Are all infinities the same?

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1. PRESENTATION OF THE RESEARCH TOPIC

The concept of infinity has long captivated the minds of mathematicians, leading to profound insights and paradoxes. As we embark on this exploration, we will venture beyond the familiar realms of finite numbers and immerse ourselves in the abstract landscape where infinite sets reside. Our focus will be on understanding the diverse sizes of infinities and the methods used to compare them.

2. BRIEF PRESENTATION OF THE CONJECTURES AND RESULTS OBTAINED

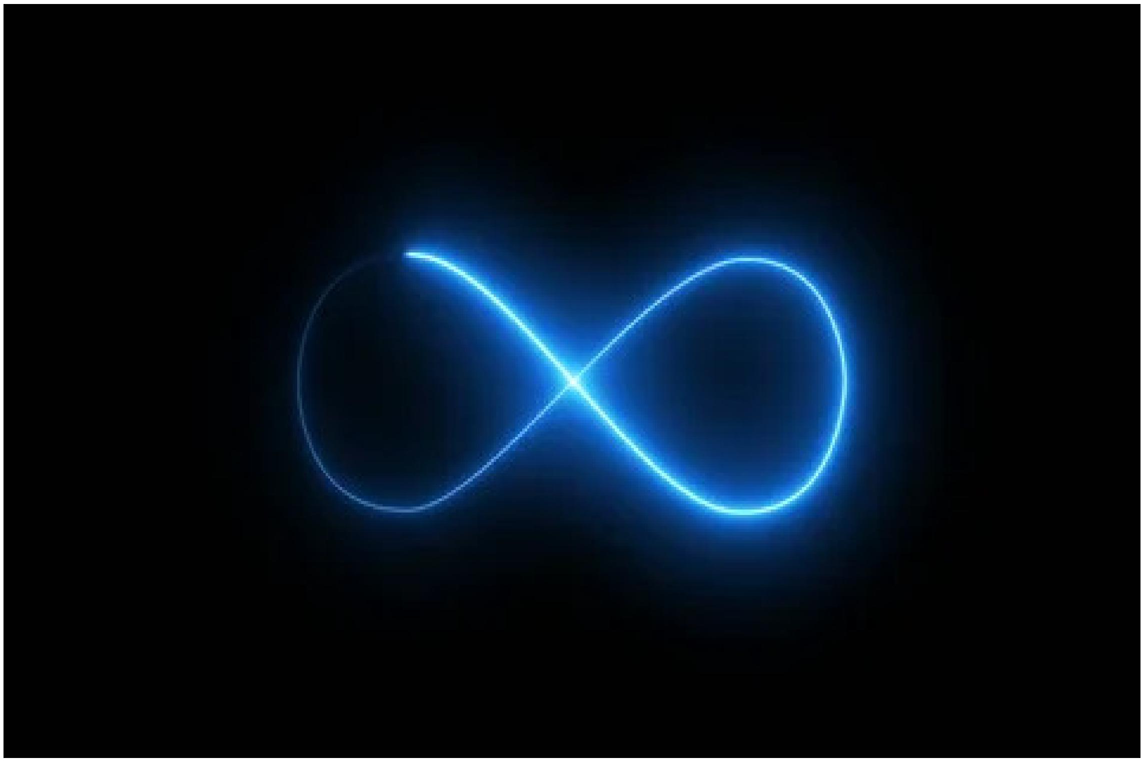
Infinity is a really important and interesting concept in mathematics. Even though it is an abstract concept, we have to think about it right from the time we start learning about natural numbers because there are just endless natural numbers. After that, we can also think about there being endless integers or even endless prime numbers. However, even if infinity might sound like the biggest way to describe something, are all infinities as big as each other? Can one infinite set be as big as another infinite set even if one includes strictly the other? Is it possible to find different types of infinities? To address these curiosities and understand this concept better, we will try to answer the following questions:

1. Is the set of prime numbers infinite?
2. Do the set of natural numbers have the same cardinality as the set of prime numbers?
3. Can we compare the cardinality of $\mathbb{N}$ with the cardinality of $\mathbb{Z}$?
4. Can we compare the cardinality of $\mathbb{N}$ and the cardinality of $\mathbb{Q}$?
5. Are there sets with a higher cardinality than $\mathbb{Q}$?

6. Does $[0,1]$ have the same cardinality as any other closed interval $[a,b]$?

7. Is it also true that the interval $(0,1)$ has the same cardinality as any other open interval (a,b)?

8. Does $[0,1]$ have the same cardinality as $\mathbb{R}$?



3. SOLUTION

Definition: A set A is called countable if there is an injection $f: A \to \mathbb{N}$.

Remark 1: Any finite set, A, is countable because one can consider $A = \{a_1, a_2, ..., a_{|A|}\}$ and let $f: A \to \mathbb{N},\ f(a_i) = i,\ \forall i \in \{1, 2, ..., |A|\}$ an injection from A to $\mathbb{N}$.

Remark 2: In literature a set of this type is called countable because one can make a list out of its elements (i.e. assign a numbered position for every element in the set).

Observation: We will consider the following theorems to be true:

Theorem 1: If there is a bijective function between a set A and set B, then the 2 sets must have the same cardinality (i.e. $|A| = |B|$).

Theorem 2: If there is an injective function between A and B and also an injective function between B and A, then A and B must have the same cardinality.

Theorem 3: If there is no surjection between A and B, this means that $|A|<|B|$.

Theorem 4: If $|A|\leq|B|$ and $|B|\leq|A|$, then $|A|=|B|$.

Remark 3: There is an important intuition of **Theorem 1.** Since a bijection assigns every element from a set to a unique distinct element from another set and covers both sets completely, it means that the 2 sets have the exact same cardinality.

Remark 4: There is an intuitive relationship between **Theorem 2** and **Theorem 4**. Creating an injection from A to B proves intuitively that $|A|\leq|B|$ because we can assign any element in A to a distinct element from B.

Remark 5: There is an intuitive explanation of **Theorem 3**. If there is no surjection between A and B, it means that one cannot cover the entire set B with the elements of A, so $|A|<|B|$.

QUESTION 1: Is the set of prime numbers infinite?

Our intuition was that the set of prime numbers is infinite. To show this, we supposed, for the sake of contradiction, that there are only finitely many prime numbers, which can be listed as: $\{p_1, p_2, p_3, ..., p_n\}$.

Now, let's construct a new number, which is one more than the product of all these primes: $N = p_1 \cdot p_2 \cdot p_3 \cdot ... \cdot p_n + 1$.

N is clearly greater than 1 because it is one more than the product of prime numbers. So, N must be either composite or prime.

We will show that N cannot be composite:

If N is composite, it should have at least on prime divisor strictly smaller than itself. However, none of the primes $p_1, p_2, p_3, ..., p_n$ can be a divisor of N, because when N is divided by any of them, there is a remainder of 1. Therefore, N must have a prime factor that is not in our original list. In consequence, we reach a contradiction which means that our assumption that the set of prime number is finite is false. Therefore, there are indeed an infinite number of prime numbers.

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QUESTION 2: Do the set of natural numbers have the same cardinality as the set of prime numbers?

We will denote by P the set of prime numbers which we already now is infinite. $P=\{p_1, p_2, p_3, ..., p_n, ...\}$, where $p_1=2<p_2=3<p_3=5...$.

We will consider the function $f:\mathbb{N}\to P$ where $f(n)=p_{n+1}$.

We will prove that f is bijective in 2 steps:

I. Firstly, we prove that f **is injective**.
Let us consider two natural numbers a and b such that $f(a)=f(b)$.
$f(a)=f(b)$ means that $p_{a+1}=p_{b+1}$. Since we considered the prime numbers to be arranged in strictly increasing order it means that $a=b$.

II. Secondly, we must prove that f is **surjective**.
To prove this, we must show that for any prime number $p\in P$ there is at least one natural number x such that $f(x)=p$. Knowing that p is prime, there exists an $n\in\mathbb{N}$ such that p is the $n+1$-th prime number in the list of prime numbers arranged in increasing order. So, there is $x=n$ such that: $f(x)=p$. This means that the function is surjective.

In conclusion, because the function is both injective and surjective, $f(n)=p_{n+1}$ is a bijection between $\mathbb{N}$ and P. So, we can conclude that $|\mathbb{N}|=|P|$.

QUESTION 3: Can we compare the cardinality of $\mathbb{N}$ with the cardinality of $\mathbb{Z}$?

We will try to find a bijective function between $\mathbb{N}$ and $\mathbb{Z}$.
Let's consider function $f:\mathbb{N}\to\mathbb{Z}$ for which we have:

1. $f(x)=-\frac{x}{2}$ for any even x.
2. $f(x)=\frac{x+1}{2}$ for any odd x.

We want to prove that function f is bijective. We will prove this in 2 steps, by proving that f is injective and surjective.

Function $f:\mathbb{N}\to\mathbb{Z}$ is injective if $f(x_1)=f(x_2)\Rightarrow x_1=x_2$ for any $x_1, x_2\in\mathbb{N}$

First of all, one can observe that f is strictly positive for odd numbers and negative for even numbers. So, we can have $f(x) = f(y)$ if only $(x-y)\vdots 2$.

If x is even: $f(x_1) = f(x_2) \Leftrightarrow -\frac{x_1}{2} = -\frac{x_2}{2} \Leftrightarrow x_1 = x_2$.

If x is odd: $f(x_1) = f(x_2) \Leftrightarrow \frac{x_1+1}{2} = \frac{x_2+1}{2} \Leftrightarrow x_1 = x_2$.

So, f is injective.

Now we move to surjectivity.

Function $f:\mathbb{N} \to \mathbb{Z}$ is surjective if, for any $y \in \mathbb{Z}$, there is an $x \in \mathbb{N}$ such that $y = f(x)$.

Any $y > 0$ can be expressed as $\frac{x+1}{2}$ by taking $x = 2y-1$.

Any $y \le 0$ can be expressed as $-\frac{x}{2}$ by taking $x = -2y$.

This shows that f is surjective.

Function f is both injective and surjective, **so f is bijective**. In conclusion, natural numbers and integers have the same cardinality.

So, from now on, we can use the cardinality of $\mathbb{N}$ and $\mathbb{Z}$ interchangeable.

Remark: The cardinality of any infinite countable set is denoted by $\aleph_0$.

QUESTION 4: Can we compare the cardinality of $\mathbb{N}$ with the cardinality of $\mathbb{Q}$?

We will try to find one injective function from $\mathbb{N}$ to $\mathbb{Q}$, and one injective function from $\mathbb{Q}$ to $\mathbb{N}$.

The function from $\mathbb{N}$ to $\mathbb{Q}$ is natural: $g:\mathbb{N} \to \mathbb{Q}$, where $g(x)=x$. This function is injective and well defined. We will mark this finding as **(1)**.

Now we will focus on the other injection. Since every rational number r can be written as $\frac{a}{b}$, where $(a,b)=1$ and $b>0$. Let $f:\mathbb{Q} \to \mathbb{N}$ defined as:

1) $f(r) = 2^a \cdot 3^b$ for $r = \frac{a}{b}$, if $a > 0$

2) $f(r) = 5^{-a} \cdot 3^b$ for $r = \frac{a}{b}$, if $a \le 0$

One can show that that $\operatorname{Im} f \subset \mathbb{N}$, because $2^a \in \mathbb{N}$ when $a > 0$, $5^{-a} \in \mathbb{N}$ when $a \le 0$, and $3^b \in \mathbb{N}$ when $b > 0$. So, f is well defined. Now, we will prove that f in is injective. Firstly, if $f(x) = f(y)$, x, y must have the same sign

So, there are two cases given their sign.

If $f(x)=2^a\cdot 3^b$, $f(y)=2^c\cdot 3^d$ and $f(x)=f(y)$ it will result that the powers of 2 will be equal (i.e. $a=c$) and the powers of 3 will be also equal (i.e. $b=d$). In a similar way, if we consider that $f(x)=5^{-a}\cdot 3^b$, $f(y)=5^{-c}\cdot 3^d$ and $f(\mathrm{x})=f(y)$ it will result that the powers of 5 will be equal (i.e. $a=c$) and the powers of 3 will be also equal (i.e. $b=d$). That shows the function f is injective because $f(x)=f(y)\Rightarrow x=y$.

Therefore, the function f **is well defined and injective**. We will mark this finding as **(2)**.

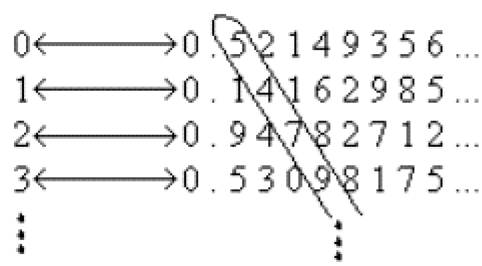
Combining **(1)** and **(2)**, with the help of **Theorem 2**, we will obtain the following important result: $|\mathbb{N}|=|\mathbb{Q}|$

QUESTION 5: Are there sets with a higher cardinality than $\mathbb{Q}$?

We will try to prove that the cardinality of $[0,1]$ is strictly higher than the cardinality of $\mathbb{Q}$.

To this end, we will prove that there is no surjective function between natural numbers and the interval $[0,1]$. This will show that the cardinality of $[0,1]$ is higher than the cardinality of $\mathbb{Q}$ because we already showed that $|\mathbb{N}|=|\mathbb{Q}|$.

Firstly, we will suppose that such a bijection exists. Then, using Cantor's Diagonalization, we will obtain a list as follows:



The number circled in the diagonal is some real number r, since it is an infinite decimal expansion. Now consider the real number s obtained by modifying every digit of r, say by replacing each digit d with $(d+5)\bmod 10$. We claim that s does not occur in our infinite list of real numbers. Suppose by sake of contradiction that it did, and that it was the n-th number in the list. Then r and s differ in the n-th digit. The n-th digit of s is the n-th digit of $(\mathrm{r}+5)\bmod 10$. So, we have a real number s that is not in the image of our supposed surjective function. This **contradicts the assertion** that f is a surjection. Thus, there is no surjection between the natural numbers and the interval $[0,1]$. So, the cardinality of $[0,1]$ **is strictly higher** than the cardinality of $\mathbb{Q}$.

Observation: The main ideas of this solution were taken from https://inst.eecs.berkeley.edu/~cs70/sp07/lec/lecture27.pdf.

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QUESTION 6: Does $[0,1]$ have the same cardinality as any other closed interval $[a,b]$?

We want to establish a bijection between the intervals $[a,b]$ and $[0,1]$ in order to prove that they have the same cardinality.

We will take the function $f:[a,b]\to[0,1]$ with $f(x)=\frac{x-a}{b-a}$. We can observe that f is well defined because $\operatorname{Im} f\subset[0,1]$.

Firstly, we will prove that the function f is injective.

We can assume that $f(x_1)=f(x_2)$ for two values $x_1,x_2\in[a,b]$. One can observe that:

$f(x_1)=f(x_2)\Leftrightarrow\frac{x_1-a}{b-a}=\frac{x_2-a}{b-a}\Leftrightarrow x_1-a=x_2-a\Leftrightarrow x_1=x_2$. So, f is **injective**. Now, we need to prove that f is surjective. For that, we have to prove that $\forall y\in[0,1]$ there is a value $x\in[a,b]$ such that $f(x)=y$. Since $f(x)=\frac{(x-a)}{(b-a)}=y\Rightarrow x-a=y\cdot(b-a)$.

In consequence, $\forall\ y\in[0,1],\ \exists\ x=y(b-a)+a\in[a,b]$ s.t. $f(x)=y$. This means that f is **surjective**.

If f is injective and surjective $\Rightarrow f$ is **bijective.**

In consequence, $[0,1]$ has the same cardinality as any other closed interval $[a,b]$.

QUESTION 7: Is it also true that $(0,1)$ has the same cardinality as any other open interval (a,b)?

Using the function, we already built at the previous point $f:[a,b]\to[0,1]$, with $f(x)=\frac{x-a}{b-a}$, we will prove that the there is a bijective function $F:(a,b)\to(0,1)$.

Knowing that $f(a)=0$ and $f(b)=1$ we will consider F to be the restriction of f on (a,b). It is clear that F will be a bijection between (a,b) and $(0,1)$.

Remark: Since $\left[\frac{a+(b-a)}{4},\ \frac{b-(b-a)}{4}\right]\subset(a,b)\subset[a-1,b+1]$ **and all closed intervals have the same cardinality, we can conclude that open intervals and closed intervals have the same cardinality due to Theorem 4.**

QUESTION 8: Does $[0,1]$ has the same cardinality with $\mathbb{R}$?

We are interested in this because we showed that $[0,1]$ has a higher cardinality than $\mathbb{Q}$.

We want to show that $[0,1]$ has the same cardinality as $\mathbb{R}$. In order to do this we will find an injection $f:[0,1]\to\mathbb{R}$, and an injection $g:\mathbb{R}\to[0,1]$.

f can be chosen as $f(x)=x$. This function is **well defined and injective**.

For $g:\mathbb{R}\to[0,1]$, we propose to take $g(x)=\frac{1}{1+2^x}$. Now we have to prove 2 things: that g is well defined (i.e. $\operatorname{Img}\subset[0,1]$) and that g is injective.

1. $\operatorname{Img}\subset[0,1]$

   It is clear that $g(x)>0, \forall\, x\in\mathbb{R}$. Also, since $2^x>0 \Rightarrow g(x)<\frac{1}{1}=1,\ \forall\, x\in\mathbb{R}$.

   So, $0<g(x)<1,\ \forall\ \mathrm{x}\in\mathbb{R}$.

2. g is injective

   Since $g(x)=g(y)\Leftrightarrow 1+2^x=1+2^y \Leftrightarrow x=y,$ we can conclude that g is injective.

   In conclusion, g is **well defined and injective**.

Having found an injection from $(0,1)$ to $\mathbb{R}$ and from $\mathbb{R}$ to $(0,1)$ we can conclude that $\mathbb{R}$ and $(0,1)$ have the same cardinality due to **Theorem 2**.

4. CONCLUSION

In order to solve this problem, we have used injective, surjective and bijective functions to compare the cardinality of different sets. The central theorems that we used are enumerated in the beginning of our solution.

In the first part we have obtained a set of results proving that there is an infinite number of prime numbers and that the set of prime numbers has the same cardinality as the set of natural numbers, integers, and rational numbers. In the second part, we managed to prove that there are infinities that have a higher cardinality than the set of rational numbers. We found that the cardinality of any closed or open interval is bigger than $\aleph_0$ and is equal to the cardinality of $\mathbb{R}$.