Toothpick arrangements

[Year 2019– 2020]

SURNAMES AND FIRST NAMES OF STUDENTS, GRADES: Adrian-Dragoș Fotea, Theodor Giurcanu, Matei-Cristian Ilie, Robert Lascăr, Tudor Mustață, Daria-Sofia Oniciuc, Bianca-Andreea Pârvan , students in 7th grade.

SCHOOL: Colegiul Naţional "Costache Negruzzi", Iaşi

TEACHER: Adrian Zanoschi

RESEARCHER: Iulian Stoleriu, Faculty of Mathematics, "Alexandru Ioan Cuza" University Iași.

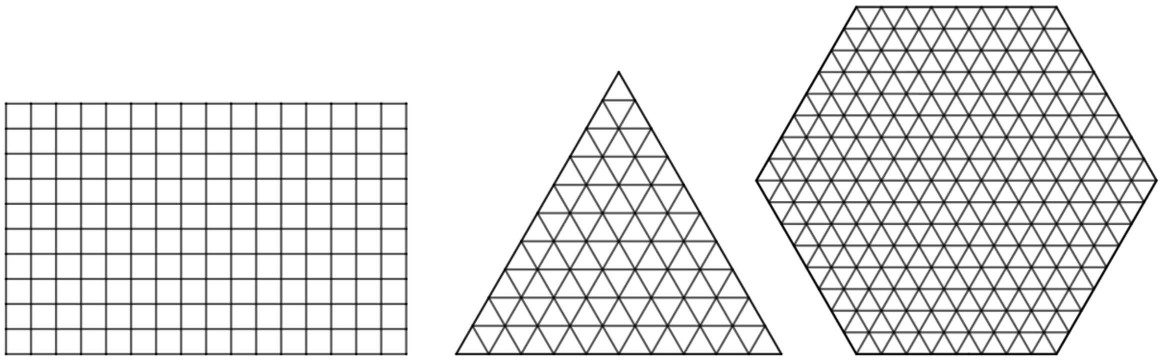
1. PRESENTATION OF THE RESEARCH TOPIC

The paper deals with counting techniques for triangles, squares and rectangles in various types of regular grids and lattices. The study of the number of such polygons or network motifs is of great interest in Graph Theory, when dealing with the complexity of large networks or with finding patterns in large scale graphs, with millions of edges. It has been shown that the distribution of the number of polygons (triangles, squares) in a large scale network can be used to create successful spam filters or to provide useful tools in assessing the content quality in social networks.

2. BRIEF PRESENTATION OF THE CONJECTURES AND RESULTS OBTAINED

a) Find how many one-unit toothpicks are needed to build a $m \times n$ rectangular grid $\left(m,\ n \in \mathbb{N}^*\right)$.

b) How many one-unit toothpicks are needed to build an equilateral triangular grid of side $n \in \mathbb{N}^*$, or a regular hexagonal grid of side *n*.

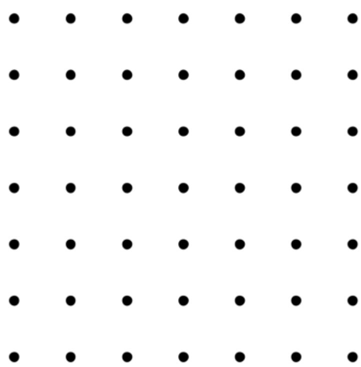


(a) (b) (c)

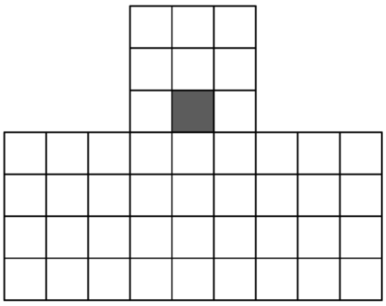
c) Consider an $m \times n$ rectangular grid. How many rectangles can you see in the figure? How many squares can you see?

d) How many equilateral triangles (of any size) can you see in an equilateral triangular grid of side *n*?

e) How many squares (of any size) can you form by joining four points of the attached $n \times n$ lattice?



f) By following the lines of the grid in the attached figure, we would like to cut-off a square parcel that does not contain the black-square. In how many ways can we do it? In how many ways can we cut a rectangular parcel that does not contain the black square?



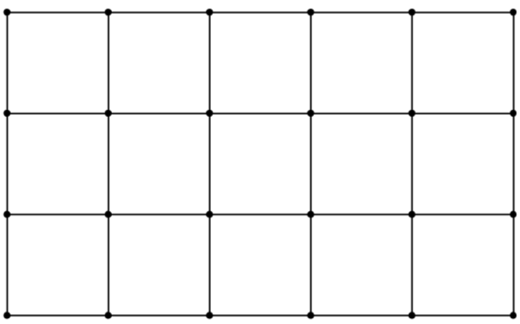
THE SOLUTION

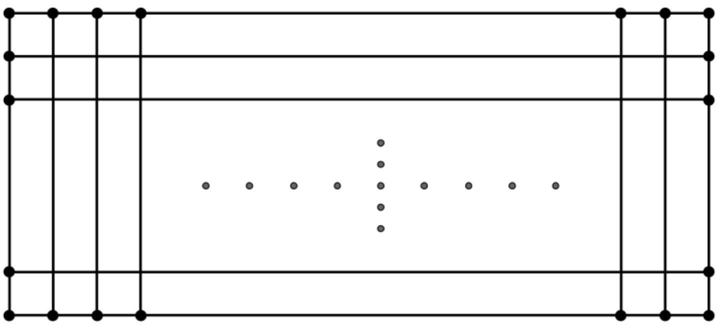
(a) Find how many one-unit toothpicks are needed to build a $m \times n$ rectangular grid $\left(m,\ n \in \mathbb{N}^*\right)$.

First, we are going to count how many toothpicks we have on the horizontal side and then how many toothpicks we have on the vertical side of the $m \times n$ rectangular grid.

Example. Let $m = 3$ and $n = 5$. We have $3 + 1 = 4$ rows with 5 toothpicks each and $5 + 1 = 6$ columns with 3 toothpicks each. In total we used $20 + 18 = 38$ toothpicks (see figure below).

General case. On the horizontal side we have $m(n+1)$ toothpicks. On the vertical side we have $n(m+1)$ toothpicks. In total, we have $m(n+1) + n(m+1) = 2mn + m + n$ toothpicks (see figure below).





(b) How many one-unit toothpicks are needed to build an equilateral triangular grid of side *n*, or a regular hexagonal grid of side *n*.

For an equilateral triangular grid of side *n*, we observe that the toothpicks can be placed in 3 ways: horizontally and on two diagonal directions. On each of the three directions we have $1+2+3+...+n=\frac{n(n+1)}{2}$ toothpicks. So, in total, we have $\frac{3n(n+1)}{2}$ toothpicks.

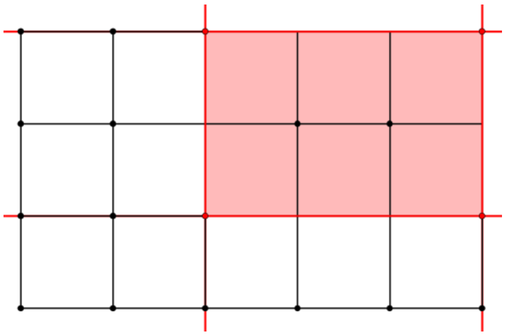
We can also count the toothpicks like that. The triangular grid can be decomposed in *n* lines of equilateral triangles, each triangle with one toothpick side. From the top to the bottom, these rows have 1 triangle, 2 triangles (in order not to count twice, we consider only the triangles which have one side on the third row), …, *n* triangles. So, in total, we have $\frac{3n(n+1)}{2}$ toothpicks.

We observe that the regular hexagon of side *n* is formed by 6 equilateral triangles of side *n*. We think of multiplying with 6 the previous result but doing that we would count twice the toothpicks colored in the drawing. We have 6 zones of overlapping, each being a side of an equilateral triangle of side *n*, so each has *n* toothpicks. The number of toothpicks in the regular hexagonal grid of side *n* is $6\left(\frac{n(n+1)}{2}\cdot 3-n\right)$.

(c) Consider a $m\times n$ rectangular grid. How many rectangles can you see in the figure?

I. First method

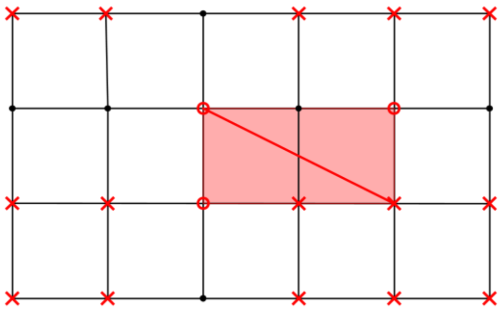
Any rectangle with the vertices on the grid is completely determined by the intersections of two columns and two rows of the grid. The two columns can be chosen in $\frac{m(m+1)}{2}$ ways and the two rows can be chosen in $\frac{n(n+1)}{2}$ ways. So, in the figure we can see $\frac{mn(m+1)(n+1)}{4}$ rectangles (see figure below).



II. Second method

Another way of counting the rectangles is this. Any rectangles with the vertices on the grid is completely determined if we choose two of his opposite vertices. Any point of the grid can be matched to another point of the grid to obtain a diagonal of an rectangle in mn ways.

But, counting like this, any rectangle is counted 4 times. Therefore, the number of rectangles is $\frac{mn(m+1)(n+1)}{4}$.



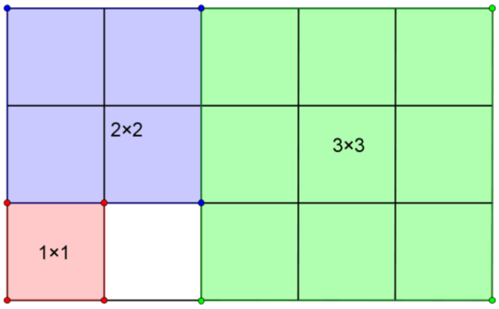
III. Third method.

We present now another version of the second method that (partially) avoids multiple counting. To an arbitrarily fixed vertex of the first row, we can associate a point of the grid to obtain a diagonal of rectangle in mn ways. So, letting the vertex free, we obtain $(m+1)mn$ rectangle with a side on the first row. But we notice that, in fact, each rectangle is counting twice. In conclusion, we obtain $\frac{(m+1)mn}{2}$ rectangle. To an arbitrary fixed vertex of the second row, we can associate a point "lower" position in the grid to obtain a diagonal of a rectangle in $m(n-1)$ ways. So, in this way, we obtain $\frac{(m+1)m(n-1)}{2}$ rectangles. We continue the process up to the n^{th} row and then, summing up, we get

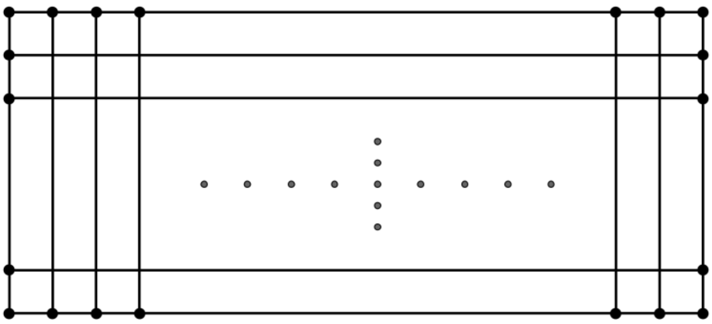
$$\frac{1}{2}\{(m+1)mn+(m+1)m(n-1)+...+(m+1)n\cdot 1\}=\frac{mn(m+1)(n+1)}{4} .$$

How many squares can you see in a $m\times n$ rectangular grid?

We firstly consider the case $m=3$ and $n=5$. There are $5\cdot 3=15$ squares of type 1×1 (length side 1), $4\cdot 2=8$ squares of type 2×2 (length side 2) and $3\cdot 1=3$ squares of type 3×3 (length side 3). The total number of all types of squares is 26.



In the general case, we can assume, without loss of generality, that $n\le m$. The biggest square is a $n\times n$ square, because $n\le m$.



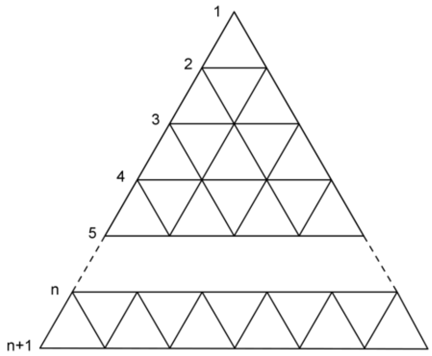
There are mn squares of type 1×1 (length side 1), $(n-1)(m-1)$ squares of type 2×2 (length side 2), $(n-2)(m-2)$ squares of type 3×3 (length side 3), … and $[n-(n-1)][m-(n-1)]$ squares of type $n\times n$ (length side *n*).

Thus the total number of squares is equal to:

$$mn+(n-1)(m-1)+(n-2)(m-2)+...+[n-(n-1)][m-(n-1)]=mn+(mn-n-m+1)+$$
$$(mn-2n-2m+4)+...+\left[mn-(n-1)n-(n-1)m+(n-1)^2\right]=mn^2-n[1+2+3+...+(n-1)]$$
$$-m[1+2+3+...+(n-1)]+\left[1^2+2^2+...+(n-1)^2\right]=\frac{n(n+1)(3m-n+1)}{6}.$$

(d) How many equilateral triangles (of any size) can you see in a equilateral triangular grid of side *n*?

We label the rows of the equilateral triangular grid from top to bottom: 1, 2, 3…, *n* and $n+1$.



There are two types of equilateral triangles: Δ first type (pointing up) and ∇ second type (pointing down).

I. Firstly, we will count the number of pointing up equilateral triangles.

The number of equilateral triangles with the up vertex on the first line is $1\cdot n$, the number of equilateral triangles with the up vertex on the second line is $2\cdot(n-1)$, the number of equilateral triangles with the up vertex on the third line is $3\cdot(n-2)$, … and the number of equilateral triangles with the up vertex on the *n* line is $n\cdot1$. So, the total number of first type equilateral triangles is:

$$S=1\cdot n+2\cdot(n-1)+3\cdot(n-2)+...+(n-1)\cdot 2+n\cdot 1=1\cdot(n+1-1)+2\cdot(n+1-2)+3\cdot(n+1-3)$$

$$+...+(n-1)\cdot(n+1-(n-1))+n\cdot(n+1-n)=(n+1)(1+2+3+...+(n-1)+n)-$$

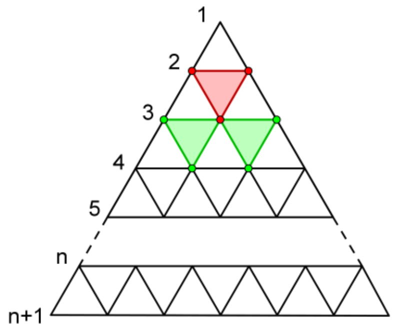
$$\left(1^2+2^2+3^2+...+(n-1)^2+n^2\right)=(n+1)\frac{n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}=\frac{n(n+1)(n+2)}{6}\ .$$

II. Now we will count the number of the down vertex equilateral triangles. We have two cases: when *n* is even and when *n* is odd.

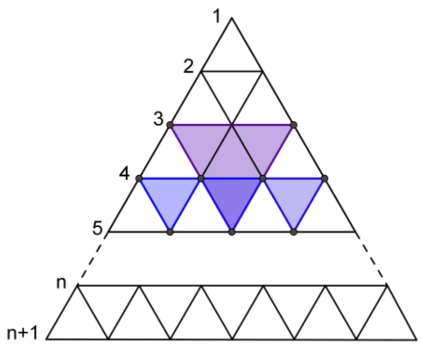
II.1. *The case when n is even.*

We do not have triangles of the second type with the vertex on the first or second row. There is only one triangle of the , which are not square type with the vertex on the third row.

There are $1+1=2$ triangles of second type on the fourth row.



There are $1+2+1=\frac{4^2}{4}$ triangles of second type on the fifth row.



There are $1+2+2+1=\frac{4\cdot 6}{4}$ triangles of second type on the sixth row.

There are $1+2+3+2+1=\frac{6^2}{4}$ triangles of second type on the seventh row.

…

There are $1+2+...+\left(\frac{n}{2}-2\right)+\left(\frac{n}{2}-1\right)+\left(\frac{n}{2}-2\right)+...+2+1=\frac{(n-2)^2}{4}$ triangles of second type in the row $n-1$.

There are $1+2+...+\left(\frac{n}{2}-1\right)+\left(\frac{n}{2}-1\right)+...+2+1=\frac{n(n-2)}{4}$ triangles of second type in the row *n*.

So, the total number of triangles of second type with *n* even is:

$$S_{even}=\frac{1}{4}\left(2^2+4^2+...+n^2\right)+\frac{1}{4}\left(2\cdot 4+4\cdot 6+...+(n-2)n\right)=\frac{1}{4}\sum_{k=1}^{n/2}(2k)^2+\frac{1}{4}\sum_{k=1}^{n/2}2k(2k-2)$$

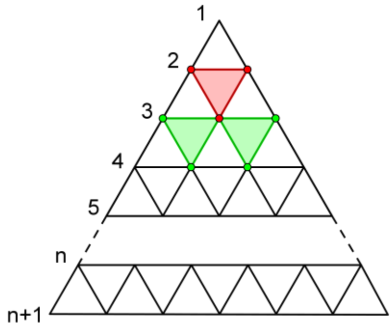
$$=2\sum_{k=1}^{n/2}k^2-\sum_{k=1}^{n/2}k=2\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)(n+1)}{6}-\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2}.$$

Thus, $S_{even}=\frac{n(n+2)(2n-1)}{24}$.

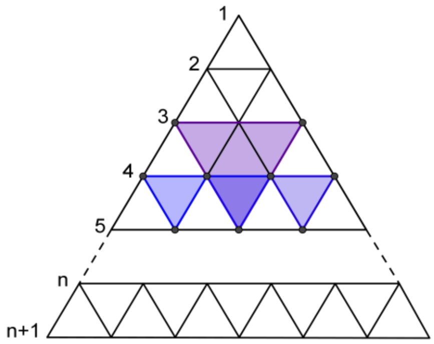
II.2. *The case when n is odd.*

There is only one triangle of the second type with the vertex on the third row

There are $1+1=2$ triangles of second type on the fourth row.



There are $1+2+1=\frac{4^2}{4}$ triangles of second type on the fifth row.



There are $1+2+2+1=\frac{4\cdot 6}{4}$ triangles of second type on the sixth row.

There are $1+2+3+2+1=\frac{6^2}{4}$ triangles of second type on the seventh row.

…

There are $1+2+...+\frac{n-3}{2}+\frac{n-3}{2}+...+2+1=\frac{(n-1)(n-3)}{4}$ triangles of second type on $n-1$ row.

There are $1+2+...+\frac{n-3}{2}+\frac{n-1}{2}+\frac{n-3}{2}+...+2+1=\frac{(n-2)^2}{4}$ triangles of second type on the n row. In total, the number of triangles of second type with n odd is:

$$S_{odd}=\frac{1}{4}\left(2^2+4^2+...+(n-3)^2+(n-1)^2\right)+\frac{1}{4}\left(2\cdot 4+4\cdot 6+...+(n-1)(n-3)\right)=\frac{1}{4}\sum_{k=1}^{(n-2)/2}(2k)^2+$$

$$\frac{1}{4}\sum_{k=1}^{(n-2)/2}(2k)(2k+2)=\frac{1}{24}\left(n^2-1\right)(2n+3).$$

So, for *n* even, the total number of triangles is:

$$S=\frac{n(n+1)(n+2)}{6}+\frac{n(n+2)(2n-1)}{24}=\frac{1}{8}\left(2n^3+5n^2+2n\right),$$

and, for *n* odd, the total number of triangles is:

$$S=\frac{n(n+1)(n+2)}{6}+\frac{\left(n^2-1\right)(2n+3)}{24}=\frac{1}{8}\left(2n^3+5n^2+2n-1\right).$$

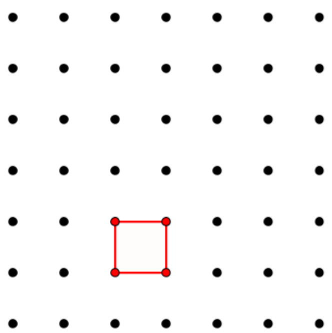
Finally, we can put these results together and write that

$$S=\left\lfloor \frac{1}{8}n(n+2)(2n+1)\right\rfloor,$$

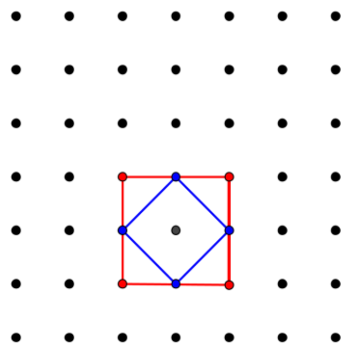
where $\lfloor x \rfloor$ is the integer part of x.

(e) How many squares (of any size) can you form by joining four points of the attached $n\times n$ lattice?

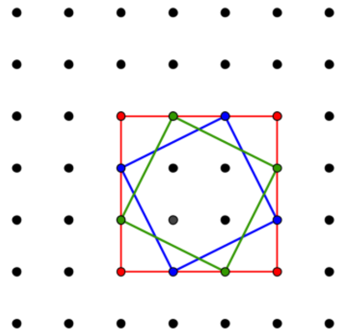
There are $(n-1)^2\cdot 1$ square with side 1 unit in a $n\times n$ lattice:



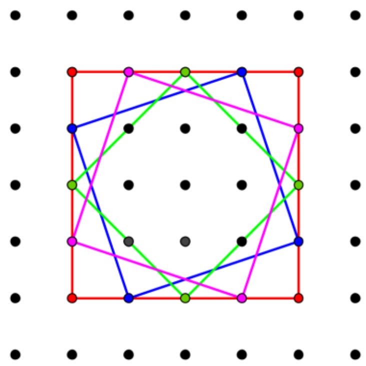
There are $(n-1)^2 \cdot 2$ square with the vertices on 2×2 square grid side:



There are $(n-3)^2 \cdot 3$ square with the vertices on 3×3 square grid side:



There are $(n-4)^2 \cdot 4$ square with the vertices on 4×4 square grid side:

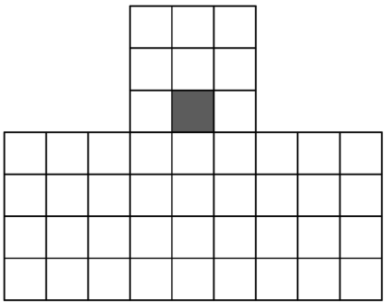


In this way, we find out that the total number of squares (of any size) which we can form by joining four points of the attached $n \times n$ lattice is:

$$N = (n-1)^2 \cdot 1 + (n-2)^2 \cdot 2 + (n-3)^2 \cdot 3 + (n-4)^2 \cdot 4 + ... + 1^2 \cdot (n-1) .$$

f) By following the lines of the grid in the attached figure, we would like to cut-off a square parcel that does not contain the black-square. In how many ways can we do it?

In how many ways can we cut a rectangular parcel that do not contain the black square?



We are going to count in three different ways.

I. *First method*

1. We shall start counting squares with side one, then two, three and four.

There are $9 \cdot 4 + (3+3+2) = 44$ squares of side 1, then $2 + 8 \cdot 3 = 26$ squares of side 2, then $7 \cdot 2 = 14$ squares of side 3 and 6 squares of side 4.

In total, we obtain **90** squares. There are no squares of side 5 or more, due to the shape of the figure.

2. We counted the squares so now we are going to count rest of the rectangles.

There are $4 + 8 \cdot 4 = 36$ rectangles of type 1×2, $2 + 7 \cdot 4 = 30$ rectangles of type 1×3, $6 \cdot 4 = 24$ rectangles of type 1×4, $5 \cdot 4 = 20$ rectangles of type 1×5, $4 \cdot 4 = 16$ rectangles of type 1×6,

$3\cdot 4=12$ rectangles of type 1×7, $2\cdot 4=8$ rectangles of type 1×8, $1\cdot 4=4$ rectangles of type 1×9.

Thus, there are **150** rectangles of type $1\times n$, which are not square.

There are $7+9\cdot 3=34$ rectangles of type 2×1, $1+7\cdot 3=22$ rectangles of type 2×3, $6\cdot 3=18$ rectangles of type 2×4, $5\cdot 3=15$ rectangles of type 2×5, $4\cdot 3=12$ rectangles of type 2×6, $3\cdot 3=9$ rectangles of type 2×7, $2\cdot 3=6$ rectangles of type 2×8 and $1\cdot 3=3$ rectangles of type 2×9.

Thus, there are **119** rectangles of type $2\times n$, which are not square.

There are $6+9\cdot 2=24$ rectangles of type 3×1, $8\cdot 2=16$ rectangles of type 3×2, $6\cdot 2=12$ rectangles of type 3×4, $5\cdot 2=10$ rectangles of type 3×5, $4\cdot 2=8$ rectangles of type 3×6, $3\cdot 2=6$ rectangles of type 3×7, $2\cdot 2=4$ rectangles of type 3×8, $1\cdot 2=2$ rectangles of type 3×9.

Thus, there are **82** rectangles of type $3\times n$, which are not square.

There are $9+6=15$ rectangles of type 4×1, 8 rectangles of type 4×2, 7 rectangles of type 4×3, 5 rectangles of type 4×5, 4 rectangles of type 4×6, 3 rectangles of type 4×7, 2 rectangles of type 4×8 and 1 rectangles of type 4×9.

Thus, there are **45** rectangles of type $4\times n$, which are not square.

There are 6 rectangles of type 5×1, 4 rectangles of type 6×1 and 2 rectangles of type 7×1.

Thus, there are **12** rectangles of type $n\times 1$, which are not square.

Therefore, the are $90+150+119+82+45+12=\mathbf{498}$ rectangles of any size which satisfy our requirements.

II. *Second method*

Consider first the bottom 4 by 9 rectangle. A rectangle contained in it can be chosen by selecting 2 of 5 horizontal lines and 2 of the 10 vertical lines as its sides. Thus, there are $10 \cdot 45 = 450$ such rectangles. Similarly, there are $3 \cdot 6 = 18$ rectangles contained in the top 2 by 3 rectangle. All remaining rectangles come from the two vertical columns adjacent to the black square, and must contain a square next (left and right) to the black square. We may add 0, 1 or 2 squares above it and 0, 1, 2, 3 or 4 squares below it. This yields $3 \cdot 5 = 15$ additional rectangles per column. The grand total is $450 + 18 + 2 \cdot 15 = \mathbf{498}$.

III. *Third method*

We can choose a rectangle from the 4×9 horizontal grid in $\frac{4 \cdot 9 \cdot 5 \cdot 10}{4} = 450$ ways. Also, we can choose a rectangle from the 7×3 vertical grid in $\frac{7 \cdot 3 \cdot 8 \cdot 4}{4} = 168$ ways. The intersection of the two grids is a 4×3 grid. From this intersection grid, we can cut a rectangle in $\frac{4 \cdot 3 \cdot 5 \cdot 4}{4} = 60$ ways.

Therefore, from the entire grid we can cut a total of $450 + 168 - 60 = 558$ rectangles. Out of these, a number of $3 \cdot 5 \cdot 2 \cdot 2 = 60$ rectangles contain the black square. It remains that we can cut exactly $558 - 60 = \mathbf{498}$ rectangles that do not contain the black square.

3. CONCLUSION

In this paper we have employed various techniques for counting the number of triangles, squares or rectangles that can be seen in regular grids or lattices. Some of the questions were solved using two or three methods. We are aware that other straightforward solutions may be provided for some of the questions, but we have adapted our solutions to the secondary school level, avoiding the use of combinatorial notions and formulae. We believe that our methods may also be adapted in counting other types of polygons in complex grid structures.