The path of ants

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1 Research topic

The problem we were presented with is:
The first ant makes 10 random moves, each having a length of one unit and, at each move, she leaves a mark. After the tenth move, she reaches food and sends a radio signal back to the anthill. A second ant goes along the first half of the first segment, then, from the middle of it to the middle of the second and so on, until she reaches the middle of the last segment, from where she goes to the food along the second half of the last segment. A third ant follows the same strategy, following the second ant's path.

a) Is the second path shorter than the first?

b) Will the ants tend to find the shortest itinerary between the two points?

What the second question actually asks for is to describe how a given polygonal chain behaves, asymptotically, when the above described algorithm is infinitely repeated.

2 Our research

2.1 Results

The path will tend towards a straight line between the two endpoints, in terms of both shape and length.[1] Throughout the proof, we have used remarkable results, such as the triangle's inequality, Pascal's triangle and Wallis' approximation.

2.2 Proof of a)

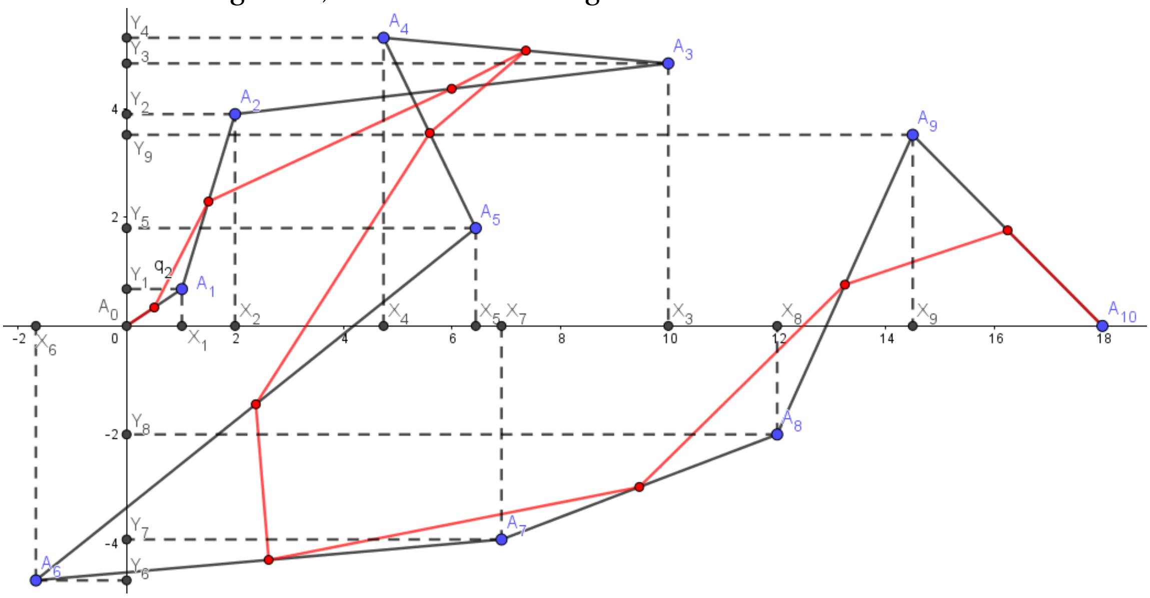
We prove the slightly more general statement:
"An ant starts moving at a point A_0 and makes n moves along the segments A_0A_1, A_1A_2, ..., $A_{n-1}A_n$. If a second ant moves along the segments A_0M_1, M_1M_2, M_2M_3, ..., $M_{n-1}M_n$, M_nA_n, where $M_1, M_2, \dots M_n$ are the midpoints of A_0A_1, A_1A_2, ..., $A_{n-1}A_n$ respectively, then, the second path will be shorter than the first one.":

$$A_0M_1 = \frac{A_0A_1}{2}$$
$$M_nA_n = \frac{A_{n-1}A_n}{2}$$
$$M_kM_{k+1} = \frac{A_{k-1}A_{k+1}}{2} \leq \frac{A_{k-1}A_k + A_kA_{k+1}}{2}$$

for every k from 1 to $n-1$. Adding these up, we get to the conclusion

2.3 Proof of b1)

We prove a weaker result: The path tends towards a line in terms of shape. That is, we consider the xOy Carthesian system, with the origin in A_0 and with $Ox = A_0A_n$ and we consider the coordinates of our points. We show that the ordinates tend to 0, and the abscissae tend to be in ascending order, inside the initial segment.



We firstly show that the assertion holds for ordinates. For this, consider the following table, where every line represents an ant(note that, from now on, unless stated otherwise, the numbering of lines and columns starts from 0):

$$\begin{array}{lllllllllllll}
line\ \ 0 & & & y_1 & & y_2 & & y_3 & \dots & & y_n & & \\
line\ \ 1 & & \frac{y_1}{2} & & \frac{y_1+y_2}{2} & & \frac{y_2+y_3}{2} & & \dots & \frac{y_{n-1}+y_n}{2} & & \frac{y_n}{2} & \\
line\ \ 2 & \frac{y_1}{4} & & \frac{2y_1+y_2}{4} & & \frac{y_1+2y_2+y_3}{4} & & \dots & & & \frac{y_{n-1}+2y_n}{4} & & \frac{y_n}{4}
\end{array}$$

...

To be able to determine a closed-form of the general term, we consider the contribution of every y_i separately. Take the following triangle, constructed by dividing every line i of Pascal's triangle by 2^i, where every term is the arithmetic mean of the two above, just as in the first table:

$$\begin{array}{llllll}
line\ \ 0 & & & 1 & & \\
line\ \ 1 & & \frac{1}{2} & & \frac{1}{2} & \\
line\ \ 2 & \frac{1}{2^2} & & \frac{2}{2^2} & & \frac{1}{2^2}
\end{array}$$

...

Now, adding up the coefficients, we obtain the formula:

$$y_{ij} = \frac{\sum_{\alpha=1}^{n} y_\alpha \binom{i}{j-\alpha+1}}{2^i}$$

Considering $M = max\{|y_1|, |y_2|, \dots |y_n|\}$ and applying a basic inequality, we get:

$$|y_{ij}| \leq Mn\frac{\binom{i}{\frac{i}{2}}}{2^i}$$

But, by Wallis' approximation:

$$\lim_{n\to\infty} \frac{\binom{2n}{n}}{4^n}\sqrt{n\pi} = 1$$
$$\lim_{i\to\infty} \frac{\binom{i}{\frac{i}{2}}}{2^i} = 0$$

Now, we move on to the abscissae. We consider the same type of table as before, the only exception being that every line is infinite, the last term repeating indefinitely [2] . The triangles of coefficients are the same as above for x_1, x_2, ..., x_{n-1}, so, by Wallis' approximation, they tend towards 0. The coefficients of x_n are given by:

$$
\begin{array}{lccccccccccc}
line & 0 & & & 1 & & 1 & & 1 & & 1 & \dots \\
line & 1 & & \frac{1}{2} & & \frac{2}{2} & & \frac{2}{2} & & \frac{2}{2} & & \dots \\
line & 2 & \frac{1}{2^2} & & \frac{3}{2^2} & & \frac{4}{2^2} & & \frac{4}{2^2} & & & \dots
\end{array}
$$

This "trapezoid" $\left(T_{i\,j}\right)$ is just an infinite sum of the above triangles, so its terms are of the form $\frac{\sum_{\alpha=0}^{j}\binom{i}{\alpha}}{2^i}$, which are clearly in ascending order on each line. That is, the statement is completely proven.

2.4 Proof of b2)

Now, we prove the strong result, that is, the length of the path tends towards the distance between the initial endpoints. We use both the notations and the results from the previous subsection. If S_n denotes the length of the n-th path (where the numbering starts from 0), we get:

$$
\begin{aligned}
S_i &= \sum_{j=-1}^{i+n-1} \sqrt{\left(x_{i\,j+1} - x_{i\,j}\right)^2 + \left(y_{i\,j+1} - y_{i\,j}\right)^2} \\
&\le \sum_{j=-1}^{i+n-1} \left(|x_{i\,j+1} - x_{i\,j}| + |y_{i\,j+1} - y_{i\,j}|\right) \\
&= \sum_{j=-1}^{i+n-1} |x_{i\,j+1} - x_{i\,j}| + \sum_{j=0}^{i+n-1} |y_{i\,j+1} - y_{i\,j}|
\end{aligned}
$$

where everything undefined (with negative indices or too large indices) is 0. We show that the first sum P_n converges to x_n and the second Q_n to 0. As before, we start with the ordinates:

$$
\begin{aligned}
Q_i &= \sum_{j=0}^{i+n-1} |y_{i\,j+1} - y_{i\,j}| \\
&= \sum_{j=-1}^{i+n-1} \frac{\left|\sum_{\alpha=1}^{n} y_\alpha \binom{i}{j-\alpha+2} - \sum_{\alpha=1}^{n} y_\alpha \binom{i}{j-\alpha+1}\right|}{2^i} \\
&= \sum_{j=-1}^{i+n-1} \frac{\left|\sum_{\alpha=1}^{n} y_\alpha \left(\binom{i}{j-\alpha+2} - \binom{i}{j-\alpha+1}\right)\right|}{2^i} \\
&\le \frac{M}{2^i} \sum_{j=-1}^{i+n-1} \sum_{\alpha=1}^{n} \left|\left(\binom{i}{j-\alpha+2} - \binom{i}{j-\alpha+1}\right)\right|
\end{aligned}
$$

Now, we evaluate the interior sums, as many terms cancel out: [3]

$$\sum_{\alpha=1}^{n}\left|\left(\binom{i}{j-\alpha+2}-\binom{i}{j-\alpha+1}\right)\right| = \begin{cases} \binom{i}{j+1}-\binom{i}{j-n+1}, & j \le \frac{i}{2}-1 \\ 2\binom{i}{\frac{i}{2}}-\binom{i}{j+1}-\binom{i}{j-n+1} & \frac{i}{2} \le j \le \frac{i}{2}+n-2 \\ \binom{i}{j-n+1}-\binom{i}{j+1} & \frac{i}{2}+n-1 \le j \end{cases}$$

We proceed to evaluate the last sum, where other terms cancel out:

$$\begin{aligned} Q_i \le \frac{M}{2^i}\Bigg(& \sum_{j=-1}^{\frac{i}{2}-1}\left(\binom{i}{j+1}-\binom{i}{j-n+1}\right)+ \\ & \sum_{j=\frac{i}{2}}^{\frac{i}{2}+n-2}\left(2\binom{i}{\frac{i}{2}}-\binom{i}{j+1}-\binom{i}{j-n+1}\right)+ \\ & \sum_{j=\frac{i}{2}+n-1}^{i+n-1}\left(\binom{i}{j-n+1}-\binom{i}{j+1}\right)\Bigg) \end{aligned}$$

$$Q_n \le 2nM\frac{\binom{i}{\frac{i}{2}}}{2^i}$$

By Wallis' approximation, the last term tends to 0, therefore, so does Q_n(Q_n is nonnegative). We move on to the abscissae and we let $x = x_n$. Recall the construction of the abscissae's table and let $x_{i\,j} = x'_{i\,j} + x''_{i\,j}$, where $x'_{i\,j}$ is a linear expression in $x_1, x_2, ..., x_{n-1}$ and $x''_{i\,j}$ is of the form $x''_{i\,j} = T_{i\,j-n+1}x$, $(T_{i\,j})$ being the "trapezoid" mentioned in the previous section. So:

$$\begin{aligned} P_i &= \sum_{j=-1}^{i+n-1}|x_{i\,j+1}-x_{i\,j}| \\ &\le \sum_{j=-1}^{i+n-1}|x'_{i\,j+1}-x'_{i\,j}| + \sum_{j=-1}^{i+n-1}|x''_{i\,j+1}-x''_{i\,j}| \end{aligned}$$

As above, the first sum converges to 0, so we compute the second sum:

$$\begin{aligned} P''_i &= \sum_{j=-1}^{i+n-1}|x''_{i\,j+1}-x''_{i\,j}| \\ &= \frac{x}{2^i}\sum_{j=0}^{i}\binom{i}{j} \\ &= x \end{aligned}$$

That is, P_i is smaller than something that tends to x and, together with the result about Q_i, we obtain S_i smaller than something that tends to x. But, as $S_i \ge x$(the shortest path between two points is the straight line), we get that S_i tends to x and the conclusion follows.

3 Conclusions and generalisations

The problem, as stated, is completely solved. Although the first variant of b) is weaker, that was the first we came up with and only after being shown why it does not imply the stronger one, did we manage to prove the latter.
In the proof of b), the index i is tacitly assumed even. Although that is not always the case, the computations for the other case are similar but, even more, given the result from a), the proof that the even subsequence converges is enough.
An immediate generalisation is to extend the euclidean plane to $\mathbb{R}^n$, where the proof is identical. The generalisation to a sphere, a torus or any other such surface is not always true, as counter-examples are easily constructed for the first two. A far more interesting generalisation is to consider a metric on $\mathbb{R}^2$ (or $\mathbb{R}^n$) different from the usual euclidean distance, and ask under what conditions we can reach the same conclusions, or if there exists a metric in which they fail. These are very general questions that we were not able to tackle yet.

Edition Notes

[1] Here, something should be said about the two notions and about the relationship between them.

[2] This sentence is not so clear. Some more explanation would be useful.

[3] These calculations are rather complex. They should be explained and developed in more detail.