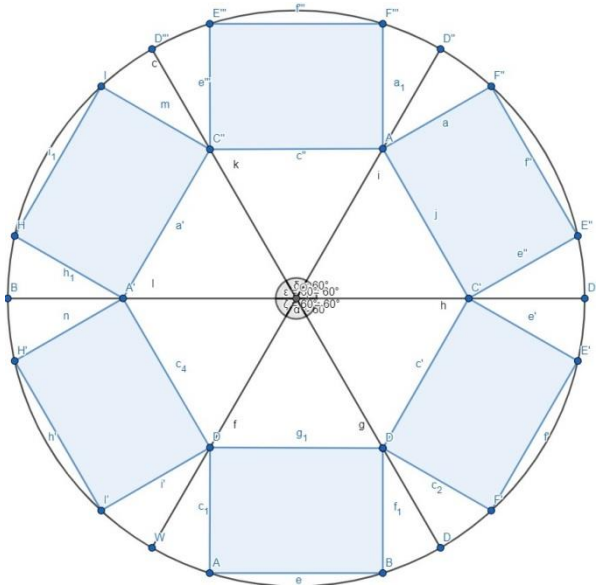
From the relations (7), (8), (9) we can deduce that $ctg^2\alpha - 3$ is the only variable, which means that $ctg^2\alpha$ is the only variable.

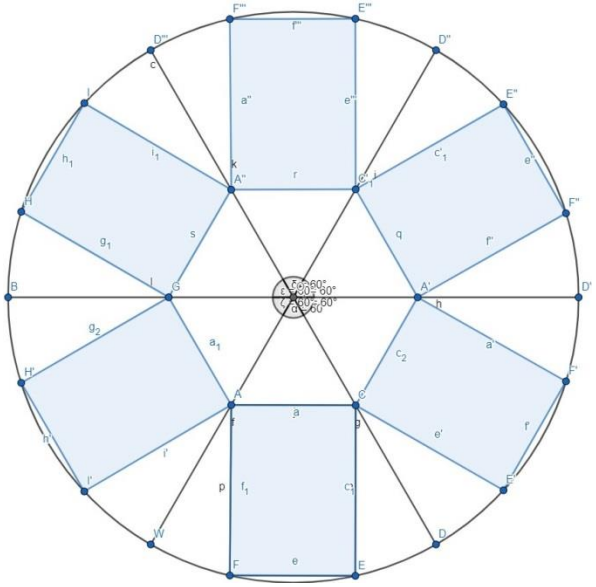
Since $2\alpha \leq 360° \Rightarrow \alpha \leq 180°$, we deduced that as long as α is a divisor of the number 180, an exact number of trays will fit on the table, which will have the value equal to $\frac{360°}{\alpha}$.

$ctg^2\alpha = 3 \iff ctg\alpha = \pm\sqrt{3}$, from which we deduced that:

1. $ctg\alpha = \sqrt{3} \Rightarrow \alpha = 30° \Rightarrow n = 6 \in \mathrm{IN}$
2. $ctg\alpha = -\sqrt{3} \Rightarrow \alpha = 150° \Rightarrow n = \frac{6}{5} \notin \mathrm{IN}$

I. $n = 6 \iff \alpha = 30° \Rightarrow$ The angle at the center is $60° \Rightarrow R^2 = r^2$, so it doesn't matter how we arrange the trays, because the size of the table remains the same in both situations



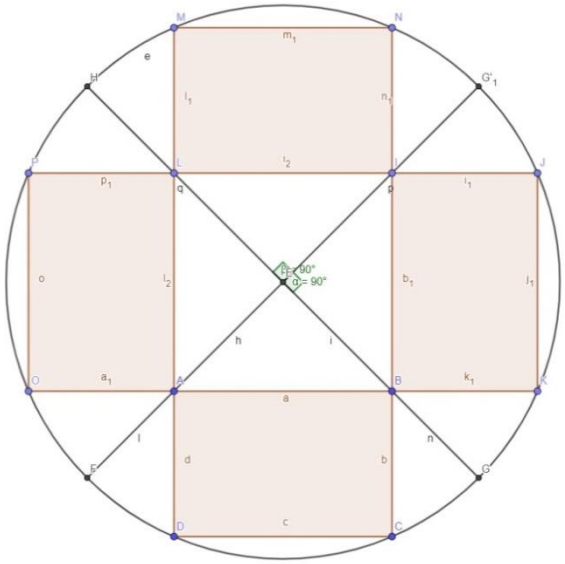


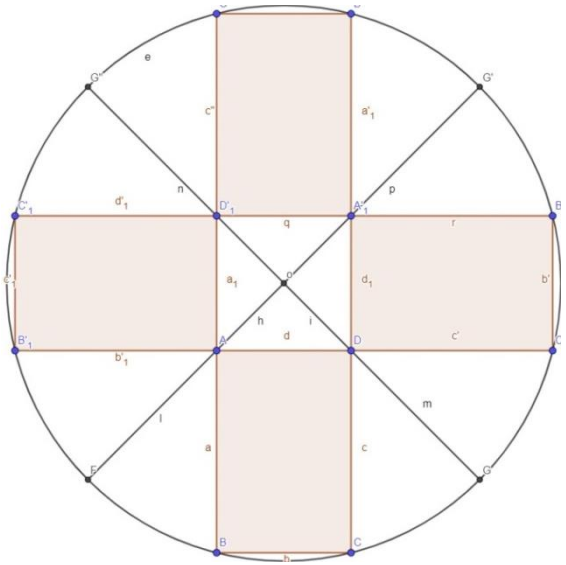
II. $n > 6 \iff 30° < \alpha < 180° \Rightarrow \mathrm{ctg} < 3 \Rightarrow r^2 > R^2$, so it is more efficient to place the trays lengthwise

III. $n < 6 \iff 30° > \alpha \Rightarrow \mathrm{ctg} > 3 \Rightarrow r^2 < R^2$, so it is more efficient to place the trays in width

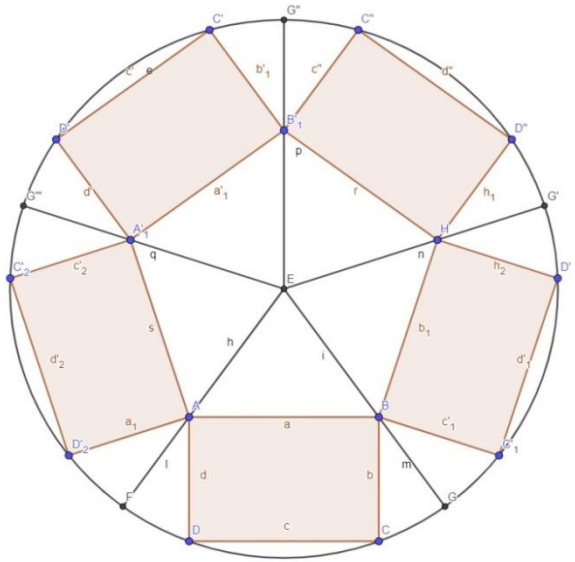
Therefore, we took some concrete examples, namely when the number of trays is equal to 4, 5, 8, respectively 9.

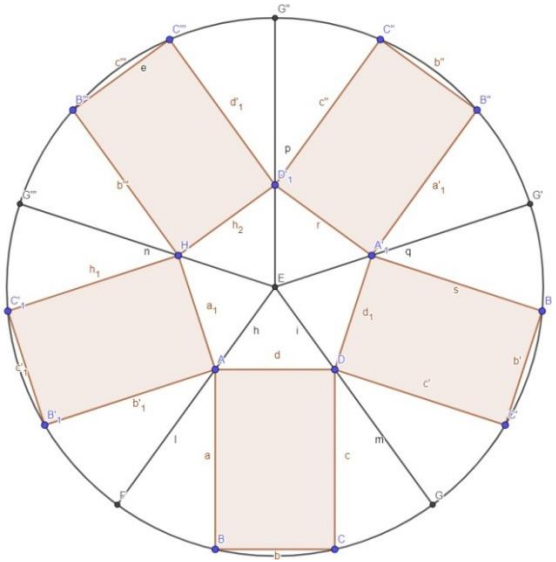
n=4:



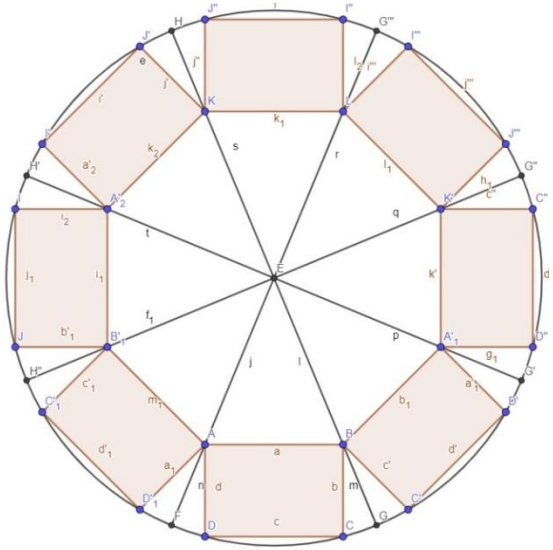


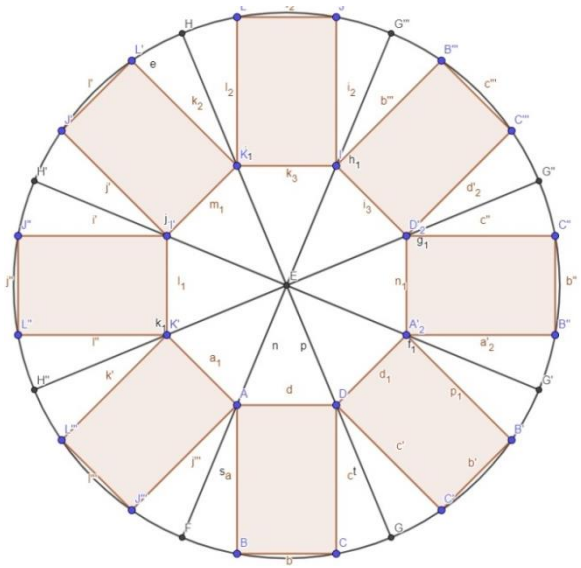
n=5:



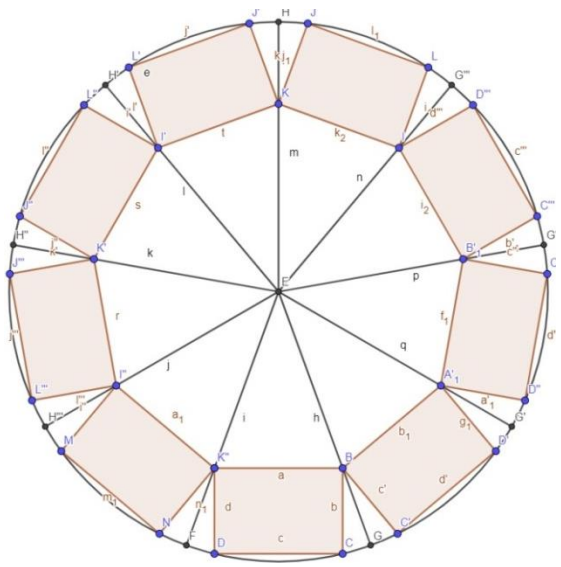


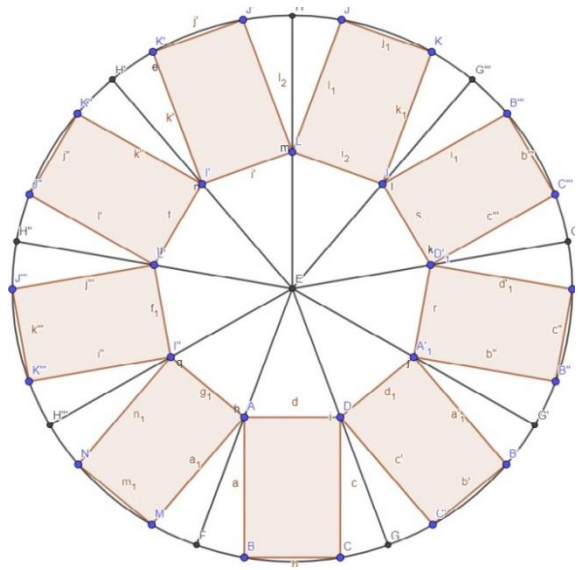
n=8:





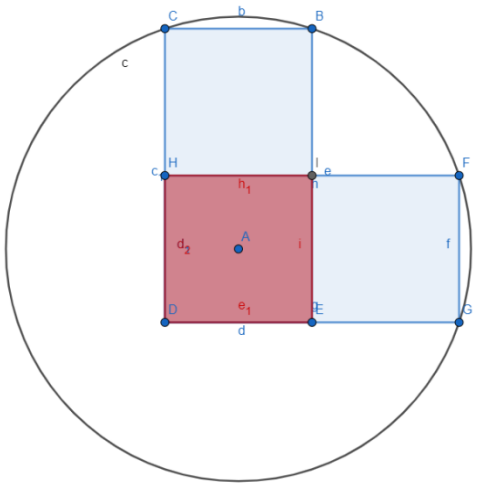
n=9:

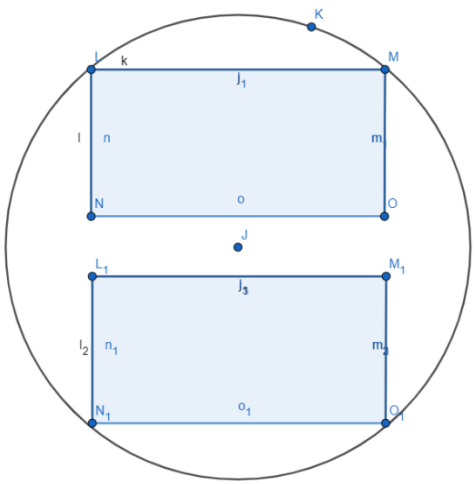


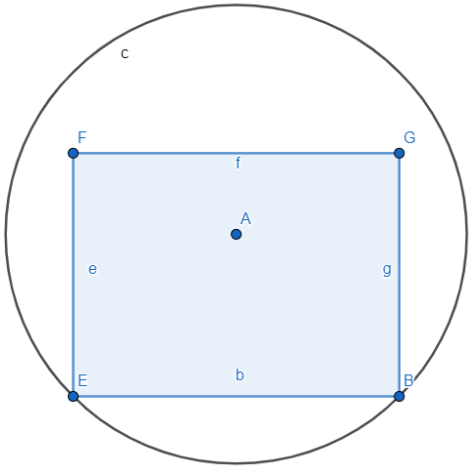


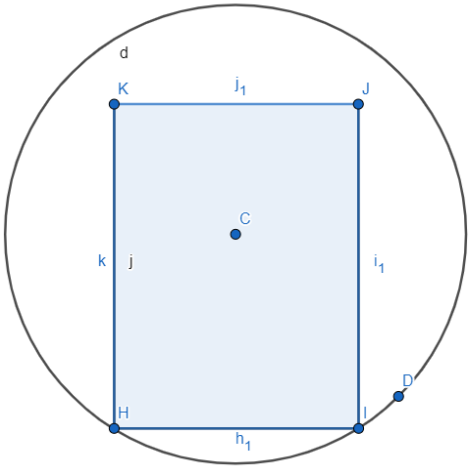
It can be observed that for any number of trays smaller than 6, the variant in which the trays are placed on the width is more efficient, and if it is larger than 6, the one in which the trays are placed on the length is more efficient.

We noticed that if the length is longer than the table radius, but the width is much smaller than the radius, it will be more efficient to place the trays on the length, not on the width, and if the width is bigger than the radius on the table it will only fit a tray.



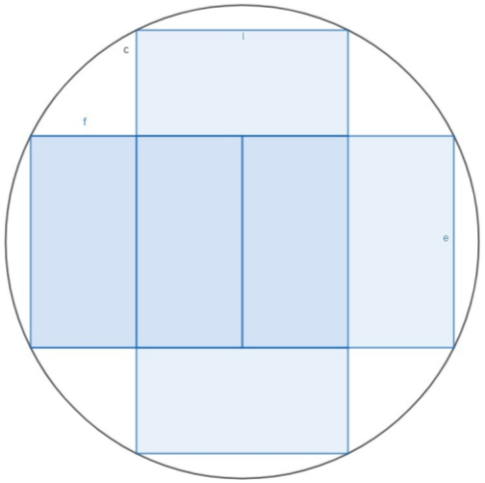


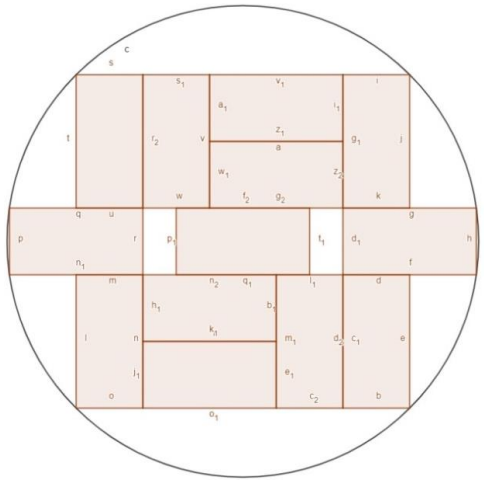


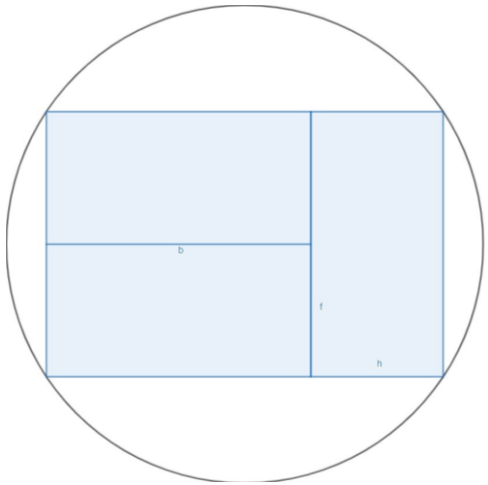


Eliminating the condition that two adjacent corners of each tray sit on the edge of the table means having more space to arrange the trays, as both the space lost between the 2 trays and the one in the center can now be filled by them.

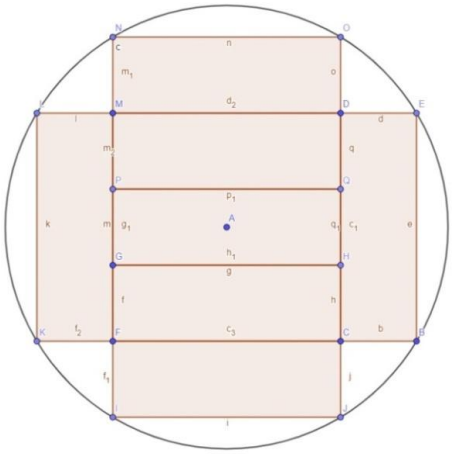
We took some concrete examples and noticed that a favorable case would be the one whose length is twice the width, so x=2y.

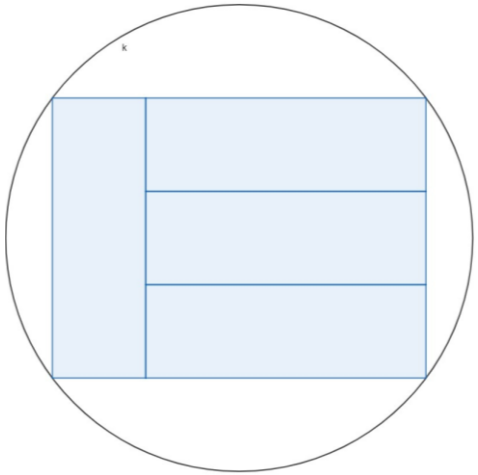


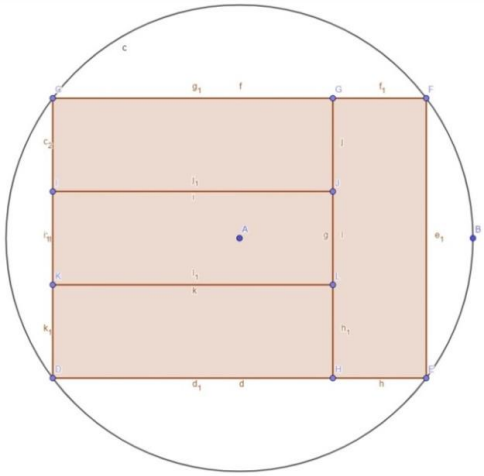




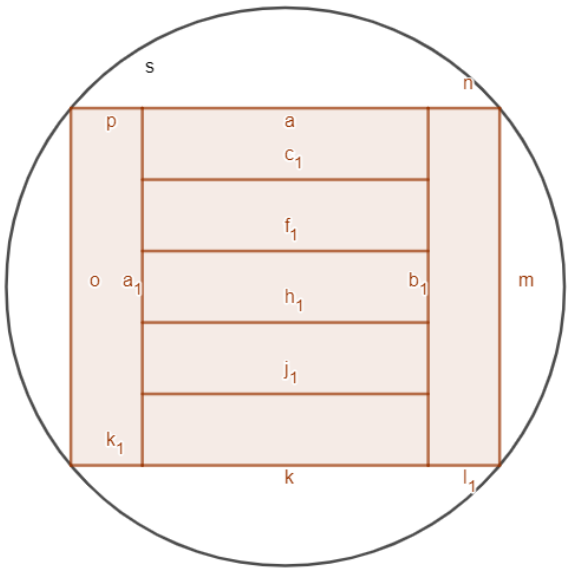
Another favorable case would be the one in which the length is three times the width, so x = 3y.

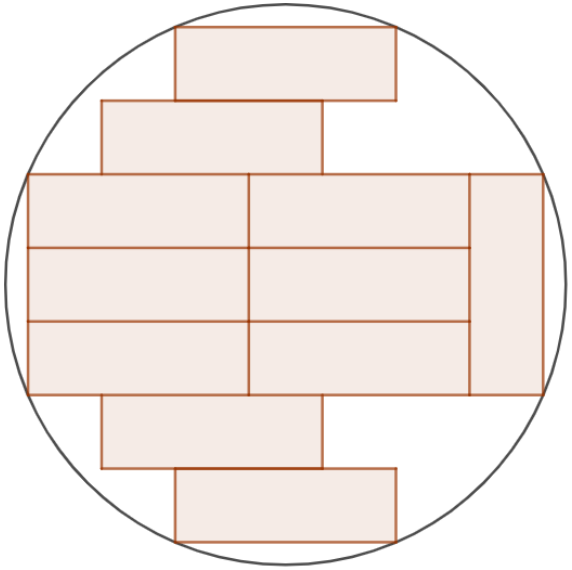


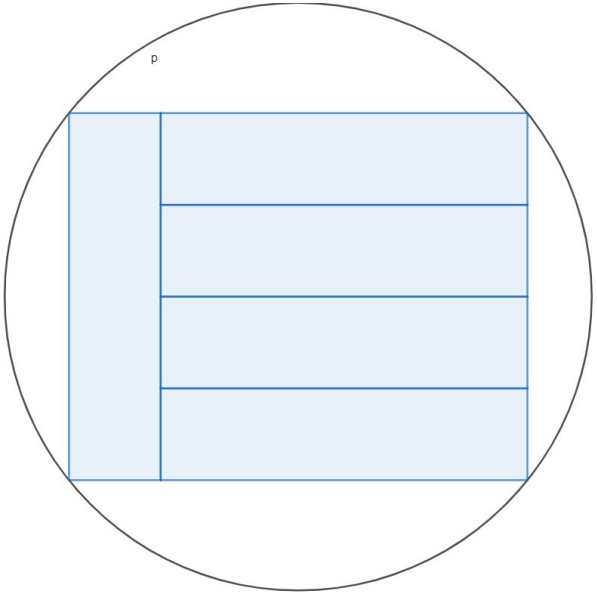




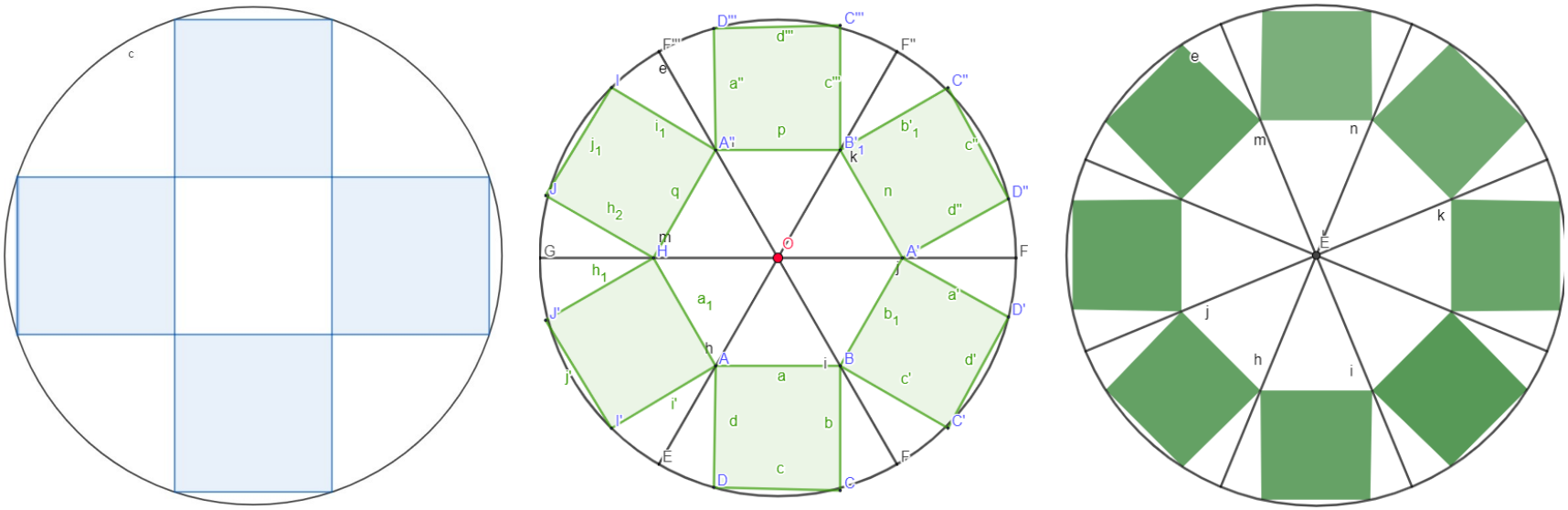
Therefore, we realized that in order to have the smallest radius, it would be useful for the length to be a multiple of the width.







If the trays are square, x will be equal to y, so it doesn't matter how we place them.



If the square trays respect the case of adjacency, we can calculate the radius of the table by processing the formula obtained in the case of rectangular trays: $R^2 = \frac{x^2}{4} + \left(\frac{x}{2} \times ctg\alpha + y\right)^2$.

Since, in the case of square trays, the length is equal to the width, the formula becomes:

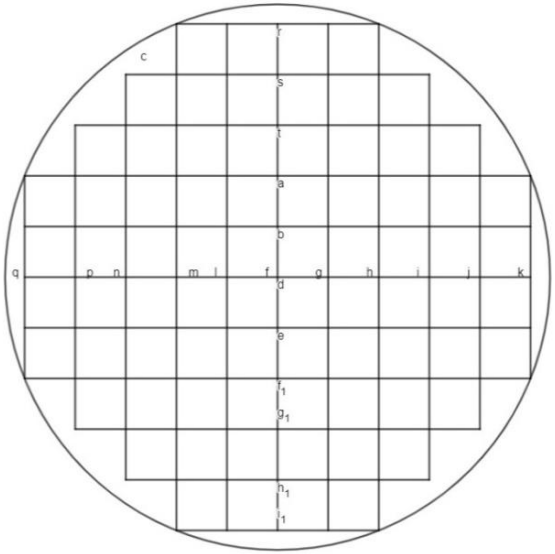
$$R^2 = \frac{x^2}{4} + \left(\frac{x}{2} \times ctg\alpha + x\right)^2$$

$$R^2 = \frac{x^2}{4} + \frac{x^2}{4} \times ctg^2\alpha + 2 \times \frac{x}{2} \times ctg\alpha \times x + x^2$$

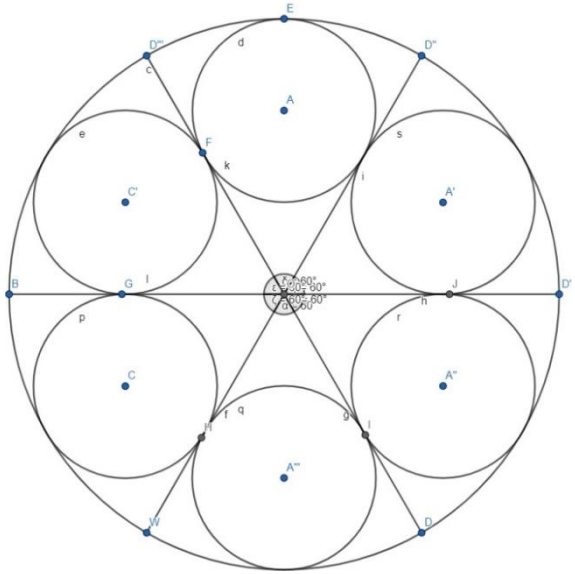
$$R^2 = x^2 \times \left(\frac{1}{4} + \frac{ctg^2\alpha}{4} + ctg\alpha + 1\right)$$

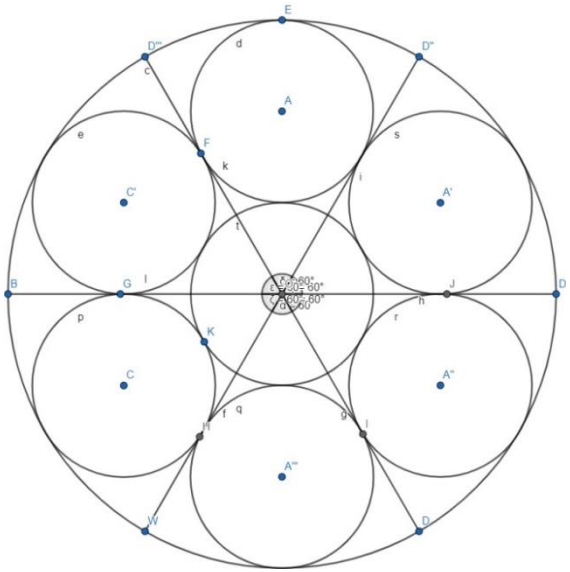
$$\boxed{R^2 = \frac{x^2}{4} \times (ctg^2\alpha + 4ctg\alpha + 5)}$$

If the trays are not placed with the adjacent corners, we will have more space to arrange them on the table, as it can be seen in the following examples:



Since it was a tender subject and we were passionate about it, we decided to do some tests if, instead of trays, we used plates. Thus, since the number of plates used is calculated by the formula $n = \frac{\pi \times R^2}{\pi \times r^2}$, where R represents the radius of the table, r the radius of the plate and n the number of plates, we could deduce the formula to find table radius:
$R^2 = n \times r^2$





Conclusion

In conclusion, with the reduction of the size of the table, fewer trays will fit on the table. In order for the tables to be as small as possible and for the number of trays to be bigger than or equal to 6, they must be placed longitudinally. If the number of trays is less than or equal to six, they must be placed wide. If the trays are square, their arrangement does not matter. However, now with this pandemic, we cannot place the trays to have an adjacent peak, because the distance of two meters between two people would not be respected.