# Equidecomposability of polygons

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## 1 Statement

Two polygons A and B of equal area are given. Can A be cut into smaller polygons so that after rearrangement they form B?

*In other words*: Our task is to prove that any polygon can be cut into smaller polygons that rearranged form another polygon of the same area [1] . To make the redaction easier, we will use the following notations: A*B, which means that A *can* be cut into polygons which rearranged form B, and A! *B, [2] which means that A *cannot* be cut into polygons which rearranged form B.

## 2 Steps

I. *The polygon A can be cut into triangles;*

II. *Any given triangle can be cut into smaller polygons that rearranged form a rectangle with equal area;*

III. *Any given rectangle can be cut into smaller polygons which rearranged form a rectangle with one side of length 1 and of equal area C;*

*IV. All these rectangles with one side of length 1 can be put together to form a bigger rectangle with one side of length 1 and with the same area as A;*

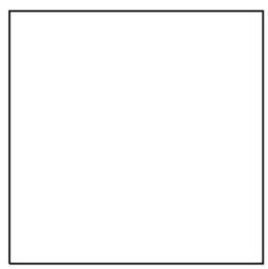
A is a polygon, which means that A*C. B is also a polygon, therefore B*C'. Since A and B have the same area, C and C' are the same rectangle. A*C, B*C => A*B.

**Here are two examples:**

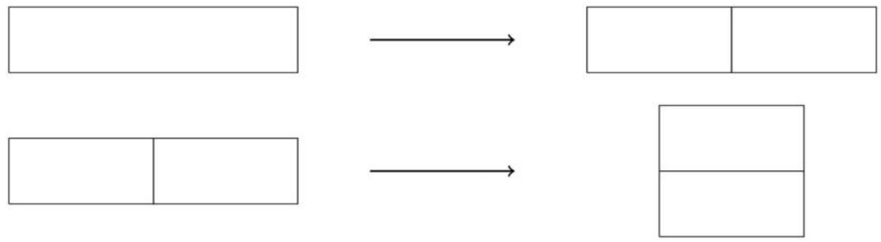
*Example 1:*

A= rectangle with sides 1 and 4

A'= square with side 2



After the following set of cuts and rearrangements, we will get that A*B.



*Example 2:*

A = square with side 2

B = square with side 3

In this case, A! *B.

*Remarks:*

- If A*B, then B*A;
- If A*C and C*B, then A*B;

From these two remarks, we get that: if A*C and B*C, then A*B.

The actions of cutting and rearranging do not affect the area of the initial polygon. In order to have A*B, it is absolutely necessary that the area of A is equal to the area of B.

**We will prove that given two polygons A and B of equal areas, there is a way to cut A into smaller polygons that rearranged form B.**

## 2.1 The triangulation of a polygon

For a convex polygon, it can be cut into triangles by choosing a vertex and tracing all of the diagonals which start from that vertex.

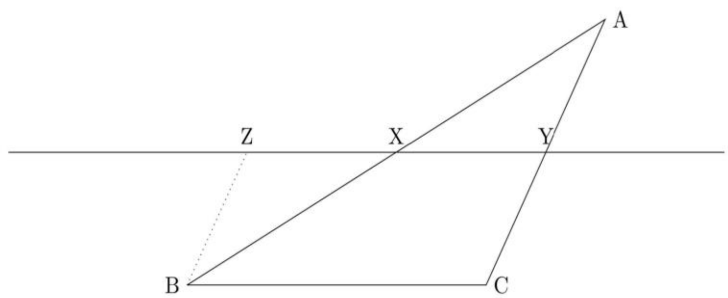
For a concave polygon, it can be cut into convex polygons, and we repeat the previous steps.

## 2.2 Transforming any given triangle into a rectangle [3]

a) We will transform the triangle ΔABC in the rectangle parallelogram ZYCB. We take X and Y as the midpoints of the line segments [AB], respectively [AC]. We take *d* as the parallel to AC, that goes through B, and Z is the intersection of *d* and XY ({Z} = d ∩ XY). Since XY is the midsegment in ΔABC, XY ∥ BC. XY ∥ BC, ZB ∥ YC →ZYCB is a parallelogram. [4] Because we know that XYCB is ZYCB intersected with ΔABC (XYCB = ZYCB ∩ ΔABC), in order to show that ΔABC*ZYCB, it suffices to prove that ΔZXB ≡ ΔYXA.

b) X is the midpoint of the line segment [AB] (X = mid [AB]) → AX ≡ BX (1),
∠ZBX and ∠YXA are opposite angles →∠ZBX ≡ ∠YXA (2);
ZB ∥ AY, AB = transversal → ∠ZBX ≡ ∠YAX (alternate interior angles) (3).
From (1), (2) and (3) ΔZXB ≡ ΔYXA (angle, side, angle - ASA).

c) We will transform the parallelogram ZYCB into a rectangle ZZ'YY'. Z' is the foot of the perpendicular from Z to BC, and Y' is the foot of the perpendicular from Y to BC.




ZZ'⊥BC; BC ∥ ZY → ZZ' ⊥ ZY → ZZ'=dist (BC, ZY) - (ZZ' is the distance from BC to ZY)

In the same way we prove that YY' is also the distance from BC to ZY. (YY'=dist(BC, ZY) ) → ZZ' = YY' (1)

α = m(∠BZY)

β = m(∠BZZ')

γ = m(∠CYY')

$\theta = m(\angle ZYC)$

BZ ∥ YC, ZY = transversal → $\alpha + \theta = 180^\circ$ (i)

$\beta = \alpha - 90^\circ$ (ii)

$\theta + \gamma = 90^\circ$ (iii)

from (i), (ii) and (iii) → $\alpha - \gamma = 90^\circ \rightarrow \gamma = \alpha - 90^\circ \rightarrow \gamma = \beta$,

∠BZZ' = ∠CYY' (2)

BZ = CY (3)

From (1), (2) and (3) → ΔBZZ' ≡ ΔCYY' (SAS– side, angle, side).


## 2.3 The transformation of a rectangle into a rectangle with one side of length 1

We will prove that

(1) any rectangle can be transformed into a square with the same area,

which implies that

(2) any square can be transformed into any rectangle of the same area

From (1) and (2) we have that any rectangle can be transformed into any other rectangle of the same area and hence any rectangle can be transformed into a rectangle of the same area and with one side of length 1.

We consider the rectangle A'B'C'D', with sides of length a' and b' (a'<b') and the square of equal area XYZT, with side of length c=$\sqrt{a' \cdot b'}$.

We will transform the rectangle A'B'C'D' into the rectangle ABCD with sides of lengths a and b, and with the following properties:

a < b;

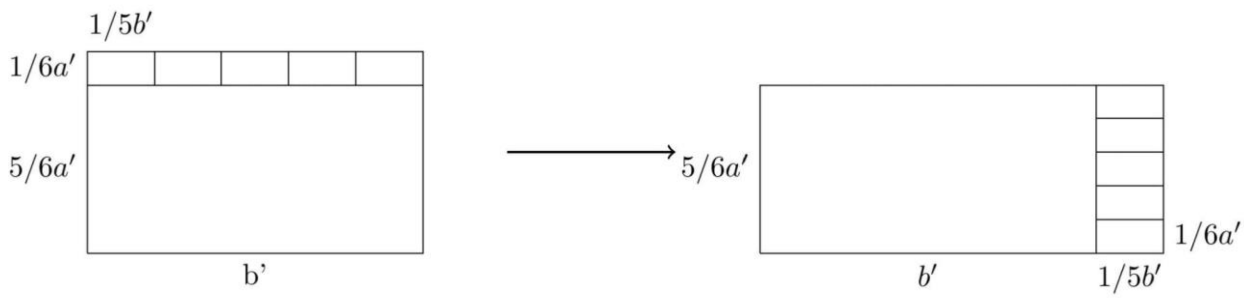
$\frac{b}{c} \in (\sqrt{5}, \sqrt{10})$

(we will explain later on why this condition is necessary)

I. $\frac{b'}{c} \leq \sqrt{5}$

We will do the following transformation α times, with $\alpha \in N$.

We cut the initial rectangle into another rectangle of sides a'$\cdot\frac{5}{6}$, respectively b' and 5 rectangles of sides a'$\cdot\frac{1}{6}$ and b' $\frac{1}{5}$. These rectangles are then rearranged to form a rectangle with sides of length a'$\cdot\frac{5}{6}$ and b'$\cdot\frac{6}{5}$.




After the α transformations, b will be equal to b'$\cdot\left(\frac{6}{5}\right)^{\alpha}$.

We will prove that $\exists\ \alpha \in N$, so that $\frac{b'}{c} \cdot (\frac{6}{5})^{\alpha} \in (\sqrt{5}, \sqrt{10})$.

Suppose that there is no $\alpha \in N$ that respects the property above $\rightarrow \exists\ \alpha' \in N$ so that $\frac{b'}{c} \cdot \left(\frac{6}{5}\right)^{\alpha'} \leq \sqrt{5}$ and $\frac{b'}{c} \cdot (\frac{6}{5})^{\alpha'+1} \geq \sqrt{10}$. [5]

Let $x$ be $\frac{b'}{c} \cdot \left(\frac{6}{5}\right)^{\alpha'}$.

$x \leq \sqrt{5} \rightarrow x \cdot \frac{6}{5} \leq \sqrt{5} \cdot \left(\frac{6}{5}\right)$ (1)

$x \cdot \frac{6}{5} \leq \sqrt{10}$ (2)

$\sqrt{5} \cdot (\frac{6}{5}) < \sqrt{10}$ (3)

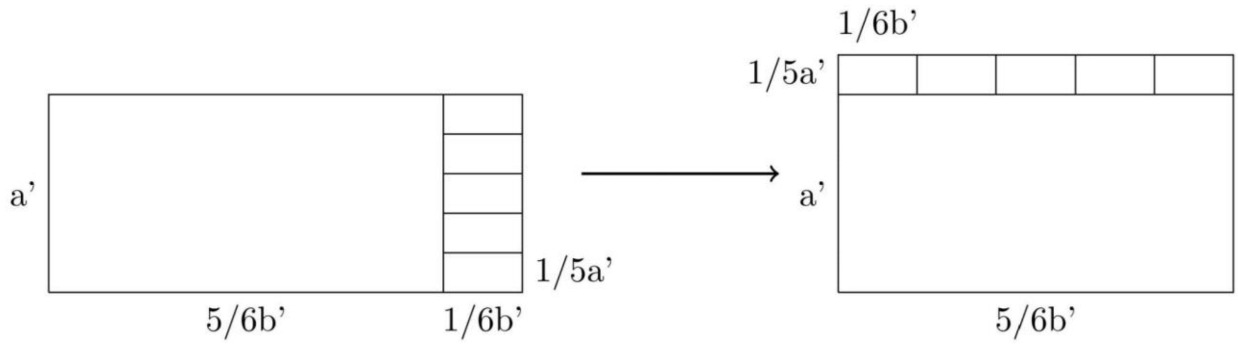
From (1), (2) and (3) → contradiction, so ∃ α ϵ *N* so that

$\frac{b'}{c} \cdot (\frac{6}{5})^{\alpha} \in (\sqrt{5}, \sqrt{10})$

II. $\frac{b'}{c} > \sqrt{10}$

We will do the following transformation α times, with α ϵ *N*

We will cut the initial rectangle into another rectangle of sides a', respectively $(\frac{5}{6})\cdot$b', and five rectangles of sides $(\frac{1}{5})\cdot$a', respectively $(\frac{1}{6})\cdot$b'. These rectangles are rearranged in order to form a rectangle with sides of lengths $(\frac{6}{5})\cdot$a', respectively $(\frac{5}{6})\cdot$b'.




After the α transformations, b will be equal to b'$\cdot (\frac{5}{6})^{\alpha}$.

We will prove that ∃ α ϵ *N* so that $(\frac{b'}{c}) \cdot (\frac{5}{6})^{\alpha} \in (\sqrt{5}, \sqrt{10})$

Suppose that there is no α with the requested property

→ ∃ α' ϵ *N* so that $(\frac{b'}{c}) \cdot (\frac{5}{6})^{\alpha'+1} \leq \sqrt{5}$, and $(\frac{b'}{c}) \cdot (\frac{5}{6})^{\alpha'} \geq \sqrt{10}$.

Let x be $(\frac{b'}{c}) \cdot (\frac{5}{6})^{\alpha'}$.

$x \geq \sqrt{10} \rightarrow x \cdot (\frac{5}{6}) \geq \sqrt{10} \cdot (\frac{5}{6})$ (1)

$x \cdot (\frac{5}{6}) \leq \sqrt{5}$ (2)

$\sqrt{10} \cdot (\frac{5}{6}) > \sqrt{5}$ (3)

From (1), (2) and (3) → contradiction. Therefore, $\exists\ \alpha \in N$ so that

$(\frac{b'}{c}) \cdot (\frac{5}{6})^{\alpha} \in (\sqrt{5}, \sqrt{10})$.

Let ABCD be a rectangle with sides of lengths a and b, respectively XYZT a square with side of length $c = \sqrt{a \cdot b}$ with the following properties:

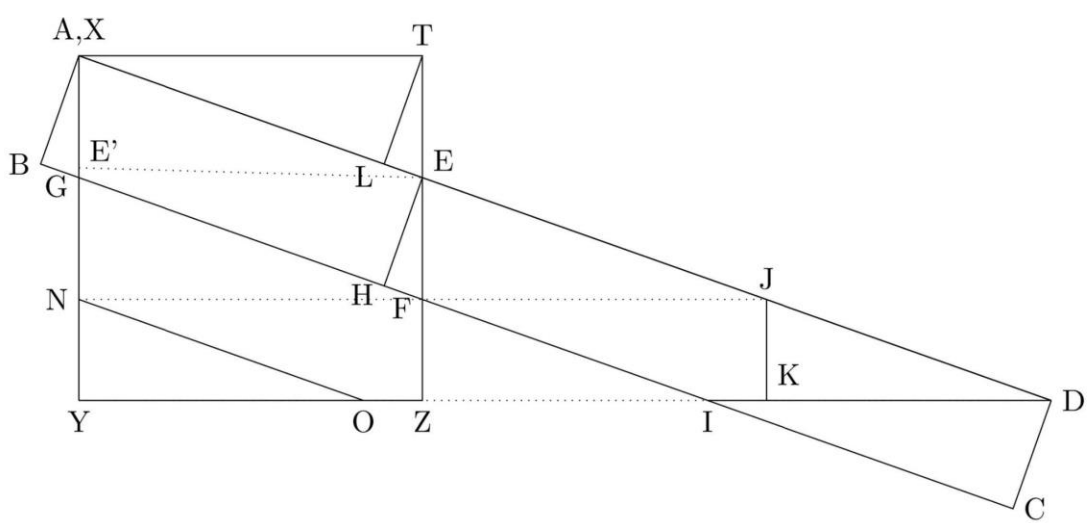
$a < b$;

$(\frac{b}{c}) \in (\sqrt{5}, \sqrt{10})$

$A_{ABCD} = A_{XYZT}$

A'B'C'D'$^{*}$ABCD

If we prove that ABCD$^{*}$XYZT → A'B'C'D' $^{*}$ XYZT




(picture found using the link https://www.cut-the-knot.org/Curriculum/Geometry/SquareRectangle.shtml)

We overlap ABCD and XYZT so that D ϵ (YZ) and A is the same as X.

In order to prove that ABCD*XYZT, we will prove that there is a way to cut the rectangle ABCD so that the resulting polygons can form XYZT after rearrangement.

For this we will cut both XYZT and ABCD and we will prove that each polygon resulted from the dissection of the square is congruent with a polygon resulted from the dissection of the rectangle.

We will use the following notations:

$y = m(\angle YDA)$

$F = BC \cap TZ$

H = the foot of the perpendicular from E to BC

N = symmetric to A with respect to G

K = the foot of the perpendicular from J to YD

L = the foot of the perpendicular from T to AD

l = the length of the line segment AE

$E = AD \cap TZ$

$G = BC \cap AY$

$I = BC \cap YD$

J = symmetric to A with respect to E

o = the parallel to BC through N

$O = o \cap YZ$

d = the length of the line segment AG

XYZT is cut into {AEFG; TLE; GFZON; NOY; TLA}

ABCD is cut into {AEFG; ABG; EJKIF; JDK; DCI}

We will prove that:

(1) TLE ≡ ABG

(2) GFZON ≡ EJKIF

(3) NOY ≡ JDK

(4) TLA ≡ DCI

(5) AEFG ≡ AEFG

From (1), (2), (3), (4) and (5) it results that there is a way to cut ABCD into polygons that rearranged form XYZT. [6]

**Proof of (2)**

AY ⊥ YD → Δ AYD is a right triangle → $\sin(y) = \frac{AY}{AD} = \frac{c}{b}$

EZ ⊥ ZD → ΔEZD right triangle → $m(\angle ZED)$=90˚- y

EH ⊥ BC, AD ∥ BC → EH ⊥ AD → $m(\angle HED)$=90˚ (2.1)

m(∠HED) = m(∠HEF) + m(∠ZED) (2.2)

m(∠HEF) = y (2.3)

EH ⊥ HF → ΔEHF is a right triangle (2.4)

From (2.1), (2.2), (2.3) and (2.4) → cos(y)= $\frac{EH}{EF}$ (i)

EH ⊥ BC, BC ∥ AD → EH = distance from BC to AD (2.5)

AB ⊥ BC, BC ∥ AD → AB = distance from BC to AD (2.6)

From (2.5) and (2.6) → AB = EH = a (ii)

EF ∥ AG, AE ∥ GF → AEFG is a parallelogram → EF = AG = d (iii)

From (i), (ii) and (iii) → cos(y)=$\frac{a}{d}$

E' = the foot of the perpendicular from E to AY

EE' ⊥ AY, ZT ∥ AY → EE' = the distance from AY to ZT → EE'=c

ZT ∥ AY, YZ ⊥ AY → YZ = the distance from AY to ZT → YZ=c

YZ = EE' = c (2.7)

ΔAE'E = right triangle (2.8)

In ΔAYD:

m(∠AYD) = 90˚, m(∠ADY) = y → m(∠YAD) = 90˚ - y = m(∠E'AE) (2.9)

From (2.8) and (2.9) → m ∠ (AEE') = y (2.10)

From (2.7), (2.8) and (2.10) → cos(y) = $\frac{GE}{AE} = \frac{c}{l}$, sin(y) = $\frac{AE\prime}{l}$

$A_{ABCD}$= a·b, $A_{XYZT}$= $c^2$ → ab = $c^2$→ $\frac{a}{c} = \frac{c}{b}$ = sin(y)

sin(y)·[cos(y)/cos(y)]= $\left(\frac{a}{c}\right) \cdot \left(\frac{c}{l}\right) \cdot \left(\frac{d}{a}\right) = \frac{d}{l}$ → sin(y) = $\frac{d}{l}$, sin(y)=$\frac{AE\prime}{l}$

→ AE' = d, and G, E' ϵ [AY] → E' = G

EG ⊥ AY → EG ∥ YD

AEFG is a parallelogram → AG = GN, but we also have EF = AG → EF = GN

AE ∥ GF, AG transversal → m(∠GAE) = m(∠NGF) = 90˚ - y,
but m(∠FEJ) = 90˚ - y → ∠ NGF ≡ ∠ FEJ

EJ ∥ FI, EF transversal → m(∠FEJ) + m(∠EFI) = 180˚ → m(∠EFI) = 90˚+ y

GF ∥ NO, GN transversal → m(∠NGF) + m(∠GNO) = 180˚ → m(∠GNO) =90˚+y

∠EFI ≡ ∠GNO

∠EFI ≡ ∠GFZ (opposite angles)

ΔJKD is a right triangle, m(∠KDJ) = y → m(∠KJD) = 90˚ - y

∠KJE and ∠KJD are supplementary angles → m(∠KJE) = 90˚ +y = m(∠EFI)

∠KJE ≡ ∠EFI, ∠EFI ≡ ∠GFZ → ∠KJE ≡ ∠GFZ

JK ⊥ YD, I ϵ YD → m(∠JKI) = 90˚

FZ ⊥ YD, O ϵ YD → m(∠FZO) = 90˚

NY ⊥ YZ, FZ ⊥ YZ → NY ∥ FZ

YZ = ZI (they're actually the same line)

NY ∥ FZ, NO ∥ FI, YO = ZI → ΔNYO ~ ΔFZI → ∠NOY ≡ ∠FIZ

∠NOZ and ∠NOY are supplementary angles, ∠FIK and ∠FIZ supplementary,

$\angle NOY \equiv \angle FIZ \rightarrow FIK \equiv NOZ$

$\angle NGF \equiv \angle FEJ$, $\angle GNO \equiv \angle EFI$, $\angle GFZ \equiv \angle EJK$,

$\angle FZO \equiv \angle JKI$, $\angle ZON \equiv \angle KIF \rightarrow EJKIF \sim GFZON$, [7]

But since EF = GN, we get that EJKIF ≡ GFZON. (2)

**Proof of (3)**

From EJKIF ≡ GFZON, we get that JK = FZ, FI = NO, IK=OZ.

$NO \parallel BC$, $BC \parallel AD \rightarrow NO \parallel AD$

$NY \perp YD$, $JK \perp YD \rightarrow NY \parallel JK$

$NO \parallel AD$, $NY \parallel JK$, YO = KD (they're the same line) → $\Delta NYO \sim \Delta JKD$, but we also have $\Delta NYO \sim \Delta FZI$, from which we get that $\Delta JKD \sim \Delta FZI$.

$JK = FZ \rightarrow \Delta JKD \equiv \Delta FZI$

$\Delta NYO \sim \Delta FZI$, $FI = NO \rightarrow \Delta NYO \equiv \Delta FZI$

From $\Delta NYO \equiv \Delta FZI$, and $\Delta JKD \equiv \Delta FZI \rightarrow \Delta NYO \equiv \Delta JKD$ (3)

**Proof of (4)**

From $\Delta NYO \equiv \Delta JKD$, we get that YO = KD.

From (2) we also have OZ ≡ KI.

From the two above, we get that c =YO + OZ = KD + IK = ID.

m(GAD) = 90˚ - y, ∠GAD and ∠DAT complementary angles → m(∠DAT) = y

ΔFZI ≡ ΔJKD → m(∠FIZ) = m(∠JDK) = y,

m(∠CID) = m(∠FIZ) (opposite angles) → m(∠CID) = y = m(∠LAT) (angles made by parallel lines, CI ∥ AD, AD=AL (same line, because L is a point on AD), then CI ∥ AL. AT ∥ ID (because AT ∥ YZ, because it's a square, and YZ is the same as ID), and since ∠LAT and ∠CID are made by the intersection of two sets of parallel lines, they have the same measure)

ΔLAT and ΔCID are right triangles

AT = ID (AT=YZ, which in turn is equal to ID, as shown above),

∠LAT ≡ ∠CID → ΔLAT ≡ ΔCID (4)

**Proof of (1)**

ABCD = rectangle → AB ≡ CD ≡ LT (from (4) we get that LT ≡ CD)

∠BGA and ∠AGF are supplementary angles, ∠LET and ∠LEF are also supplementary angles.

Because they're made of two sets of parallel lines (GF ∥ LE (again, GF is the same as BC, and LE is the same as AD, and AD ∥ BC), we get that:

m(∠AGF) = m(∠LEF) = 90˚ + y → ∠BGA ≡ ∠LET

ΔABG and ΔTLE = right triangles (AB ⊥ BG, TL ⊥ LE, the last one is from the definition of point L), AB = LT → ΔABG ≡ Δ TLE. (1)

From all the congruences proven above

(AEFG ≡ AEFG(5), ABG ≡ TLE(1), EJKIF ≡ GFZON(2), JDK ≡ NOY(3), DCI ≡ TLA(4))

we get that ABCD⁎XYZT, which also means that A'B'C'D'⁎XYZT and XYZT⁎A'B'C'D'.

Therefore, any rectangle can be transformed into a square of equal area, and a square can be transformed into any rectangle of equal area. In other words, any rectangle can be transformed into another rectangle of equal area.

In conclusion, this is the procedure (not necessarily the best one) to dissect a polygon A into smaller polygons which rearranged form B.

**Recap:**

1. We start by cutting A in triangles (we proved at I that it's possible for any given polygon)
2. We transform each triangle firstly in a parallelogram of equal area, and then in a rectangle of the same area (we proved during part 2.2 that it's possible for any triangle).
3. We transform each rectangle in a square of equal area, which will be then transformed into a rectangle with one side of length 1 (or any other length as long as it's the same throughout the whole process) (we proved at 2.3 that it's possible for any square). We put together all the rectangles with one side of length 1 to form a bigger rectangle (C) with one side of length 1 and the same area as A.
4. We repeat steps 1), 2) and 3) for polygon B, to obtain a rectangle with one side of length 1 and the same are as B (C'). Because B and A have the same area, rectangle C and rectangle C' will be the same rectangle. In order to obtain B from A, we repeat the steps in reverse order.

**Now we prove why we need $\frac{b}{c} \in (\sqrt{5}, \sqrt{10})$ .**

In order to consider the pentagon EJKIF, the point I needs to be more to the left of K (YI < YK).

In ΔGIY we observe that $YI = ctg(y) \cdot GY = ctg(y) \cdot (c-d)$, NY = JK, NY ∥ JK

→ NYKJ is a parallelogram → YK = NJ, YK ∥ NJ, YK ⊥ AY → NJ ⊥ AY

→ ΔANJ is a right triangle

m(NAJ) = 90˚ - y, m(JNA) = 90˚ → m(NJA) = y

→ NJ = cos(y)·AJ = cos(y)·2*l*, YK = cos(y)·2*l* (*l* is the length of AE and of EJ)

$\cos(y) \cdot 2l > \frac{\cos(y)}{\sin(y)}(c-d)$

$\cos(y) \cdot 2l \cdot (\frac{d}{l}) > \cos(y) \cdot (c-d)$

$2d > c-d \rightarrow 3d > c \rightarrow 3 > \frac{c}{d}$, $d = \frac{a}{\cos(y)} \rightarrow 3 > c \cdot \frac{\cos(y)}{a}$

$\frac{c}{a} = \frac{b}{c} = \frac{1}{\sin(y)} \rightarrow 3 > \frac{\cos(y)}{\sin(y)} \rightarrow 3 > \text{ctg}(y)$

In order to consider the pentagon GNOZF, F needs to be higher than Z (F ϵ [TZ]). We will consider x to be the distance (with sign) from F to Z (positive if F is higher than Z, and negative if F is lower than Z).

TE + EF +FZ = TZ

TE = EF = d, TZ = c

2d + FZ = c, and we want FZ > 0 → 2d < c → $2 < \frac{c}{d}$ and, through the same steps as in the previous case, we obtain 2 < ctg(y).

ctg(y) ϵ (2, 3), y ϵ (0˚, 90˚), ctg is a monotonically decreasing function on (0˚, 90˚) → y ϵ (arcctg(3), arcctg(2))

sin is a monotonically increasing function on (0˚, 90˚)

→ sin(y) ϵ (sin(arcctg(3)), sin(arcctg(2))), therefore sin(y) ϵ $(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{5}})$

$\sin(y) = \frac{c}{b} \rightarrow \frac{b}{c} \in (\sqrt{5}, \sqrt{10})$

**EDITION NOTES**

[1] The result presented in the work is a known as the Wallace–Bolyai–Gerwien theorem. This should have been mentioned in the paper.

[2] This notation is used only in Example 2, which is rather obvious. It is not advisable to introduce a new notation and to use it only once or twice.

[3] This result does not deserve a detailed proof. The two pictures give a precise idea of the proof and are enough. I think that very similar constructions can be found in many textbooks in basic geometry.

[4] The implication symbol → is frequently used in the work, also outside formal expressions. In these cases, it is advisable to use English language expressions like: "then", "and hence", "which proves", ....

[5] It should be explicitly observed that this conclusion holds because the sequence $\left(\frac{6}{5}\right)^{\alpha}$ is strictly increasing and has no upper bound.

More important, the proof needs to be fixed. Inequality (2) derives from (3) and hence the order of these two inequalities should be inverted. Moreover, (1), (2), and (3) do not lead to a contradiction. We get a contradiction by considering the original assumption $\frac{b'}{c} \cdot (\frac{6}{5})^{\alpha'+1} \geq \sqrt{10}$ and (2), which holds as a strict inequality because it is a consequence of (3).

A simpler proof can be the following: $\frac{b'}{c} \cdot \left(\frac{6}{5}\right)^{\alpha'} \leq \sqrt{5}$ implies $\frac{b'}{c} \cdot (\frac{6}{5})^{\alpha'+1} \leq \left(\frac{6}{5}\right) \cdot \sqrt{5}$ , and hence, since $\frac{6}{5} < \sqrt{2}$ , $\frac{b'}{c} \cdot (\frac{6}{5})^{\alpha'+1} < \sqrt{10}$ . This contradicts $\frac{b'}{c} \cdot (\frac{6}{5})^{\alpha'+1} \geq \sqrt{10}$.

Similar observations hold for Case II.

[6] The proofs below are rather complex. It is very likely that simpler versions can be found. Perhaps, a simplification can be obtained by preliminarily consider suitable classes of similar triangles and use these similarities in the proofs.

[7] The equality of angles is not a sufficient condition for the similarity of polygons with more than three sides. An explicit proof that the corresponding sides are congruent should be given. A simpler and correct proof is likely to be obtained by dividing the pentagon GFZON into three triangles: GFN, NFO, and FZO, and similarly for EJKIF.