Shortest cobweb length

Year 2022 – 2023

Matei Marcel-Rareș (11th grade), Moșneagu Magda (9th grade),
Săveanu Andreea-Florina (9th grade), Sârcu Eliza-Maria (11th grade)

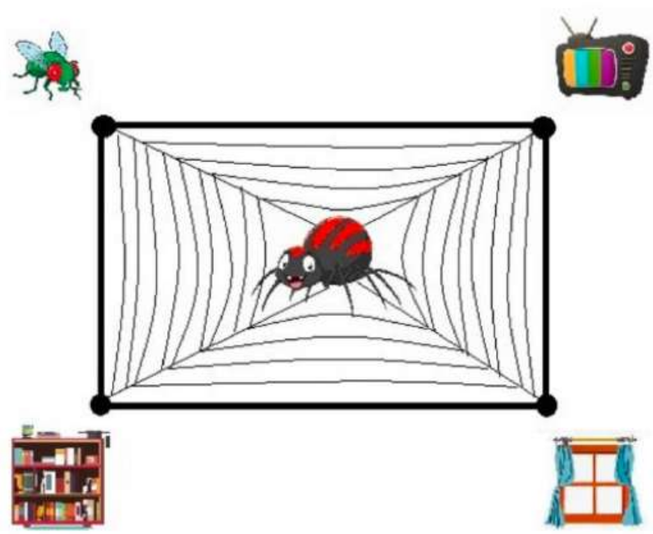
School: Colegiul National "Costache Negruzzi", Iași, Romania
Teacher: Ph.D. Ioana Cătălina Anton
Researcher: Ph.D. Iulian Stoleriu, University "Alexandru Ioan Cuza" of Iași, Romania

Abstract:

Our research deals with finding the minimum length of a cobweb that Spider Webster should build in order to connect multiple points of interest.

The problem

Spider Webster is fed up of investing resources and energy in building new cobwebs every day, and wants to be more pragmatic when building its cobweb. Webster wants to build a cobweb of minimum total length, which connects the four points of interest to it. These four points are the vertices of a rectangle. Because its Mathematics it is so dusty, it expects some help from you.



You can also investigate the case when the four points are the vertices of a parallelogram.
What if Webster has now five points of interest, which are the vertices of a regular pyramid. Can you help it build the cobweb of shortest length between these points?

Solution of the problem

We shall prove that, in the case of a rectangular shape, the minimum cobweb length is obtained when the spider constructs two specific nodes inside the rectangle. Firstly, we shall find the position of these nodes inside the rectangle, and then calculate the shortest length of the cobweb.

In order to find the minimum cobweb length for a rectangle, we shall split the rectangular shape into triangles, and then look for the minimum length for some of them.
Therefore, we also need to consider the problem of finding the shortest cobweb length that Webster can build inside a triangular shape.

The case of a triangular shape

Let ABC be a scalene triangle, with angles less than 120°. We aim to find the point P in the plane of ΔABC such that the sum $AP + BP + CP$ is minimum. In literature, the point P is known as ***the Fermat–Torricelli point***, or ***the Fermat point***.

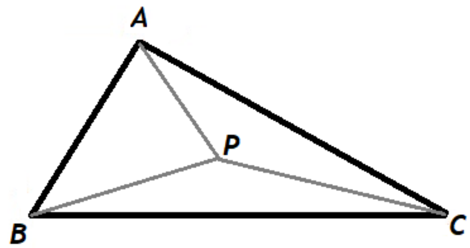


***Figure 1**. A scalene triangle ΔABC.* (1)

We begin the construction as follows. As drawn in Figure 2, we construct the equilateral triangles $A'BC$, AB'C and ABC' outside the sides of the given triangle ABC. Then, we draw the segments from A' to A, B' to B, and C' to C. Our first claim is:

Proposition 1. The segments AA', BB', CC' have all the same length, and any two of these segments meet at an angle of $120°$.

Proof. Let P be the intersection point of segments BB' and CC'. We firstly prove that BB' and CC' have the same length and the angle $B'PC'$ has $120°$.

We can easily see that the triangle $C'AC$ is obtained from the triangle BAB' by a rotation of 60° about the fixed point A. This means that the triangles $C'AC$ and BAB' are congruent and the angle $B'PC$ has 60°. Consequently, the angle $B'PC'$ has $120°$.
Similarly for AA' and BB', and for AA' and CC'. Thus, $AA' = BB' = CC'$ and any two of these segments meet at an angle of $120°$. ■

Note that, at this point, we do not know whether all three segments have the same intersection point.

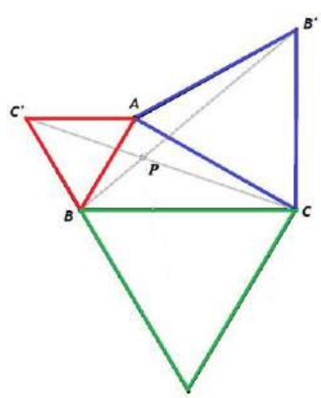


***Figure 2.** The construction of the Fermat point of ΔABC.*

Proposition 2. The circumcircles of the equilateral triangles $\Delta ABC'$, $\Delta AB'C$ and $\Delta A'BC$ intersect at point P.

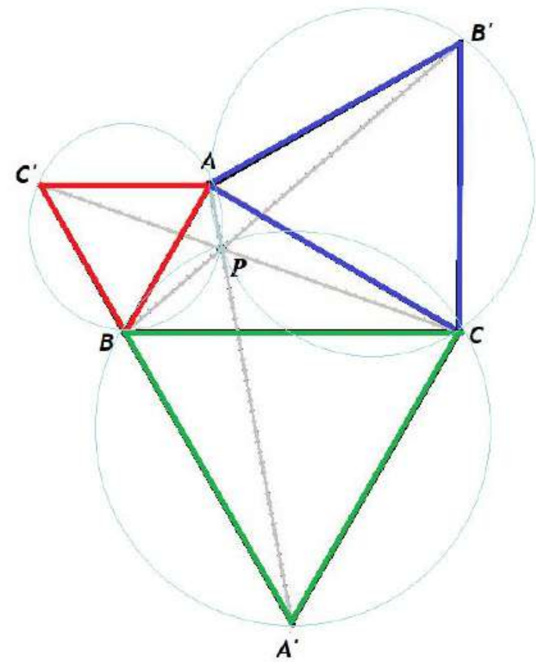


Figure 3. *The intersection of the circumcircles is the Fermat point of* ΔABC.

Proof. From the fact that $\widehat{BAC'} = \widehat{BPC'} = 60°$, we deduce that the quadrilateral $APBC'$ is cyclic, and thus the point P lies on the circumcircle of $\Delta ABC'$. Similarly, as $\widehat{B'AC} = \widehat{B'PC} = 60°$, we deduce that the quadrilateral $AB'CP$ is cyclic, and thus the point P lies also on the circumcircle of $\Delta AB'C$. For the moment, we know that P is one of the intersection points between the circumcircles of $\Delta ABC'$ and $\Delta AB'C$. However, as $\widehat{BPC} = 120°$, and $\widehat{BA'C} = 60°$,we get that these are supplementary angles in the quadrilateral $A'BPC$, thus this quadrilateral is cyclic. In conclusion, the point P lies also on the circumcircle of $\Delta A'BC$. Therefore, the point P is the intersection of the circumcircles to the equilateral triangles $\Delta ABC'$, $\Delta AB'C$ and $\Delta A'BC$ (see Figure 3). ■

Theorem 1. The point of intersection of the segments AA', BB' and CC' is the Fermat point of ΔABC, which means that $AP + BP + CP$ is minimum.

Proof. We know that the point P is the intersection point of segments BB' and CC', and we have proved that P is also the intersection of the circumcircles to the equilateral triangles $\Delta ABC'$, $\Delta AB'C$ and $\Delta A'BC$. We shall prove that P lies also on AA'.
Let us say that Q is the intersection of AA' and BB'. Then, using the previous two propositions, we could prove that Q is the concurrence point of the three circumcircles to the equilateral triangles $\Delta ABC'$, $\Delta AB'C$ and $\Delta A'BC$. But this means $P = Q$, and thus P lies on AA' as well. (2) ■

Remark. If the triangle ABC has an obtuse angle equal to or larger than 120°, then the Fermat point lies at the vertex with the angle of 120° degrees or larger.

The case of a rectangular shape

From the beginning we were thinking of a diagonal traversal of the 4 points of interest (as in Figure 4), because, as Pythagoras said, "the shortest path is the straight line that holds them together".

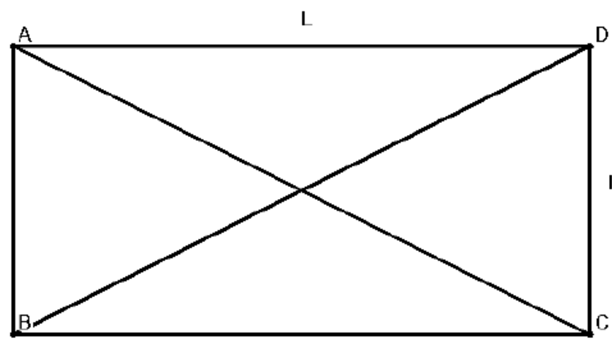


Figure 4. *First attempt to find a solution*

However, we didn't think that the problem was so simple, so we tried several options and noticed that there are shorter ways to join all four points, and the most advantageous one is shown in Figure 5.

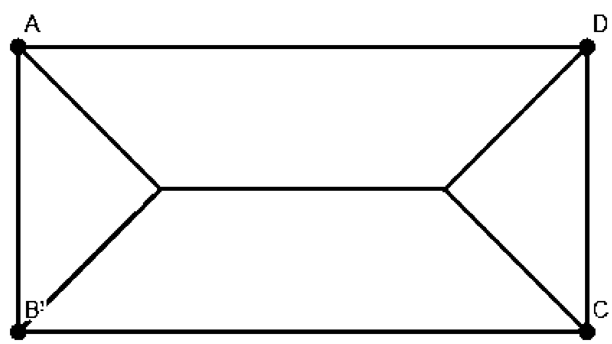


Figure 5. *Second attempt to find a solution*

We are now ready to proceed in finding the shortest path length between the four vertices of a rectangle.

Proposition 3. If Webster decides to have only one node (point of intersection) inside the rectangle, then the minimum cobweb length is the sum of diagonals.

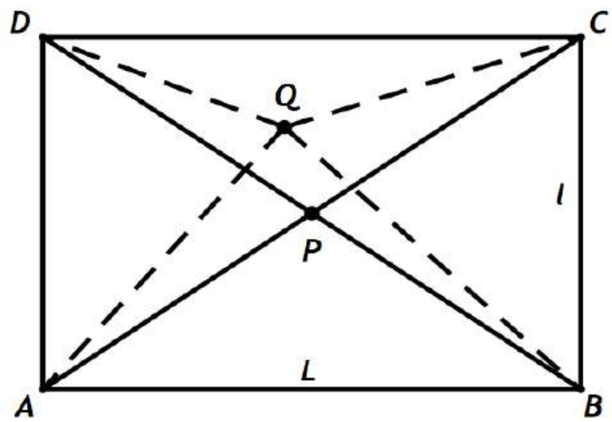


Figure 6. *Only one point inside the rectangle*

Proof. Let P the intersection of diagonals of the rectangle, as drawn in Figure 6. If we suppose that the node inside the rectangle is different from P, say Q, then, by the triangle inequality, we have:

$$AQ + CQ > AC$$

and

$$BQ + DQ > BD.$$

By adding these two inequalities, we get that

$$AQ + BQ + CQ + DQ > AC + BD.$$

Thus, the sum of diagonals minimizes the sum of distances from an interior point of the rectangle to its vertices. ■

We suppose now that there are more than one node inside the rectangle (see Figure 7).

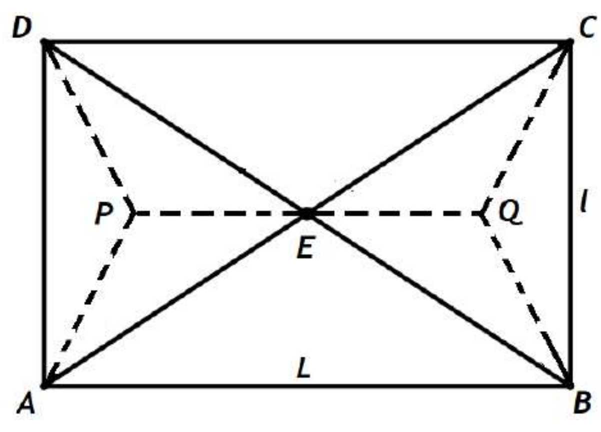


Figure 7. *Two nodes inside the rectangle*

Proposition 4. If there are two nodes inside the rectangular room, then the minimum cobweb length that Webster can construct passing through these two points will be less than the sum of diagonals.

Proof. Without losing the generality, assume that $\widehat{AED} < 90°$. Otherwise, we consider the angle $\widehat{AEB}$. Since both angles $\widehat{AED}$ and $\widehat{BEC}$ are smaller than 120°, one can always find the Fermat points P and Q for the triangles AED and BEC. This means that

$$AP + DP + EP < AE + DE$$

and

$$BQ + CQ + EQ < BE + CE.$$

By adding these two inequalities, we get that

$$AP + DP + BQ + CQ + PQ < AE + BE + CE + DE = AC + BD.$$

Note that, for the time being, the points P, E and Q are not necessarily collinear (3). However, if we suppose that they are not collinear, then, in the triangle PEQ, we shall write the triangle inequality:

$$PQ < PE + QE,$$

and the cobweb length will be even larger. Therefore, the points P, E and Q must be collinear, in order to guarantee the minimum length of the cobweb.
Thus, the minimum cobweb that Webster can construct passing through these two points will be even less than the sum of diagonals. ■

Proposition 5. If there are $n \geq 2$ nodes inside the rectangular room, then the minimum cobweb length that Webster can construct passing through these$n$points will be minimized for $n = 2$.

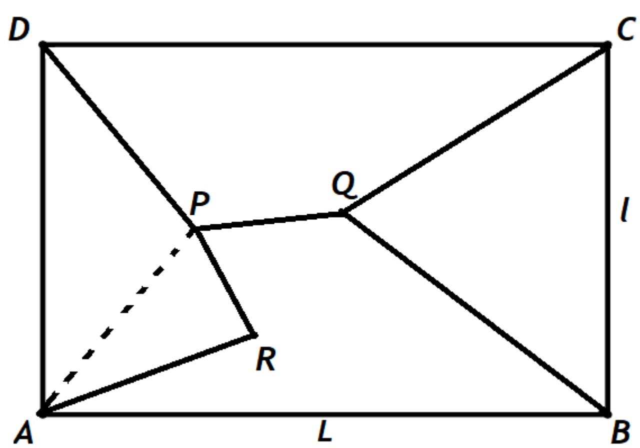


Figure 8. *Three nodes inside the rectangle*

Proof. Let us suppose that P, Q and R are three inside nodes, as in Figure 8. We can easily see that, in order to minimize the total cobweb length, we can disregard the point R and consider a path that goes only through the nodes P and Q (the dotted line). Indeed, by the triangle inequality in $\Delta A'BC$, we have that $AP < AR + PR$.
Similar arguments will hold if we have more that three nodes inside the rectangular, by ignoring the extra point. ■

Theorem 2. In any given rectangle $ABCD$, the total cobweb length is minimized when the two points P and Q are located in such a way that both are on segment which connects the two mid-points of the shorter sides of the rectangular (see Figure 9).

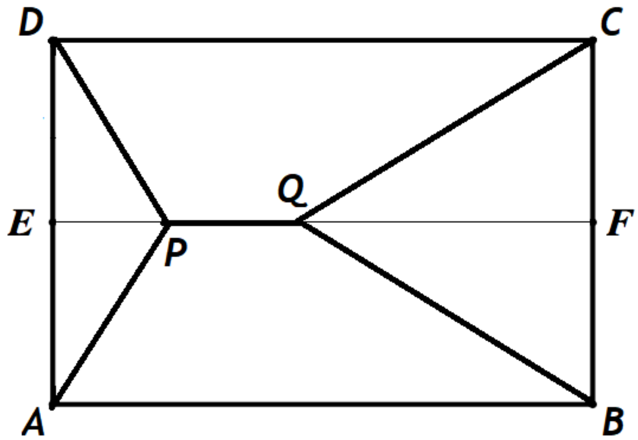


Figure 9. *Optimal positions of P and Q inside the rectangle*

Proof. Suppose that $AB > AD$. Denote by E and F the two mid-points of the shorter sides of the rectangular.
For the moment, let us suppose that the node Q is fixed inside the rectangle, which means that the sum $BQ + CQ$ is also fixed. If the point P is variable inside the rectangle, such that the sum $AP + DP$ remains fixed (4), then the minimum cobweb length is obtained when the length of PQ is minimum. We must search for the position of node P that minimizes PQ, with Q fixed and $AP + DP = constant$ (see Figure 10).

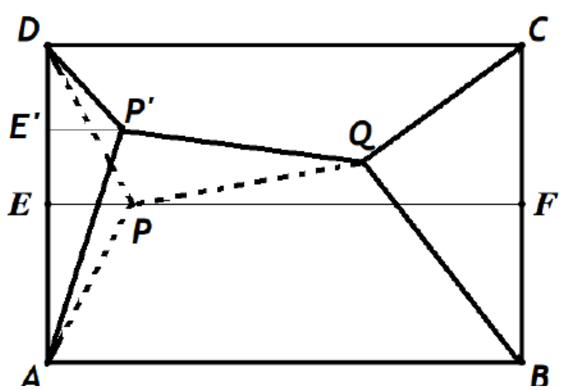


Figure 10. *Optimal positions of P inside the rectangle for fixed Q*

If we want to minimize PQ, then the distance from P to the side AD must be maximum (5).
We claim that the optimal position of point P (such that $AP + DP = constant$ and $d(P, AD) = maximum$) is obtained when the triangle APD is isosceles. Indeed, if the perimeter of a triangle is given (here, $AP + PD + AD$) and AD is fixed, then the triangle that has the maximum area (and, thus, the maximum height from P) is the isosceles triangle.
Thus, the point P must be on the segment EF.

If we argue similarly for Q (when P is fixed and $BQ + CQ \ = constant$), then we find that Q must also be on the segment EF.
In conclusion, both interior points P and Q must lie on the segment EF (as in Figure 9). ■

Our aim now is to find the exact positions of P and Q on the segment EF.
Regardless the positions of these two points on EF, we can always find a point G on the segment PQ, such that at least one of the triangles APD and BGC has all angles less than 120°. If, for example, triangle BGC has this property, then consider Q' the Fermat point for this triangle (see Figure 11). This means that $BQ' + CQ' + GQ' = minimum$.

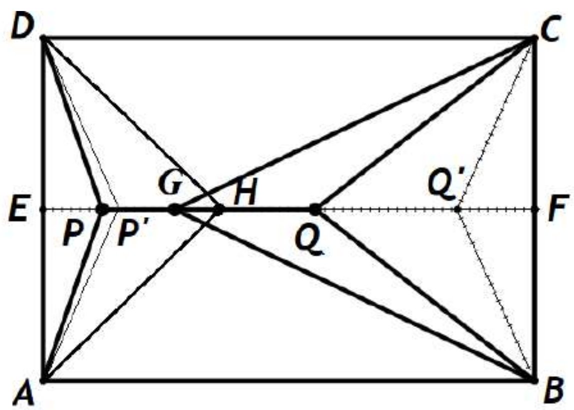


Figure 11. *Optimal position of point Q*

As the triangle BCG is isosceles, the point Q' lies on its altitude, thus $Q' \in EF$.
Similarly, for the new middle segment PQ', one can find a point H on it, such that at least one of the triangles AHD and BHC has all angles less than 120°. For this triangle, we can construct the Fermat point P'. As a consequence, we can assert that:

Proposition 5. The optimal positions of P and Q are the constructed Fermat points P' and Q' in the previous paragraph.

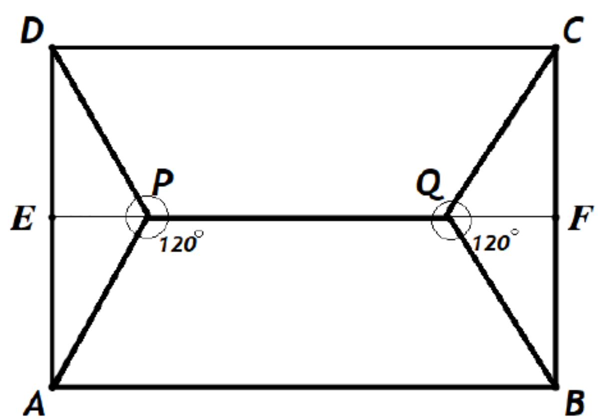


Figure 12. *The optimal positions of P and Q*

Theorem 3. The positions on EF of the points P and Q, that guarantee the minimum cobweb length, are such that $\widehat{APD} = \widehat{BQC} = 120°$ (see Figure 12).

Proof. (6)

First Method:

However, what is the optimal form? To find out the minimum length of the spider's web we will use x and l, two notations to facilitate the calculations, as can be seen in the Figure 13:

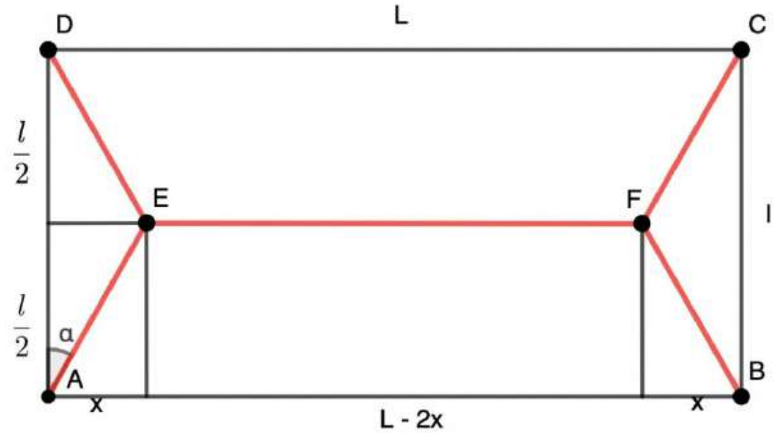


Figure 13. *The minimum length of a cobweb*

We consider $AB = L$ and $AD = l$. The cobweb length is therefore

$$L(x) = L - 2x + 4\sqrt{x^2 + \frac{l^2}{4}}$$

Because $x = \frac{l}{2}\tan\alpha$, with $\alpha \in \left[0, \frac{\pi}{4}\right]$, then(7)

$$L(x) = L(\alpha) = L + l\left(\frac{2}{\cos\alpha} - \tan\alpha\right).$$

Denoting by $f(\alpha) = \left(\frac{2}{\cos\alpha} - \tan\alpha\right)$, we observe that we will get the minimum for $L(\alpha)$, when $f(\alpha)$ is minimum.

For $\alpha \in \left[0, \frac{\pi}{4}\right]$, $\sin\alpha \geq 0$, $\cos\alpha > 0$, and $m = \min f(\alpha) > 0$. Since $f(\alpha) \geq m$, it results that

$\sin\alpha + m\cos\alpha \leq 2$. We point out that it exists $\cos\varphi = \frac{1}{\sqrt{1+m^2}}$ and $\sin\varphi = \frac{m}{\sqrt{1+m^2}}$ such that

$$\cos\varphi\sin\alpha + \sin\varphi\cos\alpha \leq \frac{2}{\sqrt{1+m^2}}.$$

This suggests we should look for $m = \sqrt{3}$, since $\sin(\varphi + \alpha) \leq 1$.

Let us prove now that $\sin\alpha+\sqrt{3}\cos\alpha\le 2$. Because $\sin\alpha=\sqrt{\frac{1-c}{2}}$ and $\cos\alpha=\sqrt{\frac{1+c}{2}}$, with $c=\cos 2\alpha\in[0,1]$, we get that $\sqrt{\frac{1-c}{2}}+\sqrt{3}\sqrt{\frac{1+c}{2}}\le 2\Leftrightarrow(1-2c)^2\ge 0$, which is true (8), and $\min\ f(\alpha)$ is obtained for $c=\frac{1}{2}$. Therefore, $2\alpha=\frac{\pi}{3}$, and $\alpha=\frac{\pi}{6}$.

The length of minimum cobweb is $L\left(\frac{\pi}{6}\right)=L+l\sqrt{3}$.

Second Method:

We can rewrite $L(x)=L(\alpha)=L+l\left(\frac{2}{\cos\alpha}-\tan\alpha\right)=L+l\left(\frac{2-\sin\alpha}{\cos\alpha}\right)$. Because L and l are positive constants, will get the minimum for $L(\alpha)$, when $\left(\frac{2-\sin\alpha}{\cos\alpha}\right)$ is minimum.

We denote by $t=\tan\frac{\alpha}{2}$. Because $\alpha\in\left[0,\frac{\pi}{4}\right]$, then $t\in\left[0,\tan\frac{\pi}{8}\right]=\left[0,\sqrt{2}-1\right]$. We get that (9)

$$\frac{2-\sin\alpha}{\cos\alpha}=-2+2\frac{t-2}{t^2-1}$$

We minimize now $f(t)=-2+2\frac{t-2}{t^2-1}$ on $[0,\sqrt{2}-1]$. We consider $n=\frac{t-2}{t^2-1}$ and the corresponding equation $nt^2-t+2-n=0^2$. We compute $\Delta=4n^2-8n+1$ and we impose that $\Delta=0$ (10). The values of *n* can be calculated now as $n_{1,2}=1\pm\frac{\sqrt{3}}{2}$ and consequently $t=2\pm\sqrt{3}$. Only the solution $t=2-\sqrt{3}$ fits, because $t\in[0,\sqrt{2}-1]$.

Since $\sin\alpha=\frac{2t}{1+t^2}=\frac{1}{2}$ and $\alpha\in\left[0,\frac{\pi}{4}\right]$, then $\alpha=\frac{\pi}{6}$.

Third Method:

We rewrite the length of the cobweb as

$$L(\alpha)=L-l\tan\alpha+\frac{2l}{\cos\alpha}\text{, with }\alpha\in\left[0,\frac{\pi}{2}\right).$$

We know that $L(0)=L+2l$ and $\lim\limits_{\alpha\nearrow\frac{\pi}{2}}L(\alpha)=+\infty$.

In order to calculate the minimum of this function we are going to derive it twice (it is possible because it is a continuous function as it is a composition of elementary functions):

$$L'(\alpha)=\frac{l(2\sin\alpha-1)}{\cos^2\alpha}\text{ with }L'(0)=l\text{ and }\lim_{\alpha\nearrow\frac{\pi}{2}}L'(\alpha)=+\infty$$

$$L''(\alpha)=\frac{2l(\sin^2\alpha-\sin\alpha+1)}{\cos^3\alpha}$$ with $L''(0)=2l$ and $\lim_{\alpha\nearrow\frac{\pi}{2}} L''(\alpha)=+\infty$

In order to find the critical points of the continuous function L, we look for the points where its first derivative is zero:

$L'(\alpha)=0 \Leftrightarrow \frac{l(2\sin\alpha-1)}{\cos^2\alpha}=0$, therefore $\sin\alpha=\frac{1}{2}$. Since $\alpha\in\left[0,\frac{\pi}{2}\right)$, it results that $\alpha=\frac{\pi}{6}$.

We evaluate the second derivative in the critical point:

$L''(\alpha)=\frac{2l(\sin^2\alpha-\sin\alpha+1)}{\cos^3\alpha}=\frac{2l(1/4-1/2+1)}{3\sqrt{3}/8}=\frac{4l}{\sqrt{3}}>0$, since $l>0$, so the critical point $\alpha=\frac{\pi}{6}$ is a minimum point for the function L.

. $L\left(\frac{\pi}{6}\right)=L-l\tan\frac{\pi}{6}+\frac{2l}{\cos\frac{\pi}{6}}=L+l\sqrt{3}$, therefore the minimum length of the cobweb, in this case, is reached when $\alpha=\frac{\pi}{6}$ and has a length $L+l\sqrt{3}$. (11) ■

The case of a parallelogram shape

We considered the case when the 4 points are the vertices of a parallelogram.

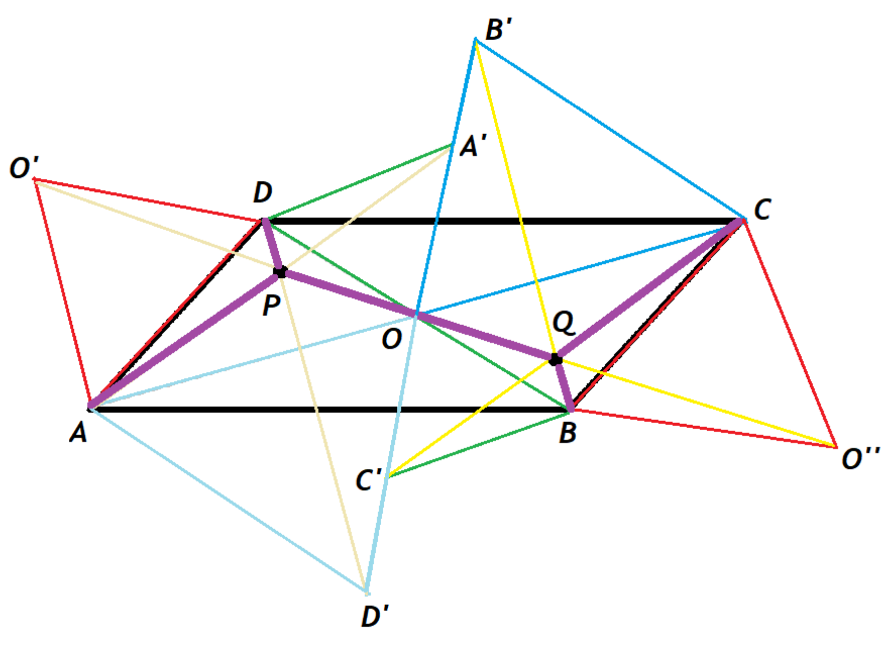


Figure 14. *Minimum cobweb length (drawn with **magenta**) for a parallelogram*

Having the parallelogram $ABCD$, where we have noted with O the intersection of the diagonals, we construct the equilateral triangles $AO'D$; AOD'; $A'OD$; $BO"C$; $B'OC$; BOC', as shown in Figure 14.

We consider the two triangles formed by the diagonals in the parallelogram that have all angles less than $120°$. Let P be the intersection of the segments AA', DD', $O'O"$, representing the Fermat point for the triangle AOD. Thus, $AP + PO + PD$ is minimum sum for the triangle AOD. Let Q be the intersection of segments BB', CC', $O'O"$, representing the Fermat point for the triangle BOC. Thus, $BQ + OQ + CQ$ is minimum sum for the triangle BOC.

We claim that:

Theorem 4. The positions of the points P and Q that guarantee the minimum cobweb length inside a parallelogram are given by Fermat points for the triangles AOD and BOC, respectively.

Proof.

As P is Fermat point in the triangle AOD, we have that

$$AP + PO + PD < AO + OD.$$

Similarly, because Q is Fermat point in the triangle *BOC,* we have that

$$BQ + OQ + CQ < BO + CO.$$

By adding the last two relations, we obtain

$$AP + PD + PQ + BQ + CQ < AC + BD.$$

If we argue in a similar way to the case of the rectangular room, we conclude that $AP + PD + PQ + BQ + CQ$ has the minimum cobweb length. (12) ■

The case of a square-base regular pyramid

We are now looking for the position of a point X inside the pyramid (as shown in Figure 15), such that the total sum from X to the vertices of the pyramid is minimum (13). If this point X would be situate on the plane of $ABCD$, then clearly X must be the same with the point O, which is the intersection of rectangular diagonals.

Therefore, we believe that this point must belong to the altitude from V, as the sum of the distances from V to the four vertices situated on the base is minimized when these distances are equal.
We make the following notation:

$$AB = l, VO = h, XO = x.$$

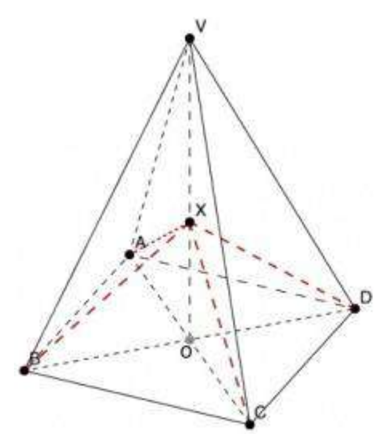


Figure 15. *A square-base regular pyramid VABCD*

Then, the total sum from X to the vertices of the pyramid is:

$$m = VX + XA + XB + XC + XD = VX + 4XA$$

thus,

$$m = 4\sqrt{\left(\frac{l\sqrt{2}}{2}\right)^2 + x^2} + h - x.$$

We attach a function to this formula,

$$f(x) = 4\sqrt{\left(\frac{l\sqrt{2}}{2}\right)^2 + x^2} + h - x.$$

In order to find the critical points of the continuous function f, we look for the points where its first derivative is zero:

$$f'(x) = 0 \Rightarrow \frac{4x}{\sqrt{x^2 + \frac{l^2}{2}}} - 1 = 0 \Leftrightarrow 4x = \sqrt{x^2 + \frac{l^2}{2}} \Leftrightarrow 16x^2 = x^2 + \frac{l^2}{2} \Leftrightarrow x^2 = \frac{l^2}{30} \Rightarrow x = \frac{l}{\sqrt{30}}$$

(the value of *x* with negative sign has no geometrical relevance, as it must be a length).

We evaluate the second derivative in the critical point:

$$f''(x) = \frac{2l^2}{\sqrt{\left(x^2 + \frac{l^2}{2}\right)^3}} \Rightarrow f''\left(\frac{l}{\sqrt{30}}\right) = \frac{2l^2}{\sqrt{\left(\left(\frac{l}{\sqrt{30}}\right)^2 + \frac{l^2}{2}\right)^3}} = \frac{15\sqrt{30}}{16l} > 0, \text{ since } l > 0.$$

We conclude that the critical point $x = \frac{l}{\sqrt{30}}$ is a minimum point for the function *f*.

Thus, the point X is situated on the altitude VO, at the distance x from O. The minimum length of the cobweb is:

$$f\left(\frac{l}{\sqrt{30}}\right) = 4\sqrt{\left(\frac{l\sqrt{2}}{2}\right)^2 + \left(\frac{l}{\sqrt{30}}\right)^2} + h - \frac{l}{\sqrt{30}}.$$

Conclusions

At first a first glance, we thought that it would be easy to find the minimum length of a cobweb that Spider Webster should build in order to connect his multiple points of interest. However, we discovered that the cobweb of minimum length is not actually as Pythagoras said, "the straight line that holds them together", and that it has, in fact, a unique shape (14). Therefore, we needed to calculate a certain angle, which gives us the best configuration, so that we could help Webster achieve his goal.

References

- Greg Petrics, Fermat Point of a Triangle with Geogebra, online notes
- Art of Problem Solving, Fermat point, AoPS online
- Dario Gonzalez Martinez, The Fermat point, University of Georgia, online notes

Editing notes

(1) On this figure P is clearly not the Fermat point.

(2) There, the authors focus on the good definition of P, i.e. that the 3 segments are concurrent. But the proof of "The Fermat point P minimizes the length $AP + BP + CP$" is still missing. This is unfortunate because the proof is affordable.

(3) Yes they are collinear, due to the horizontal and vertical symmetry of Figure 7.

(4) Indeed, if P minimizes $AP + DP + PQ$, then among all the points P' with fixed $length\ AP' + DP' = AP + DP$, P minimizes $P'Q$.

(5) This is not true. Indeed, consider a case with Q is very close to A. Then in order to minimize PQ, P should move towards the bottom of the rectangle. The sequel of the proof is not valid. Here is a way to circumvent the problem: Imagine that Q is closer to the line (EF) than P (if not, exchange the names of vertices). By minimality of $AP + DP + PQ$, P is the Fermat point of triangle ADQ. As it lies on the line formed by Q and the third vertex of the equilateral triangle based on AD, is necessarily strictly closer to (EF) than Q, which is in contradiction to the hypothesis. So P and Q are both on EF.

(6) Here the authors provide other analytical proofs.

(7) Strictly speaking, we need the function to be minimized on $[0, \pi/2]$.

(8) The previous equivalence requires some tedious calculations. Here it was enough to notice that $\sin\alpha + \sqrt{3}\cos\alpha = 2\sin(\alpha + \frac{\pi}{3})$, so immediately is ≤ 2.

(9) The formula is obtained thanks to the tangent half-angle formulae

(10) Why imposing this condition? the argument is missing.

(11) A discussion on the uniqueness of the construction would have been welcome! Indeed, if the rectangle is a square, there are two ways of building the cobweb, one with $[PQ]$ horizontal, and another one with $[PQ]$ vertical, so there is no uniqueness. But what happens for the case of a (non-square) rectangle for which the diagonals meet with all angles less than $120°$? Again there are two ways of building the cobweb but are they both optimal?

(12) Note that O, P and Q are aligned (because the figure is invariant under the central symmetry around O), so $PQ = OP + OQ$ and the proof works.

(13) Here the authors make the assumption of a single node. But has the optimal solution a single node?

(14) The term "unique shape" is not clear. Indeed, each shape will have its own solution, or its own solutions.