Possible Problems for MathEnJeans 2015

Difficulty levels are marked in the end of the problem formulation by [1]-[4] with [4] most difficult.

(1) Please tell how many ways you have to subtract two integers. Take an example 623-465. If you cannot come up with new ideas, try to solve the following problem. Your handball team will participate in a tournament abroad. So each player is recommended to make some money to cover partial costs for the tournament by selling flowers, sausages or socks. First, you have to buy the stuff. This costs 465kr each. Then you can sell them for 623kr. Now you want to figure out how much money you'll make. But there's one problem - you have to buy things BEFORE you can sell them. So you borrow 1000kr from Mom and Dad. From that 1000kr you spend 465kr to buy the materials. Could you find the relation to the subtraction 623-465? Experiment you results to any other two integers. Explain why it works mathematically. [1]

(2) Standing at the river's edge, the Emperor Yu-Huang watched the mighty Huang-He (Yellow River) rush before him. Evening's darkness was soon to come and this was good. The emperor had had a very difficult day.

Today, Emperor Yu had dealt with taxes, an underpaid army, and his angry wife, who said that she never saw enough of him. Looking out across the broad, dark back of Huang-He, Emperor Yu could feel his problems slip away. It was as if the rapid river were dragging the emperor's concerns along with it.

Emperor Yu enjoyed the river. He wished that he could visit it more often. Tonight, he was glad that he had walked to the river's edge alone. Huang-He (Yellow River) was something to visit by yourself. Looking out to the opposite side of the river, Emperor Yu slowly allowed his gaze to drop until he was looking at the river's edge right below his feet. It was at that moment that he saw the divine turtle.

Emperor Yu had seen the divine turtle before, but as a pattern in the stars, never this close. Every night, right before he went to bed, Emperor Yu would look out his bedroom window and see the turtle in the night sky. The emperor knew the Lo River story and believed that the turtle was a symbol of good luck. Just before he went to sleep the emperor would look at the turtle to ensure continued good luck. Now it was right before the emperor, swimming slowly at the river's edge.

Wanting to get a better look at the magic turtle, the emperor took one step closer. The turtle didn't notice the emperor and continued to move its legs slowly in the clear water. There was no mistaking the divine animal, for Emperor Yu had watched it with great care for many years. The emperor was familiar with the shape of the creature, but the detail of the shell that Emperor Yu now saw was new to him.

A turtle's hard back is half of the tough house that protects its body from enemies. The roof of this house looks like puzzle pieces glued together to form two circles around a rectangle. Emperor Yu looked long at these shapes on the turtle's back and noticed a pattern of dots etched on them.

Starting next to the turtle's right leg was a square formed by four linked dots. Traveling around the shell as the hands of a clock travel, the emperor came next to nine dots in a row. At the five o'clock position of a clock face there were two dots. At the bottom or six o'clock position was a row of seven linked dots. Next came a rectangle etched by six dots, and then a solitary dot at the nine o'clock spot. A long rectangle of eight dots followed, and at the top was a short line of three dots. In the center of all these dots was the intersection of two lines sharing five dots.

What did this all mean, the emperor wondered. Was the divine turtle giving a signal? Help the emperor solve the puzzle.

Discover the pattern. Find all possible solutions for these numbers. Construct any 3×3 magic squares. Discover more "magic" properties. Explore all properties. [2]

[The answer is the magic square: sum of rows, columns and the diagonals are 15 like this

4	9	2
3	5	7
8	1	6

Pattern can be recognized by symmetry, e.g. draw a line through the diagonal to find another additional properties: the difference between the center number 5 and the neighborhood numbers Then we can change columns or rows according to the symmetry.

In general for $n \geq 4$

n+3	n-4	n+1
n-2	n	n+2
n-1	n+4	n-3

An affine transformation carries a square with magic properties to another one with magic properties.

Or by raising 2 (or any other integer) to the power of each element, you'll get a new magic square.Why?]

(3) If you have read Dan Brown's novel "The lost symbol" you may recognize the following square played large roles. Please explain why it is so magic (and of course why it appeared in the novel).

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Assume now that you were to make a magic square with this year's date [2015], what size grid would you need to fill? [2]

[Some of the properties: The sum 34 can be found in the rows, columns, diagonals, each of the quadrants, the center four squares, and the corner squares(of the 4×4 as well as the four contained 3×3 grids). This sum can also be found in the four outer numbers clockwise from the corners (3+8+14+9) and likewise the four counter-clockwise, the two sets of four symmetrical numbers (2+8+9+15 and 3+5+12+14), the sum of the middle two entries of the two outer columns and rows (5+9+8+12 and 3+2+15+14), and in four kite or cross shaped quartets (3+5+11+15, 2+10+8+14, 3+9+7+15, and 2+6+12+14). The two numbers in the middle of the bottom row give the date of the engraving: 1514. The numbers 1 and 4 at either side of the date correspond to the letters 'A' and 'D' which are the initials of the artist. and raise the power...]

(4) Investigate the following numbers. Some questions for example, Can you continue the patterns? Are there any relations between them? Are there any numbers you know have these representations? Prove your statements.

$$3+\cfrac{1}{6+\cfrac{9}{6+\cfrac{25}{6+\cfrac{49}{6+\cfrac{81}{6+\cfrac{121}{6+\cdots}}}}}}$$

$$1-\cfrac{1}{3-\cfrac{6}{1-\cfrac{2}{3-\cfrac{20}{1-\cfrac{12}{3-\cfrac{42}{1-\cfrac{39}{3-\cdots}}}}}}}$$

$$1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cfrac{81}{2+\cfrac{121}{2+\cdots}}}}}}$$

$$1+\cfrac{1}{3+\cfrac{4}{5+\cfrac{9}{7+\cfrac{16}{9+\cfrac{25}{11+\cfrac{36}{13+\cdots}}}}}}$$

$$1+\cfrac{1}{3+\cfrac{16}{5+\cfrac{81}{7+\cfrac{256}{9+\cfrac{625}{11+\cfrac{1296}{13+\cdots}}}}}}$$

$$1+\cfrac{1}{1+\cfrac{2}{1+\cfrac{4}{1+\cfrac{6}{1+\cfrac{9}{1+\cfrac{12}{1+\cdots}}}}}}$$

and the beautiful series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...$$

[4]

(5) Soit ABC un triangle dont tous les angles sont aigus, avec $AB > AC$. Soit Γ son cercle circonscrit, H son orthocentre et F le pied de sa hauteur issue de A. On désigne par M le milieu du segment $[BC]$. Soit Q le point de Γ tel que $\widehat{HQA} = 90°$ et soit K le point de Γ tel que $\widehat{HKQ} = 90°$. On suppose que les points A, B, C, K et Q sont tour distincts et dans cet ordre sur Γ. Observez la relation entre les cercles circonscrits des triangles KQH et FKM. Prouvez votre observation. Combien de façons existe-t-il pour le démontrer. [4]

(6) Soit ABC un triangle de cercle circonscrit Ω, et O le centre de Ω. Un cercle Γ de centre A rencontre le segment $[BC]$ aux points D et E, de sorte que B, D, E et C sont distincts et dans cet ordre sur la droite (BC). On note F et G les points d'intersection de Γ et Ω, de sorte que A, F, B, C et G sont dans cet ordre sut Ω. Soit K le second point d'intersection du cercle circonscrit aux triantle CGE avec le segment $[CA]$. On suppose que les droites (FK) et (GL) ne sont pas confondues et qu'elles se rencontrent au point X. [3]

(7) Soit $\mathbb{R}$ l'ensemble des nombres réels. Détermininer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ qui vérifient l'equation

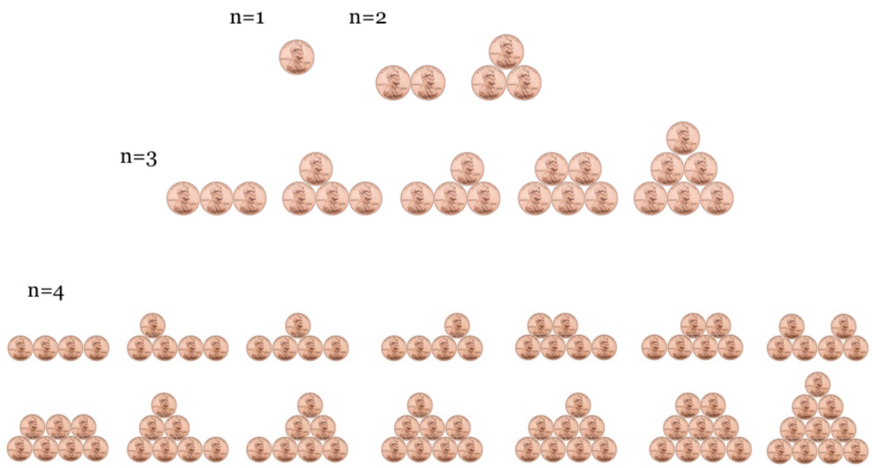
$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

pours tours réels x et y. [4]

(8) Could you count?

Consider the following problems

(a) Stacking Coins How many ways do you have to stack coins on a bottom row that consists of n consecutive coins in a plane, such that no coins are allowed to be put on the two sides of the bottom coins and every additional coin must be above two other coins?



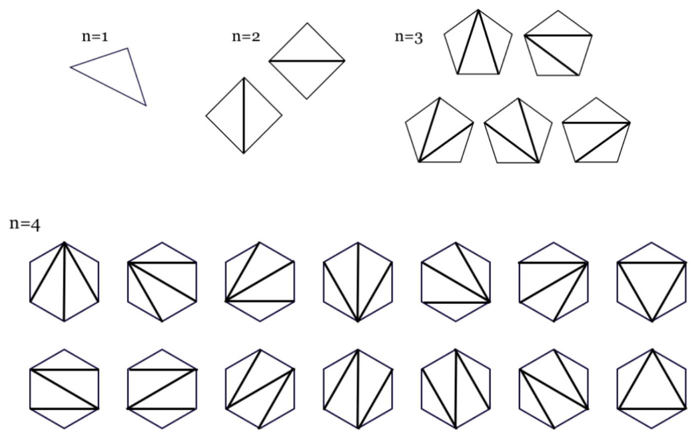
(b) Balanced Parentheses How many ways do you find to group a string of n pairs of parentheses, such that each open parenthesis has a matching closed parenthesis? For example, (( )( )) is valid, but)( ))((and ( ))( )(are not.

$n = 0$	Do Nothing.	1way
$n = 1$	( )	1way
$n = 2$	(( )), ( )( )	2 ways
$n = 3$	( )( )( ), ( ) (( )), (( )) ( ), (( )( )), ((( )))	5 ways

(c) Mountain ranges How many ways can you form mountain ranges on a line with n upstrokes and n downstrokes, such that each upstroke has a matching downstroke and the path does not go below the starting point?

n=0	Do Nothing.	1 way
n=1	/\	1 way
n=2	/\ /\/\, / \	2 ways
n=3	/\ /\ /\ /\/\ / \ /\/\/\, /\/ \, / \/\, / \, / \	5 ways

(d) Polygon triangulation Count how many ways to cut an $n + 2$-sided convex polygon in a plane into triangles by connecting vertices with straight, non-intersecting lines. This is the application in which Euler was interested. A convex polygon satisfies the following two properties: i) each interior angle is less than or equal to 180 degrees, and ii) each line segment connecting two of the vertices must remain inside the boundary of the polygon. As the term suggests, the vertices of a convex polygon point ?outward? from the center of the polygon.



Now, what have you discovered by counting the above numbers? Can you say what next numbers are?

Let's consider the Pascal's triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

Can you find the numbers you have found in this triangle? Does this help you to find all numbers you need in the above problems?

[3-4]

[Answer:

$$\binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n} = C_n$$

Proposed by Yishao Zhou// September 16, 2015