Environmental Dobble Game
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1. Introduction

One version of the Dobble game, which is the classic version, is a 55-cards set, every cards have 8 different symbols. No matter the way you play (there's different rules), the goal is always to spot as fast as possible the symbols in common between two cards. Indeed, any two cards of a Dobble game have one and only one symbols in common.
Here are the aspects we have been following during our research: How is the game built? How many symbols do you need to make a 10-card set? Or conversely, how many cards for a 10-symbol game? Can you make a Dobble game with environment-related symbols? Find a relationship between the number of cards and the number of symbols. How does this relationship change if two cards have two, three, etc… symbols in common?

Notations for a particular version of Dobble Game:
- x = number of symbols per card
- y = number of different symbols in the game
- c = number of cards
- p = symbols in common between two cards
- r = number of a symbol's repetition
- t = total number of symbols

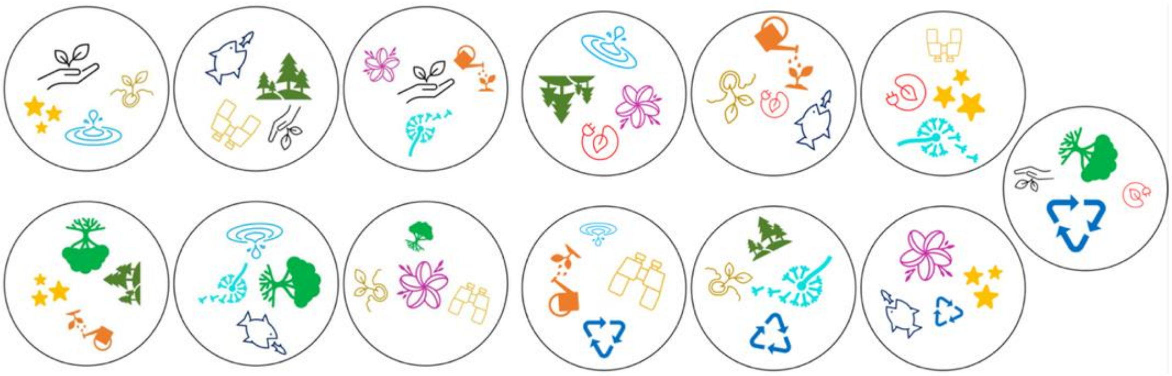
2. Results

In the table below the reader can consult the results of different comparisons we have made according to the data we have. The table works for the classic game.

Symbols per card x	Number of cards c	Different symbols y	Total number of symbols t
2	3	3	6
3	7	7	21
4	13	13	52
5	21	21	105
6	31	31	186
7	43	43	301
8	55	57	456/440
9	73	73	657
10	91	91	910

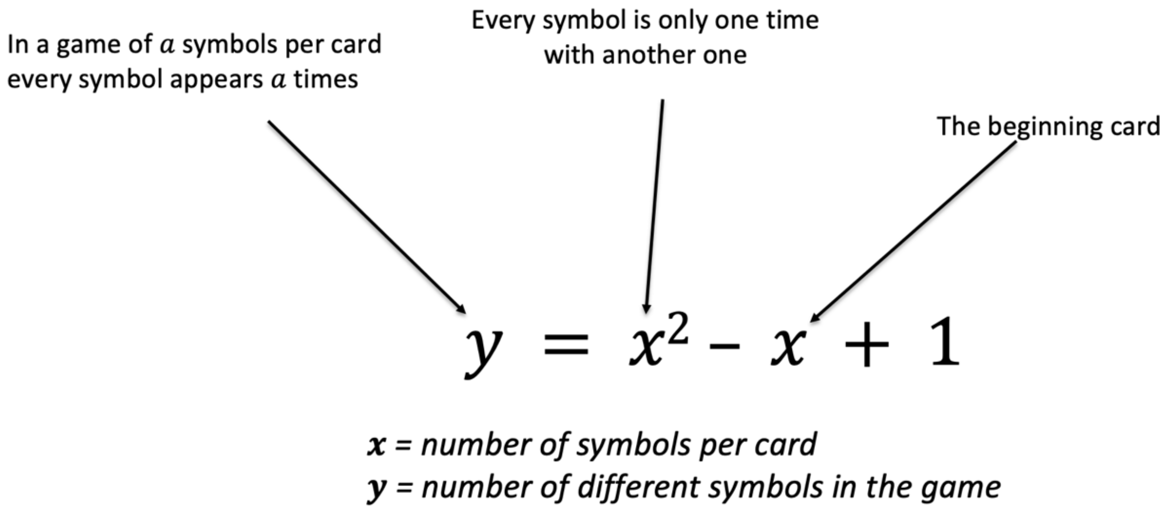
Doc.1: Results table about different optimized Dobble game

As asked, we create an environmental Dobble game of 4 symbols per card (and 13 cards).



Doc.2: Environmental Dobble Game

By a classic Dobble Game we mean a game who respect this following rule: only one symbol appears on every two cards.
If we are in the configuration of a classic game, the formula underneath allows us to have a relationship between the number of symbols per card and the number of different symbols in the game.



Doc.3: Relationship between the number of symbols per card and the number of different symbols

We found another formula :
For found the number of symbols “t” we have two way :

$$t = xc \qquad t = yr$$

So $\boldsymbol{xc = tr}$

We can also calculate the number of pairs in the game”p”

$$\boldsymbol{p = c(c-1)} \qquad \boldsymbol{p = \ y \ r(y-1)}$$

So $c\ (c-1)$ = y r (r-1)

Combining the two formulas, we get :

c(c-1)= y r(r-1), c(c-1)= c x (r-1), c - 1 = x (r - 1),

c = x(r-1)+1

In a "perfect" game, we must have y=c and so x=r

c= x(r-1)+1, c = x(x-1)+1,

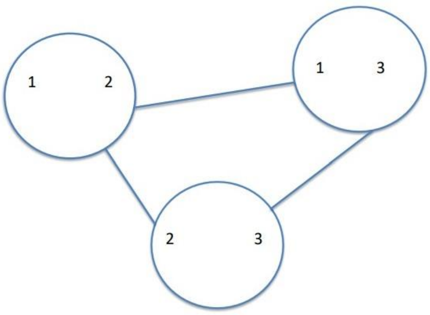
c = x^2 - x +1

3. Method

3.1. How is the game built?

As there are several groups of students working on this project, our approaches were different.

For this research, the students don't base their work on an optimized game, which won't be the same for the following part.
We started from the number of the cards to find the number of symbols. We used a schematisation :
Example for 3 cards :
- The numbers represent Dobble Symbols



Doc.4: Example for three cards

We made a chart in order to recap our results. The information that appears stands for the number of symbols per card, the number of different symbols and the number of cards.

Number of symbols per card	2	3	4	5	6	5
Number of different symbols	3	6	10	16	21	20
Number of cards	3	4	5	6	7	10

Doc.5: Chart of the different results

Problem : we don't find any link between the numbers in the table. At this time of our research, we think that we have forgotten some rules of the Dobble.

We resume the schematisation with new rules:
-Each card must have the same number of symbols We start from the number of symbols per card The chart has changed :

Number of symbols per card	3	4	5	6	7	8	9
Number of different symbols	6	10	15	21	28	36	45
Number of cards	4	5	6	7	8	9	10

Doc.6: Chart with the new results and the new rules

We can observe that if we use a random number from the column « different symbols » that we add to the number from the following column « symbols per card », we find a different number of symbols. Example :

Number of symbols per card		4
Number of different symbols	6	6 + 4 = 10

Doc.7: Observation

There is another link. Between the number of symbols per card and the number of cards.

number of symbols per card + 1 = number of cards

But there is a problem. The number of different symbols quickly becomes very high. For 10 cards we end up with 45 different symbols. Yet in the Dobble game the number of different symbols is approximately equal to 55 for 55 cards. We come to the conclusion that we have forgotten a rule.

Another approach - the optimized game: to begin, we looked for the optimized game's rules, the constraints that all Dobble games undergo. We found four notables who helped and guided us afterwards:

- Only one symbol appears on every two cards
- For every two different symbols S_1 and S_2, there is exactly one card C_12 which contains both
- Every symbol appears on the same number, say r, of cards
- In a game of a symbols per card, every symbol appears a times

All over the research, we try to respect the particular case of an optimized game. An optimized game is a game where the number of cards is maximal, and the number of symbols is minimal. In an optimized game, we have

$$s = c$$

The total number of symbols (without the repetitions of this symbol), which is t, is equal to ck. But we've also got $t = sr$

So, $ck = sr \iff k = r$

With, $k \geq p + 1$

Other approach based on the optimized game : We began our research by observing the original game. This one contains 55 cards but can have 57. There are also 57 symbols. It is a game that we have called a perfect game. A perfect game is an optimized game, it is said to be perfect when it has as many symbols as cards. We could therefore not build other cards or add additional symbols. You cannot build a perfect deck with any number of cards and symbols. For example, it is impossible to build a perfect deck comprising 10 cards and 10 symbols. We first looked for the list of games that can be perfect such as 3 cards 3 symbols with 2 symbols per card, then 7 cards 7 symbols with 3 symbols per card… We put the perfect games in the form of a table like this underneath

Each of the letters corresponds to a card and a number is equal to a symbol. This method easily exposes us to your careless mistakes when the numbers get bigger. We tried to establish links between each of these data which gave us the following results

We have seen that the dobble game as sold is a perfect game of 57 cards and 57 symbols, but for marketing reasons the manufacturers have decided to remove 2 cards. We can see an imbalance, indeed some symbols are more represented than others, for example there are only 6 times the symbol "snowmen" for 8 representation of the symbol "bombs".

Then, we build Dobble games with different symbols per card. We made comparisons of the different data we have found. We list them in the Doc.1 (results section).

In this first period of research, we also tried probability and geometry, but it was inconclusive.

Thanks to the data and comparisons, and after a few tries, we found a formula.
So, $\boldsymbol{y = x^2 - x + 1}$
For example, in a game of 3 symbols per card, we can say thanks to the formula, that we need a 7-cards game.
Thanks to the rules found, we can justify the formula:

- x squared: because in a game of a symbols per card, every symbol appears a times
- minus x: because every symbol can be only one time with another
- plus one: it stands for the beginning card

The relationship between the number of symbols per card and the number of different symbols in the game is depicted and explained in Doc.3 (results section).

In this section, we follow the approach in [2], section 1. We mention that this contribution is not the result of our work, but we decided to include it here for completeness.
Another approach for the formula (for the classic game with no rules for number of appearances of symbols) is:

At the beginning of the study we generated different cases, as well as the simplest set containing three cards and three symbols. For a geometric view, I turned the set into a triangle.

For the case with 3 symbols on a card, we will need 7 cards. As the numbers increase, so do the drawings. Thus, we resort to notations. In the principle of finite geometry, any line intersects (including parallel lines), we replace the point with the symbol, and the line with the card. We thus arrive at the following statements:

- Any two cards have a common symbol. ⇔ Any two lines meet at a point.
- Any two symbols belong to exactly one card. ⇔ Any two points determine a line.

Following the two statements, the axiom is strengthened: Any line meets. We call it "finite" because it allows us to start and finish counting the points and lines that make up our project plan. On the other hand, if it were infinite, we could not stop counting, and we do not want this to happen because we do not have an infinite number of cards.

A finite geometry is any geometric system that has only a finite number of points. The Euclidean geometry family is not finite, because a Euclidean line contains infinitely many points. A geometry based on graphics displayed on a computer screen, where pixels are considered to be points, would be finite geometry. Although there are many systems that could be called finite geometries, attention is paid especially to finite projective and affine spaces due to their regularity and simplicity. Other significant types of finite geometry are finite Möbius or inversion planes and Laguerre planes, which are examples of the general type called Benz planes, and their larger analogues, such as higher finite inverted geometries.

The following remarks apply only to finite plans. There are two main types of finite plane geometry: affine and projective. In an affine plane, the normal direction of the parallel lines is applied. In a projective plane, on the other hand, any two lines intersect at a single point, so there are no parallel lines. Both finite affine plane geometry and finite projective plane geometry can be described by fairly simple axioms.

An affine plane geometry is an empty set X (whose elements are called "points"), together with an empty L collection of subsets of X (whose elements are called "lines").

(PP1) Every two points are incident with a unique line.

(PP2) Every two lines are incident with a unique point.

(PP3) There are four points, no three collinear.

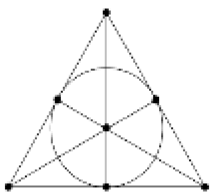
(PP4) Every point in a projective plane is incident with a constant n + 1 lines. Dually, every line is incident with n + 1 points.

The last axiom ensures that the geometry is not trivial (either empty or too simple to be of interest, such as a single line with an arbitrary number of dots on it), while the first two specify the nature of the geometry.

The simplest affine plan contains only four points; is called the 2nd order affine plane. (The order of an affine plane is the number of points on any line.) Since none of the three is collinear, any pair of points determines a single line, so this plane contains six lines. It corresponds to a tetrahedron in which the non-intersecting edges are considered "parallel" or a square in which not only the opposite sides but also the diagonals are considered "parallel". More generally, an affine finite plane of order n has n^2 points and $n^2 + n$ lines; each line contains n points and each point is on n + 1 lines. The 3rd order affine plan is known as the Hesse configuration.

Let V (n + 1, q) be a vector space of dimension n + 1 over GF(q). The projective space P G(n, q) is the geometry whose points, lines, planes, . . . , hyperplanes are the subspaces of V (n + 1, q) of rank 1, 2, 3, . . . , n. The dimension of a subspace of P G(n, q) is one less than the rank of a subspace of V (n + 1, q). The incidence structure in Figure 1.1 is what we get if we put n = q = 2. It has a group of 168 automorphisms, isomorphic to the 3×3 non-singular matrices whose elements come from GF(2).

As in linear algebra <(x0, x1, . . . , xn),(y0, y1, . . . , yn), . . . ,(z0, z1, . . . , zn)> is the space spanned by the vectors x = (x0, x1, . . . , xn), y = (y0, y1, . . . , yn) and z = (z0, z1, . . . , zn). A hyperplane is a subspace of co-dimension 1. If H a hyperplane and l is a line not contained in H then H ∩ l is a point. The geometry P G(2, q) has the property that every two lines are incident in a (unique) point. The rank of the vector space V (3, q) is 3 and the lines U and V are subspaces of rank 2. Hence the rank of U ∩ V is 1, so U ∩ V is a point.



Doc.10: PG(2,2)...The Fano plane

The number of subspaces of rank k in V (n, q) is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q^n - 1)(q^n - q)\ldots(q^n - q^{k-1})}{(q^k - 1)(q^k - q)\ldots(q^k - q^{k-1})}$$

We observe the axioms of an incidence structure that mimics those properties that P G(2, q) has. This is a common occurrence in the study of incidence structures. We will often take a naturally occurring object and define a more general object with properties it possesses. A projective plane is an incidence structure of points and lines with the following properties:

(PP1) Every two points are incident with a unique line.

(PP2) Every two lines are incident with a unique point.

(PP3) There are four points, no three collinear.

Note that the axioms (PP1)-(PP3) are self-dual. Hence the dual of a projective plane is also a projective plane. So if we prove a theorem for points in a projective plane then the dual result holds automatically for lines. We have already seen that the geometry P G(2, q) is an incidence structure satisfying these properties. It is called the Desarguesian projective plane because of the following theorem:

Every point in a projective plane is incident with a constant n + 1 lines. Dually, every line is incident with n + 1 points.

A projective plane of order n has n^2 + n + 1 points and n^2 + n + 1 lines and for our game the formula is: $y = x^2+x+1$

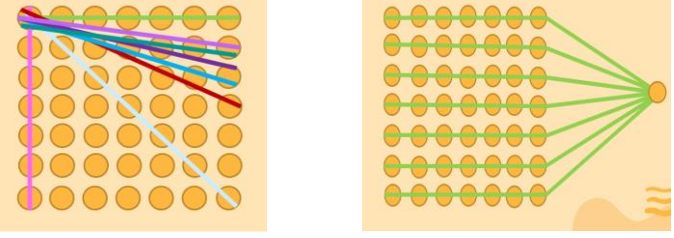
Proof: Let P be a point of a projective plane. There are n + 1 lines incident with P and each is incident with n other points. Hence the number of points in a projective plane of order n is n(n + 1) + 1. The number of lines follows from the dual argument.

There are many projective planes known that are not isomorphic to P G(2, q), however all known examples have prime power order. This brings us to the first, and probably the most famous, of the many unsolved problems in finite geometry.

Conjecture: *The order of a projective plane is the power of a prime.*

Taking the sold Dobble game as an example, we place the cards on 7 rows and 7 columns, so that the 7 cards on each row or column have one common symbol. Also, there are only one common symbol for the diagonals, which we can group in 2 in 2 cards, 3 in 3, etc. Given the principle of finite geometry, we get another 8 cards that also form a line, so they have a common symbol.

Lines of different colors represent all possible cases of parallel lines. Having a matrix of 7x7, we have 7 • 7 = 49 cards. Knowing that the 8 parallel lines (the principle of finite geometry) meet at one point, we still have 8 (7 + 1) cards.



Doc.11: A graphic representation of the demonstration

However, this principle does not only work for the powers of prime numbers, which means that for the rest of the numbers we will refer to the smallest multiple of a prime number greater than the number sought.

C++ CODE:

```
#include <bits/stdc++.h>
#define NMAX 2600
#define SMAX 50
using namespace std;
int main(){
  int n,s;
  int cards[NMAX+5][SMAX+5];
  cin>>s; ///s is the number of symbols on a card
  n=s-1; /// n must be a prime number
  for(int i=1;i<=n+1;i++){ ///generating the first group
     cards[i][1]=1;
     for(int j=2;j<=s;j++){
        cards[i][j]=(i-1)*n+j;
     }
  }
  for(int i=1;i<=n;i++){ /// generating the other groups
     for(int j=1;j<=n;j++){
        cards[n+1+(i-1)*n+j][1]=i+1;
        for(int k=2;k<=s;k++){
          cards[n+1+(i-1)*n+j][k]=n+1+n*(k-2)+((i-1)*(k-2)+j+n-1)%n +1;
           /// cyclic permutations of symbols' indexes
        }}
  }
  for(int i=1;i<=n*n+n+1;i++){
     for(int j=1;j<=s;j++){
        cout<<cards[i][j]<<' ';
     }
     cout<<'\n';
  }}
```

Doc.12: C++ code

3.2. Find a relationship between the number of cards and the number of symbols

As for finding a relationship between the number of cards and the number of symbols, our comparison shows us that the number of different symbols in the same that the number of cards.
So, $y = c$ = number of different symbols = number of cards.
It’s because that **in a game of x symbols per card every symbol appears x times**.
In other words, multiplying the number of symbols per card and the number of different symbols allows us to find the total number of symbols.

Symbols per card x different symbols = total number of cards

However, in a game where all the cards have the same number of symbols per card, this total number of symbols can also be calculated by multiplying the number of cards and symbols per card. This is why the number of cards is equal to the number of different symbols.

Symbols per card x different symbols = total number of cards
In a game of x symbols per card every symbol appears x times,
So, the total number of cards $= x(x^2 - x + 1) = x^3 - x^2 + x$

We also base several hypothesis on prime numbers, but we don't prove them.
Hypothesis on prime number: if we decompose the total number of symbols in the game into prime factors, then the last factor (the most important) or the multiplication of two of these factors is equal to the number of cards, but also to the number of different symbols. We just have to remove the number of symbols per card from this multiplication.

3.3. How many symbols do you need to make a 10-card set?

Thus, to create a 10-card set, we need to have 4 symbols per card:

$$x^2 - x + 1 = 10$$
$$\Leftrightarrow x \approx 3{,}54$$

By rounding by excess, we have $x = 4$ and $c = 4^2 - 4 + 1 = 13$

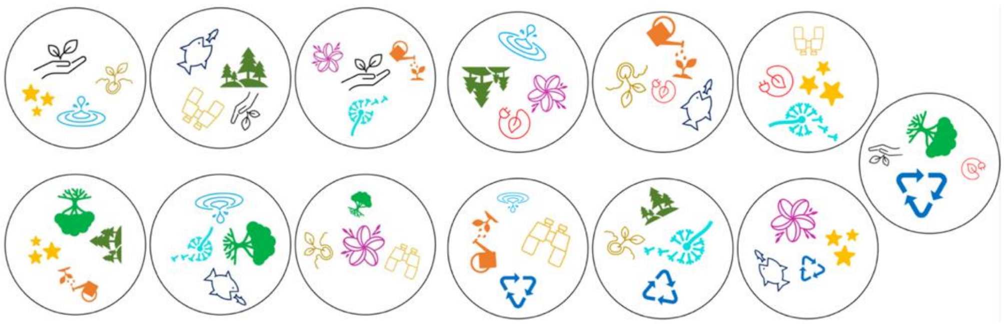
We will have to modify the game like the classic one with 8-symbols-per-card which should have 57 cards but have only 55, a game of 13 cards can be play with only 10 (the game of 13 cards is optimized).

3.4. How many cards for a 10-symbol game?

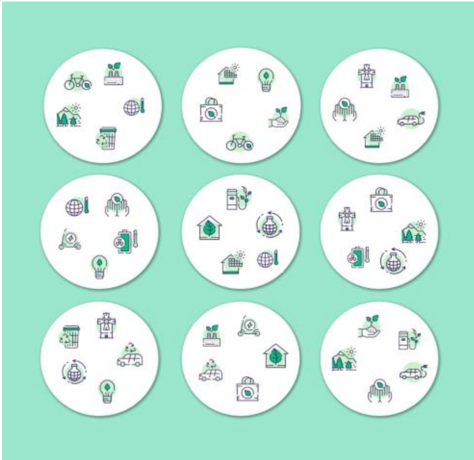
With 10 symbols per card: $c = 10^2 - 10 + 1 = 91$
So, we could create a 91-card set with 10 symbols per card.

3.5. Can you make a Dobble game with environment-related symbols?

We created our Dobble game with environment-related symbols on the model of the game of a 10- card set (by subtracting 3 cards among the 13 underneath).



Doc.13: Environmental Dobble Game(French Team)



Doc.14: Environmental Dobble Game(Romanian Team)

3.6. How does this relationship (between the number of cards and the number of symbols) change if two cards have two, three, etc… symbols in common?

Let $a \in \mathbb{N}$ and $y_0 = x_0 = p$. We have thus one only card in the game. But, for example, if $y_0 = x_0 = 2$, thus the only card will be (1 ; 2), the symbols are here the numbers.
If we had a symbol per card based on the game underneath, we could create a game of two cards (1 ; 2 ; 3) and (1 ; 2 ; 4). We have $y_1 = 4 = y_0 + 2$.
From that analysis, the following involvement is demonstrated: $x_{n+1} = x_n + a \Rightarrow y_{n+1} = y_n + 2a$
We denote by x_n the number of rang x is, which means the number of times we add 1 to x_0. That is the same for y : y_n stands for the rang.

Rang n	x	y
0	2	2
1	3	4
2	4	6
3	5	8
4	6	10
5	7	12

Rang n	x	y
0	3	3
1	4	5
2	5	7
3	6	9
4	7	11
5	8	13

Rang n	x	y
0	4	4
1	5	6
2	6	8
3	7	10
4	8	12
5	9	14

Doc.15: Data table

If the game is optimized, thus the involvement is true. The reciprocal isn't: we can't create an optimized game thanks to this rule.

4. Conclusion

To go further, an interesting research direction would be to create a game with symbols in common between cards in an optimized game, in other words, have a reciprocal rule. We'd also like finding other conclusive proof for the general formula.

5. Bibliography

[1] Finite Geometries in Modern Developments in Geometry
[2] Geertrui Van de Voorde, An Introduction to Finite Geometry