The Largest Building

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1 Problem Statement

The researched problem brings into the spotlight one of the ways of building construction.

First of all, there are disposal building blocks of rectangular parallelepipedic shapes and of dimensions L = 60 cm and l = 20 cm and some depth h = 10 cm.

The main questions are:

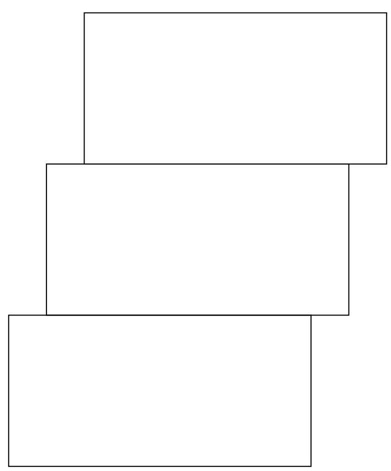
Can we place n blocks, on a planar floor, on top of each other without them collapsing such that the length L_n of this construction is 10 meters long horizontally?

If yes, how many pieces do we need for this?

Is it possible to achieve a length L_n of 100 meters without the blocks collapsing?

What is the minimal number of pieces n necessary for such a construction?

What would be the vertical height of such constructions?



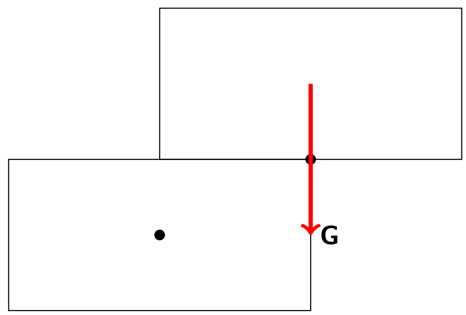
2 Conditions for Equilibrium

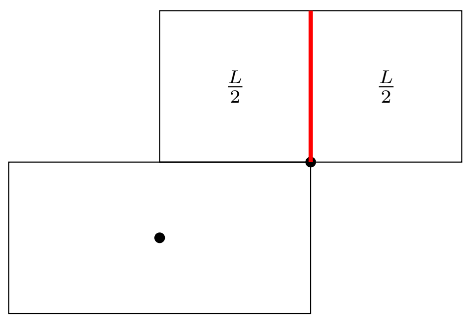
In the first place, it is important to notice the necessary conditions for the equilibrium of the bricks.

As all bricks have the same sizes and have an homogenous mass, the center of gravity is identical to the center of mass.

Consider a stack of $n + 1$ blocks, each of length L, disregarding the negligible heights 1 and h for simplification. To achieve a balanced state, the center of gravity must coincide with the center of mass for the structure. Therefore, all blocks possess an identical mass distribution.

Furthermore, in order to maximize the overhang, the center of gravity of the construction formed by the $n - 2$ blocks above must be positioned directly above the bottom block. This intriguing condition allows us to introduce additional stability to the structure.





3 Extending the Overhang

To enhance the stability of the stacked blocks, we introduce one more block at the bottom, positioned precisely under the last block. By doing so, we can shift the entire top $n-1$ blocks to the right, resulting in an overhang of $x = \frac{L}{2(n-1)}$ units. This lateral displacement balances the increased mass on the right side of the structure, maintaining equilibrium.

4 Maximum Length of Stacked Blocks

Now, let us denote by $s(n)$ the maximum length of an n-block building while in equilibrium. Starting from a single block ($n = 1$), $s(1)$ is simply equal to L. For $n = 2$, $s(2)$ becomes $L + \frac{L}{2}$. This can be verified by the previous overhang extension, where the structure consists of two blocks with a half-length overhang each.

Moving forward, we find the recursive relation for the maximum length of n-block building: $s(n) = s(n-1) + \frac{L}{2(n-1)}$. This result provides a method to determine the maximum achievable length of a stacked block structure with overhang.

5 Mathematical construction rule

5.1 Mathematical Formula

To construct a stack of bricks with n elements in the most efficient way, we introduce the following mathematical formula:

$$s(n) - s(n-1) = \frac{1}{2n}$$

Here, s represents the maximum overhang, and n denotes the number of bricks in the stack.

5.2 Formula's Demonstration

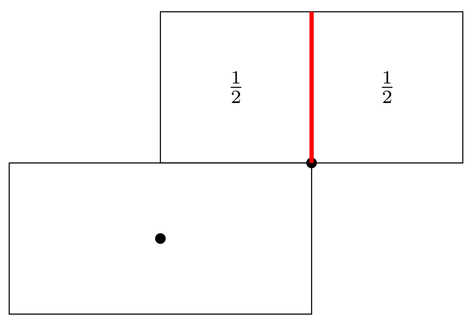
The validity of the formula can be established using mathematical induction. The maximum overhang can be interpreted as the sum of overhangs between each pair of consecutive bricks. We begin with the base case and then demonstrate the formula's validity for the inductive step.

The statement $p(n)$ will be:

$$p(n) : s(n) - s(n-1) = \frac{1}{2n}$$

5.2.1 Base Case

Let's start with the simplest scenario of two bricks stacked one above the other. To achieve the maximum overhang, the gravitational force of the top brick must be precisely aligned with the edge of the bottom brick.



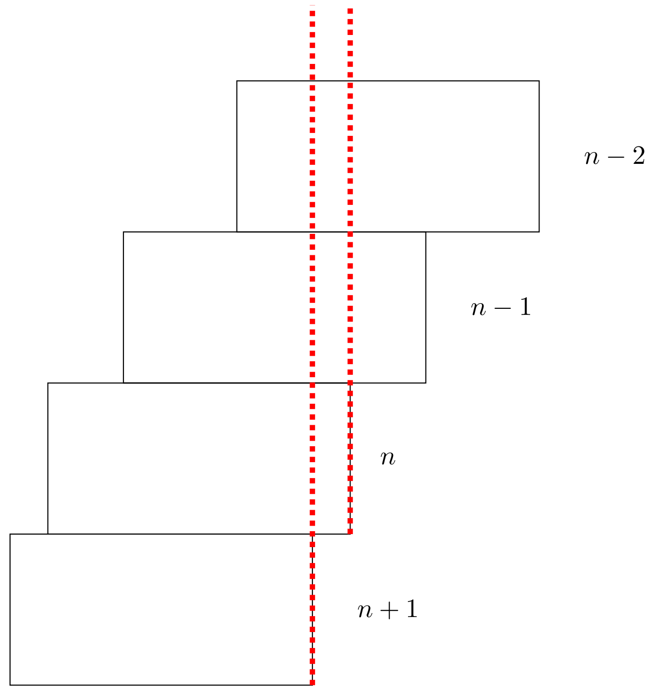
For this configuration of two bricks, the formula holds:

$$s(2) - s(1) = \frac{1}{2}$$

5.2.2 Inductive Step

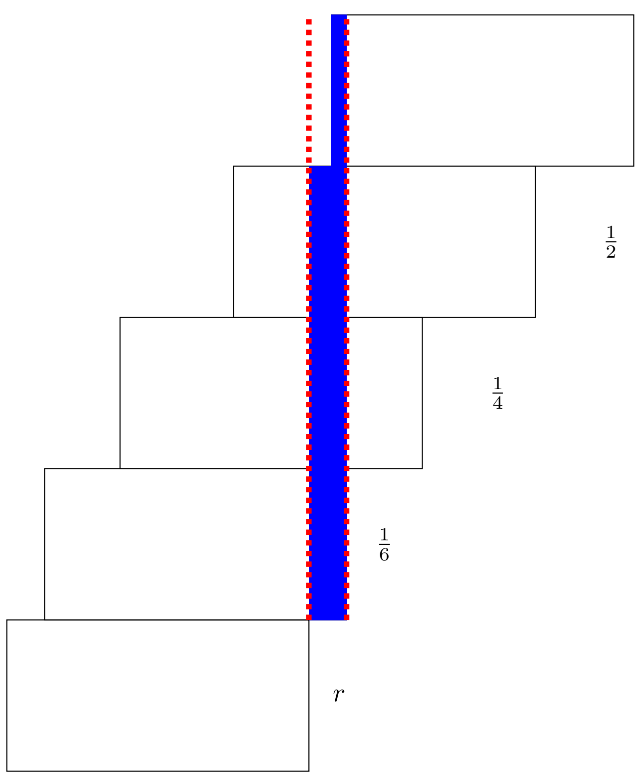
Assume that the formula $p(n) : s(n) - s(n-1) = \frac{1}{2n}$ is valid for $n-1$ bricks, and we want to prove it for n bricks.

Consider a stack of $n+1$ bricks, and let's separate the constructions made out of n and $n-1$ bricks, respectively, using vertical lines. The "brick surface" between these two lines, which divides the structure into two equal surfaces, must have an overhang of $\frac{1}{2}$ for equilibrium (by simple math, the left-hand side gains one surface unit and has to give up $\frac{1}{2}$ of a surface unit from its own surface back to the right-hand side to balance).



Hence, initially, we have:

$$s(n) - s(n-1) = \text{newly generated overhang}$$
$$= \frac{\frac{1}{2}}{n} = \frac{1}{2n}$$



However, for $n \geq 5$, the number of bricks within the two vertical lines does not always remain $n-1$. The quantity r represents the overhang of the $(n-2)$ brick:

$$r + \left(1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \ldots - \frac{1}{n}\right) = \frac{1}{2}$$

Solving for r, we get:

$$\begin{aligned} r &= \frac{\frac{1}{2} - \left(\frac{1}{2} \cdot \frac{1}{n}\right)}{n-2} \\ &= \frac{\frac{n-1}{2n}}{n-2} \\ &= \frac{1}{2n(n-2)} \end{aligned}$$

Thus, the new generated overhang is modified to:

$$\begin{aligned} \text{newly generated overhang} &= \frac{\frac{1}{2}}{n} + r \\ &= \frac{\frac{1}{2}}{n} + \frac{1}{2n(n-2)} \\ &= \frac{1}{2n}\left(1 + \frac{1}{n-2}\right) \end{aligned}$$

Finally, we have:

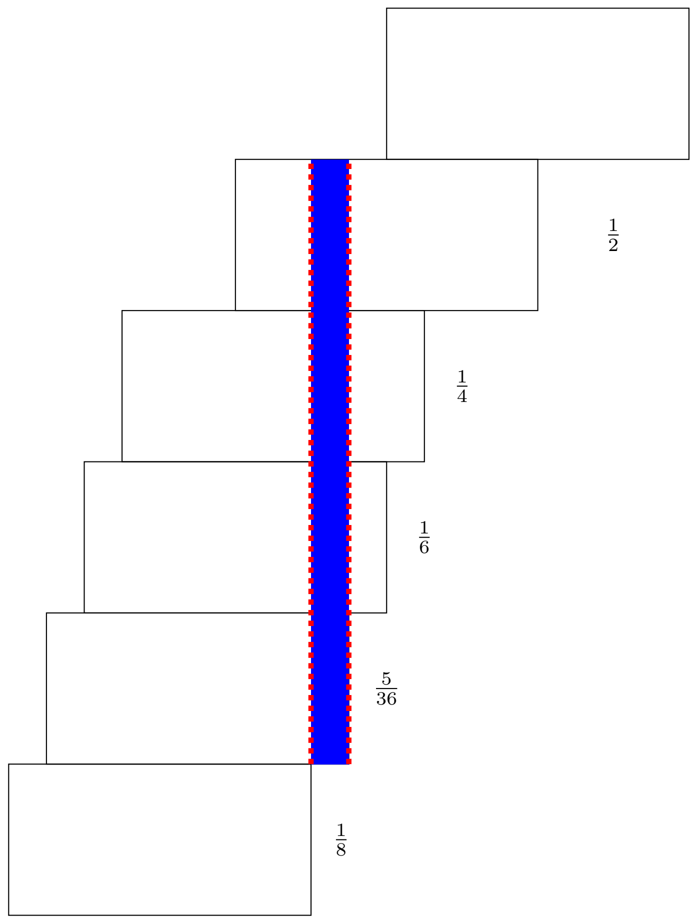
$$s(n) - s(n-1) = \frac{1}{2n}\left(1 + \frac{1}{n-2}\right)$$

which indeed satisfies the formula for $p(n)$.

$$3r + \left(1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6}\right) = \frac{1}{2}$$

$$r = \frac{\frac{1}{2} - \frac{1}{12}}{3}$$

$$r = \frac{5}{36}$$



5.2.3 Conclusion

Based on the invalidation of the inductive step, it can be said that the initial statement is not true.

It can be seen that the terms in the real amount can be approximate to those used in the initial formula. So even if the initial formula is not fully true, it can be used as a very close approximation of what is happening in reality.

6 The length of the construction

For calculating the maximum overhang of such type of construction, it is essential to calculate the limit of the sum of bricks length. The sum:

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

This sum it's known as the harmonic series and it has the property of being greater than or equal to any natural number, for a great enough n.

For finding the limit of this sum, it is enough to increase all denominators that are different from a power of two to the smallest power of two that is bigger than the denominator. In this way, it is obtained a sum which is smaller than the initial one.

$$h(n) \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \dots$$

So, it can be said that:

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \dots\right) = 1 + \frac{1}{2} + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 4 + \dots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \to \infty$$

This means that any amount of overhang can be achieved.

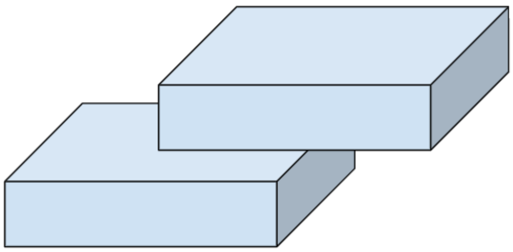
7 Another approach

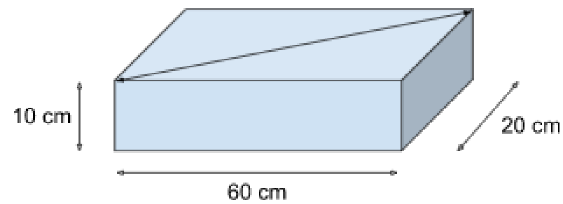
It can be noticed the interesting property the use of three-dimensional bricks allows for. Because of the fact that it's enough for the gravitational force to rest on the corner of the brick below, the bricks can be placed in a diagonal manner. So, using the length of the diagonal as the unit for the brick length.

$$L = \sqrt{60 \cdot 2 + 20 \cdot 2} = 63.245$$

,

it can be obtained:





For calculating the maximum horizontal length of a construction, it is required to use the following formula:

$$L(n) = L \cdot (s(n) + 1)$$

If the desired length is equal to 10 meters, then:

$$\begin{aligned} s(n) &= \frac{L(n)}{L} - 1 \\ &= \frac{10 \cdot 100}{63.245} - 1 \\ &= 14.811 \end{aligned}$$

There's a big chance that the sum won't exactly equal that value, for any n. Therefore, it is enough to find the smallest value of n such that

$$s(n) > 14.811$$

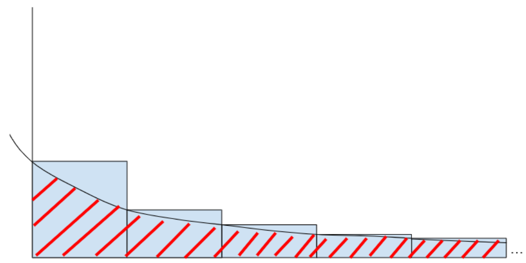
$$s(n) > 14.811 \Leftrightarrow h(n) > 14.811 \cdot 2 = 29.622$$

7.1 Harmonic Series

It can be observed that h(n) divided by the natural logarithm of n, when n approaches infinity, is equal to 1.

This means that, for great enough values of n, the harmonic series with n elements and the natural logarithm of n actually tend to be equal with one another.

This can also be intuited using the function graphic



The blue rectangles are constructed such that their heights are equal to 1 unit, $\frac{1}{2}$ units, $\frac{1}{3}$ units and so on, with the first rectangle being a square.

The sum of the blue areas will equal the harmonic series.

Moreso, the excluded blue area will be equal to the Euler-Mascheroni constant, which is about 0.577215.

In order to generate a as-close-to-the-real-value-as-possible solution to our problem (

$$h(n) = 29.622$$

), we need to subtract the Euler-Mascheroni constant from 29.622 and raise e to that power.

4 114 592 580 543 is the value needed.

This means that a construction built using

$$60 \times 20 \times 10$$

bricks with a 10 meter overhang will require 4 114 592 580 543 bricks and will be 411459258054,3 meters high, comparable to the size of R Doradus, the star with the second largest apparent size, after the Sun.

For $L(n) = 100m$, the construction would most likely be unfathomably large, maybe even larger than the universe itself, but that doesn't mean that a natural value for n wouldn't exist (it would be around the value of 16 470 366 520 057 984 503 527 314 857 155 023 784 848 798 406 018 179 865 851 203 749 620 497 020 980 193 828 295 256 672 205 140 728 010 613 775 281 655 797 777 645 546 089 587 277, which is a rather big value).

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Harmonic Series:

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