Game of differences
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Research topic:

There are 4 natural numbers a, b, c, d written on a row. The differences |a-b|, |b-c|, |c-d|, |d-a| are then written on the next row. The process is continued:

7	3	9	2
4	6	7	5
2	1	2	1
1	1	1	1
0	0	0	0

It can be observed that the null row 0, 0, 0, 0 has been obtained.

- Is this a coincidence or is the null row obtained for any natural numbers a, b, c, d?
- What happens if the rows don't have 4 numbers, but 3, 5, 6,...?
- What happens if we have 4 rational numbers
- What happens if we have 4 real numbers

Brief presentation:

- We solved the natural numbers' case using remainders modulo 2.
- We also treated the cases of 3, 4, 5, 6, 7, 8, … natural numbers on the first row, obtaining different conclusions, afterwards we made a C++ program.
- We extended the case of integers, followed by rational numbers.
- Subsequently, we prove significant results for real numbers.

Solution :

What happens if we have:

1. **4 natural numbers**
2. **5, 6, 7,.. natural numbers**
3. **4 rational numbers**
4. **4 real numbers**

1.

1.1 Solution 1:

We replace the numbers with their remainders modulo 2. There are 2^4=16 ways to distribute the remainders 0 and 1 on the first row. All these posibilities are consisted in the following table:

1	0	1	1	1	1	0	1	1	1	1	0	0	1	1	1
or				or				or				or			
0	1	0	0	0	0	1	0	0	0	0	1	1	0	0	0
1	1	0	0	0	1	1	0	0	0	1	1	1	0	0	1
0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

So, after at most 4 steps, the obtained numbers will be divisible by 2. The remainders at the division by 2^2=4 when we divide the last numbers obtained are 0 or 2. If we make similar tables with the one made before, but instead of using the pair (0,1) we used the pair (0,2) of remainders, we notice that, after at most another 4 steps, all the numbers obtained will be divisible by 4.

We continue the process. **After at most 4k steps, all the numbers on the last row will be divisible by 2^k.**

So, the numbers on the last row will be divisible with an arbitrarily big power of 2. In conclusion, after a certain number of steps, we will reach the zero row: 0, 0, 0, 0.

1.2 Solution 2:

Let $a_1 = a, b_1 = b, c_1 = c, d_1 = d$ and $a_n, b_n, c_n, d_n \in \mathbb{N}$, $a_n = |a_{n-1} - b_{n-1}|$, $b_n = |b_{n-1} - c_{n-1}|$, $c_n = |c_{n-1} - d_{n-1}|$, $d_n = |d_{n-1} - a_{n-1}|$.

Let $x_n = \max\{a_n, b_n, c_n, d_n\}$. Taking into consideration the way sequences are defined and the fact that we are working with positive integers, it can be demonstrated through mathematical induction that the sequence x_n is decreasing .To demonstrate that, on the last row, at some point, there will only be values of zero, we should prove that the sequence cannot stagnate for positive values.

Let $a_{n+1} = x_n$. We will demonstrate that the maximum cannot stagnate for more than 4 steps for positive values.

If $a_{n+5} < a_{n+1}$, the conclusion is true.

If $a_{n+5} = a_{n+1}$, as $a_{n+5} = |a_{n+4} - b_{n+4}|$, with $0 \le a_{n+4} \le a_{n+1}$ and $0 \le b_{n+4} \le a_{n+1}$ (natural numbers), it necessarily ensues that $a_{n+4} = a_{n+1}$ and $b_{n+4} = 0$, or $a_{n+4} = 0$.

I. If $a_{n+4} = a_{n+1}$ and $b_{n+4} = 0$:

But how $b_{n+4} = |b_{n+3} - c_{n+3}|$ and $a_{n+4} = |a_{n+3} - b_{n+3}|$, it results that $b_{n+3} = c_{n+3}$ and $|a_{n+3} - b_{n+3}| = a_{n+1}$, and as $a_{n+3}, b_{n+3} \in \mathbb{N}, a_{n+3}, b_{n+3} \le a_{n+1}$, we can deduce two subcases:

1. $a_{n+3} = a_{n+1}, b_{n+3} = 0$ or
2. $a_{n+3} = 0, b_{n+3} = a_{n+1}$

We study each subcase:

1. $a_{n+3} = a_{n+1} \Rightarrow |a_{n+2} - b_{n+2}| = a_{n+1}$, and $b_{n+3} = 0 \Rightarrow b_{n+2} = c_{n+2}$ and it results $c_{n+3} = 0 \Rightarrow c_{n+2} = d_{n+2}$.
   $|a_{n+2} - b_{n+2}| = a_{n+1}$ ensues:
   i. $a_{n+2} = a_{n+1}$ and $b_{n+2} = c_{n+2} = d_{n+2} = 0$, which means that all numbers will become 0.
   ii. $a_{n+1} = b_{n+1}$ and $b_{n+2} = c_{n+2} = d_{n+2} = a_{n+1}$, which means that all numbers will become 0.

2. $a_{n+3} = 0 \Rightarrow a_{n+2} = d_{n+2}$ and $b_{n+3} = a_{n+1} \Rightarrow |b_{n+2} - c_{n+2}| = a_{n+1}$ and $c_{n+3} = a_{n+1} \Rightarrow |c_{n+2} - d_{n+2}| = a_{n+1}$.
   i. $b_{n+2} = a_{n+1}, c_{n+2} = 0$ and $d_{n+2} = a_{n+2} = a_{n+1}$, which leads to 0.
   ii. $c_{n+2} = a_{n+1}, b_{n+2} = 0$ and $c_{n+2} = a_{n+1}, d_{n+2} = 0$, which leads to 0.

If $b_{n+4} = a_{n+1}, a_{n+4} = 0$. Analogous.

2.What happens if on the first line there are not four numbers, but three, five , six or seven?

We will show you that it is possible that, for certain configurations of the first line, to not achieve the zero configuration at the end.
But ,as said before, what is going to happen if we choose to have a different amount of numbers on the first line?
We are going to validate that for some ways of choosing the configurations on the first line, we will enter in a cycle. So it's obvious that no metter what we won't achieve the zeroes.

If there are 3 numbers on a row:

0	1	1
1	0	1
1	1	0
0	1	1

- cycle of length 3 -

If there are 5 numbers on a row:

0	0	0	1	1
0	0	1	0	1
0	1	1	1	1
or				
1	0	0	0	0
1	0	0	0	1
or				
0	1	1	1	0
1	0	0	1	0
or				
0	1	1	0	1
1	0	1	1	1
or				
0	1	0	0	0
1	1	0	0	0
or				
0	0	1	1	1
0	1	0	0	1
or				
1	0	1	1	0
0	1	1	0	0
or				
1	0	0	1	1

1	0	1	0	0
or				
0	1	0	1	1
1	1	1	0	1
or				
0	0	0	1	0
or				
0	0	1	1	0
0	1	0	1	0
or				
1	0	1	0	1
1	1	1	1	0
0	0	0	1	1

- cycle of length14 -

If there are 6 numbers on a row:

0	0	0	1	0	1
0	0	1	1	1	1
0	1	0	0	0	1
1	1	0	0	1	1
0	1	0	1	0	0
1	1	1	1	0	0
0	0	0	0	0	1

- cycle of length 6 -

If there are 7 numbers on a row:

0	0	0	0	0	1	1
0	0	0	0	1	0	1
0	0	0	1	1	1	1
0	0	1	0	0	0	1
0	1	1	0	0	1	1
1	0	1	0	1	0	1
1	1	1	1	1	1	0
0	0	0	0	0	1	1

- cycle of length 7 -

If on the first line there are eight numbers, they will become 0 after a number of steps as well.

To prove this, we have written a C++ program that demonstrates that, no matter the order of the zeros and ones on the first line of the table, they will become zeros in the end. Using the idea from the first solution, we replace the numbers with their remainder after the division by 2. In maximum of 8 steps they will become zero, so the numbers will be divisible by 2. Continuing the reasoning, it results that the numbers will become 0 after a maximum of 8k steps.
The program generates a first the numbers between 0 and 255 and converts them to binary to obtain all the 8 digit numbers that represent the possible combinations of 0 and 1. Then, it implements the algorithm described in the rubric, until it reaches a null row or until it becomes a cycle. To be able to check at the end if all the combinations check the condition, we have taken a variable ‘ok’, which increases each time a combination checks the condition.
Result: The condition is true for all combinations.

3.What happens if on the first line there are four rational numbers?

The set of elements M = (a, b, c, d) and tM = (ta, tb, tc, td) have the same lifespan. If we start from four rational numbers, we multiply them by the lowest common multiple of denominators and we get a first row of four integers, which, after a certain number of steps will consist of just zeros.

Let us consider $X=b_1\times b_2\times b_3\times b_4$, $n\neq0$ and $b_1,b_2,b_3,b_4\neq0$.

$\frac{a_1}{b_1}$	$\frac{a_2}{b_2}$	$\frac{a_3}{b_3}$	$\frac{a_4}{b_4}$
d_1	d_2	d_3	d_4
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
x_1	x_2	x_3	x_4

We multiply the numbers from the first column and we get the following table:

$a_1\times b_2\times c_3\times d_4\in$N or Z
$a_2\times b_1\times b_3\times b_4\in$N or Z
$a_3\times b_1\times b_2\times b_4\in$N or Z
$a_4\times b_1\times b_2\times b_3\in$N or Z

$a_1\times b_2\times b_3\times b_4$	$a_2\times b_1\times b_3\times b_4$	$a_3\times b_1\times b_2\times b4$	$a_4\times b_1\times b_2\times b_3$
$n\times d_1$	$n\times d_2$	$n\times d_3$	$n\times d_4$
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
$n\times x_1$	$n\times x_2$	$n\times x_3$	$n\times x_4$

Knowing now that we start from 4 integers, following the remainder judgment of the division by 2, we will obtain four 0.

$\rightarrow n\times x_1 = n\times x_2 = n\times x_3 = n\times x_4 = 0$

$n\neq0$

$\rightarrow x_1= x_2= x_3= x_4 = 0 \rightarrow$ and in the first table , after a certain number of steps, we will obtain just 0.

4. Is our result true if we choose some random four real numbers for the first line?

The answer is ***NO***! We can choose combinations of irrational numbers so that we will enter in a cycle and consequently, it's never going to end.
Let us consider $t > 1$ and the numbers $M = \left(1, t, t^2, t^3\right)$ for the first line. By performing the operation, we obtain on the second line $\left(t-1, t\left(t-1\right), t^2\left(t-1\right), \left(t^2+t+1\right)\left(t-1\right)\right)$. The equation $t^3 = t^2 + t + 1$ has an irrational solution t_0, as our theoretical survey (cubic equation) says, and, for this value of t_0, it results that the second line is equal with $\left(t_0 - 1\right)M$. Inductively, the (n+1)th line of the table will be equal with $\left(t_0 - 1\right)^{n-1} M$,for any natural number *n*.

Line	**Numbers**				
1	**M**	**1**	t_0	${t_0}^2$	${t_0}^3$
2	$(t_0 - 1)$M	$t_0 - 1$	$t_0(t_0 - 1)$	${t_0}^2(t_0 - 1)$	$({t_0}^2 + t_0 + 1)(t_0 - 1)$
.	.	.	.	.	.
.	.	.	.	.	.
n+1	$(t_0 - 1)^{n-1}M$	$1(t_0 - 1)^{n-1}$	$t_0(t_0 - 1)^{n-1}$	${t_0}^2(t_0 - 1)^{n-1}$	${t_0}^3(t_0 - 1)^{n-1}$

The result is that we will not get to have a line *n* made up only of zeros.

Theoretical Survey

Cubic Equation

The discriminant of a cubic equation ax^3+bx^2+cx+d=0 is given by:
Δ_3=b^2c^2-4ac^3-4b^3d-27a^2d^2+18abcd.
If Δ_3<0, then the equation has one real root and two non-complex conjugate roots.
Regarding our equation, it becomes t^3-t^2-t-1=0 with $\Delta = -44$<0, the only real solution is:
$t_0 \approx 1.8393 > 1$.

Method of infinite descend

Let *k* be a positive integer. Suppose that whenever *P(m)* holds for some *m*>*k* thenthereexists a positive integer *j* such that *m*>*j*>*k* and *P(j)* holds. Then *P(n)* is false for all positive integers *n*.
Intuition: If there exists an *n* for which *P(n)* was true, one could construct an infinite sequence n>n_1>n_2>... all of which would be greater than *k* but this infinity is impossible.
This tehnique is known as the *Method of infinite descend.*

Monotony of sequences by induction

Leta_1=*a*, b_1=*b*, c_1=*c*, d_1=*d* and let a_n, b_n, c_n, d_n ∈ N, with $a_n = |a_{n-1} - b_{n-1}|$,
$b_n = |b_{n-1} - c_{n-1}|$, $c_n = |c_{n-1} - d_{n-1}|$, $d_n = |d_{n-1} - a_{n-1}|$.
Define x_n=*max{*a_n, b_n, c_n, d_n*}*. We apply the induction to prove that x_n is a descending sequence.

Annex 1

main.cpp - Code::Blocks 16.01

```cpp
#include <iostream>
#include <vector>

using namespace std;

vector <int> combinatii_8,copie_init;

int main()
{
    int ok=0;
    int baza2,f,cif,nr_cif,nr_0,primul,ci,nrpasi;
    for (int i=0;i<256;i++)
    {
        baza2=0;
        nr_0=0;
        nr_cif=0;
        f=1;
        ci=i;
        while (ci!=0)
        {
            cif=ci%2;
            baza2=baza2+cif*f;
            f=f*10;
            ci=ci/2;
            nr_cif++;
        }
        for (int j=1;j<=8-nr_cif;j++)
            combinatii_8.push_back(0);

        while (baza2!=0)
        {
            cif=baza2%10;
            combinatii_8.push_back(cif);
            baza2=baza2/10;
        }
        copie_init=combinatii_8;
        do
        {
            primul=combinatii_8[0];
            nr_0=0;
            for (int j=0;j<7;j++)
            {
                if (combinatii_8[j]>combinatii_8[j+1])
                    combinatii_8[j]=combinatii_8[j]-combinatii_8[j+1];
                else
                    combinatii_8[j]=combinatii_8[j+1]-combinatii_8[j];
            }
            if (combinatii_8[7]>primul)
                combinatii_8[7]=combinatii_8[7]-primul;
            else
                combinatii_8[7]=primul-combinatii_8[7];
            for (int j=0;j<8;j++)
                if (combinatii_8[j]==0)
                    nr_0++;
            nrpasi++;

        } while(nr_0<8 && combinatii_8!=copie_init);
        if (nr_0==8)
            ok++;
        else
            i=256;
    }
    if (ok==256)
        cout<<"Conditia este adevarata pentru toate combinatiile."<<endl;
    else
    {
        cout<<"Conditia nu este adevarata pentru toate combinatiile:"<<' ';
        for (int i=0;i<8;i++)
            cout<<combinatii_8[i]<<' ';

    }
    cout<<nrpasi;

    return 0;
}
```

D:\Cristina\codeblocks\problema cluj\main.cpp

Conclusions:

Our research topic was to prove that we can obtain a null row in these different conditions.
We solved the natural four numbers' case using remainders modulo 2 and we proved that after a certain number of steps, we will reach the zero row: 0, 0, 0, 0.
We also treated the cases of 3, 4, 5, 6, 7, 8, … natural numbers on the first row, obtaining different conclusions, afterwards we made a C++ program.
We extended the case of integers, followed by rational numbers.
Subsequently, we prove significant results for real numbers.
Despite the difficulties we had in solving the problem, it was a challenge for us and an opportunity to improve our knowledge, to discover new fields in mathematics, to make a research work and to write a Mathematical paper, to emphasize one with each other, to develop our teamworking skills, to commit deadlines.
We intend to go further in our research taking into consideration the complex numbers and different applications of the problem in the Cryptography.

References:

”Cubic Equation” - https://www.wolframalpha.com/examples/EquationSolving.html
“Method of infinite descend” - https://proofwiki.org/wiki/Method_of_Infinite_Descent