Gridland: travelling and planning using Manhattan geometry and related problems

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Abstract

Since ancient times, the grid street map has been widely used in the urban planning of cities. Examples include ancient Giza, Babylon, Rome, as well as modern cities like Manhattan, Barcelona, and Lyon. In these cities, the streets run at right angles to each other, forming a grid. The frequent intersections and orthogonal geometry facilitate movement, orientation, and wayfinding.

In this article, we address some problems associated with such a grid street map.

The first chapter deals with the shortest path problem. When all segments of the grid are of uniform length, there are multiple shortest paths between two points, all with the same total length. The challenge here is to count the number of these shortest paths. When the segments have different lengths, the paths will generally also have different lengths, and the problem becomes finding the shortest path.

In the second chapter we model the grid as a metric space using Manhattan distance, considering only the intersections as points of interest. After developing some classical geometric notions in this setting, we apply them to address some urban planning issues. In particular, we introduce a new definition for the straight line and provide two algorithms to compute the resulting point-line distance. We then study conics and their properties in this geometry, applying them to solve some planning issues. With another geometric locus, the axis of two points, we address the problem of dividing the city between hospitals and determining the minimum number of hospitals needed to cover the city to ensure rapid medical aid.

Finally, in the last chapter we attempt to generalize the space by considering all points on the grid as points of interest.

Contents

Chapter 1

Minimal Paths

1.1 Counting the number of minimal paths

Our first task is to compute the possible ways to get from a point A to a point B in a city that has only streets perpendicular to each other, where the intersections of the streets represent the points of tourist interest of the city.

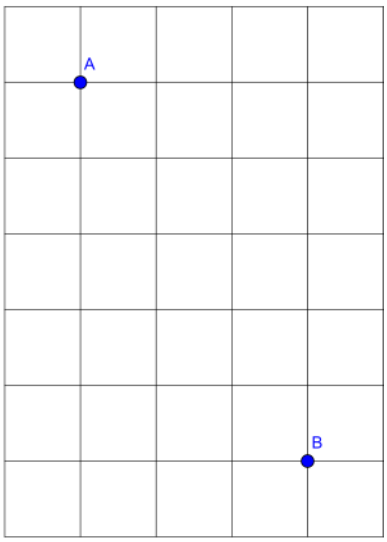


Figure 1.1: Example of the grid and two points on it

We found a way to answer the following questions:

Question 1 How many different ways are there to get from a point A to a point B?

Question 2 How many ways to get from a point A to a point B going through a point C?

Question 3 How many ways to get from a point A to a point B going through a street CD?

Question 4 How many ways to get from a point A to a point B without crossing a point C?

Question 5 How many ways to get from a point A to a point B without crossing a street CD?

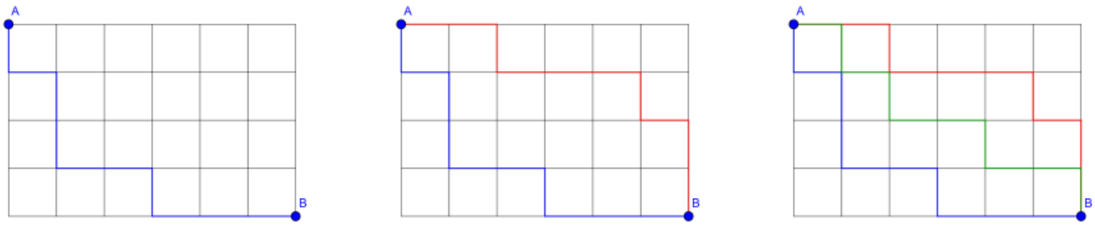
Question 6 How many possible ways to get from A to B withoutcrossing the points C and D?

Question 7 How many ways to get from a point A to a point B crossing a diagonal CD?

1.1.1 Question 1: How many different ways are there to get from a point A to a point B?

We found two methods to answer to this question.

- **First idea:** Looking at the Figure 1.2, starting from the top-left point A suppose we want to arrive at the bottom-right point B. Provided that going up or left surely leads to a non minimal path, we noticed that there are multiple ways to arrive at the point B and that each path is described by a permutation with repetitions of the letters D *(down)* and R *(right)* like in the following images:



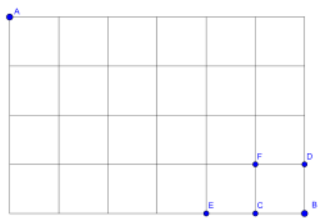
DRDDRRDRRR *RRDRRRDRDD* *RDRDRRDRRD*

Figure 1.2: Examples of ways to connect two points

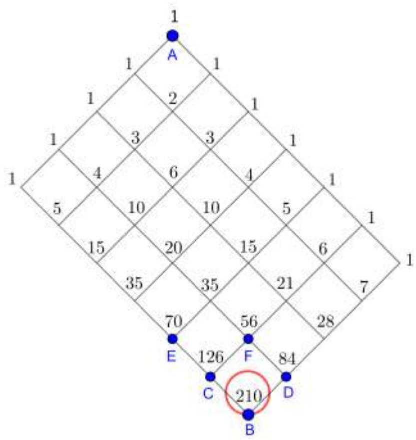
So, given a city with v vertical blocks and h horizontal blocks, the numbers of different minimal paths to get from A to B are

$$n_{AB} = \binom{v+h}{v} = \frac{(v+h)!}{v! \cdot h!}$$

- **Second Idea:** We calculated all the paths that arrived from A to C and from A to D. Then there is only one way to pass from C to B, and the same from D to B. Similarly, to compute n_{AC} we have to add $n_{AE} + n_{AF}$ and so $n_{AB} = n_{AC} + n_{AD}$.



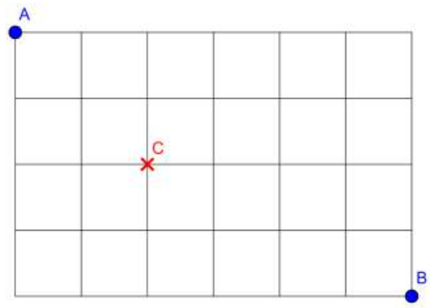
The idea is the same as that of the Pascals triangle:



1.1.2 Question 2: How many ways are there to get from a point A to a point B going through a point C?

In order to understand how many possible ways there are to go from a point A to a point B passing through a point C we realized that every path that goes from A to C can be continued with every paths that goes from C to B. That is why we use the multiplication to get the total number paths:

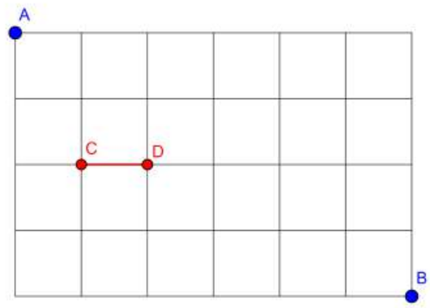
$$n_{ACB} = n_{AC} \cdot n_{CB}$$



1.1.3 Question 3: How many ways are there to get from a point A to a point B going through a street CD?

Then we tried to imagine to have in our "Gridland" a street where we are obliged to pass through. How many possible paths there are to or from a point A to a point B passing through a street CD? The criterion is the same of the previous one but in this case we have to multiply the paths that go from A to C (the start of the path) for the paths that go from D (the end of the path) to B.

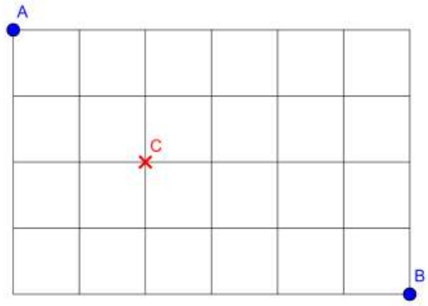
$$n_{ACDB} = n_{AC} \cdot n_{DB}$$



1.1.4 Question 4: How many ways are there to get from a point A to a point B without going through a point C?

Once we found the possible paths to go from a point A to a point B without passing through a point C, we tried to calculate the paths that go from A to B without passing through a point C. We immediately get

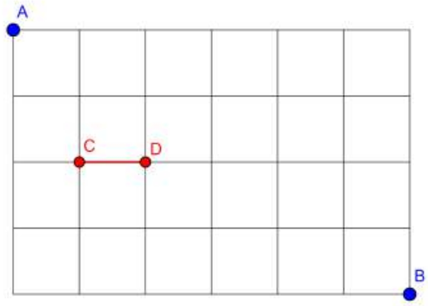
$$n_{AB\setminus C} = n_{AB} - n_{AC} \cdot n_{CB}$$



1.1.5 Question 5: How many ways are there to get from a point A to a point B without going through a street CD?

The criterion is the same of the previous one, and we have

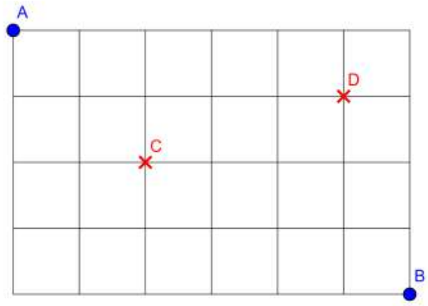
$$n_{AB\setminus CD} = n_{AB} - n_{AC} \cdot n_{DB}$$



1.1.6 Question 6: How many possible ways are there to get from A to B without going through the points C and D?

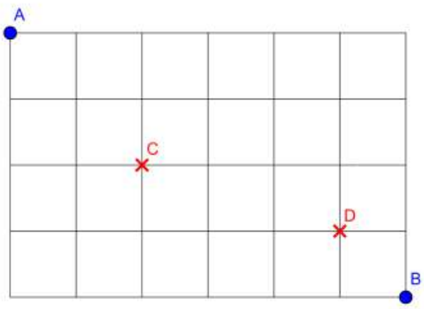
Now the case becomes more difficult because we have to differentiate the problem into two cases: In the first one there are no minimal paths that cross both the two points. In this case we can proceed with the same logic used to answer the question to do not pass trough a point, using this formula:

$$n_{AB\setminus C,D} = n_{AB} - n_{AC} \cdot n_{CB} - n_{AD} \cdot n_{DB}$$



The other case happens when there is some minimal path that goes from A to B crossing both the points C and D. In this case we have to do a series of passages to find out how many possible paths there are to go from A to B without passing through C and D. First we have to multiplicate the paths that go from A to C for the paths that go from C to B. Second we have to multiplicate the paths that go from A to D for the paths that go from D to B. Then we have to add some paths that we have removed more that one times. So we have to add the paths find by the multiplication between the possible paths that go from A to C, the paths that go from C to D and the paths that go from D to B. So we have summarized this case in this formula:

$$n_{AB\setminus\{C,D\}} = n_{AB} - n_{AC} \cdot n_{CB} - n_{AD} \cdot n_{DB} + n_{AC} \cdot n_{CD} \cdot n_{DB}$$

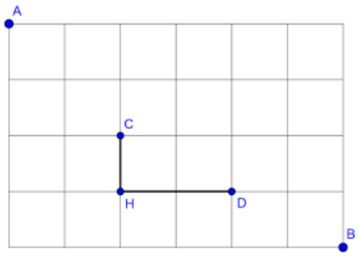


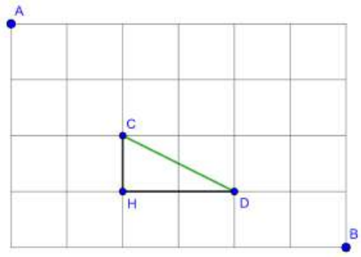
1.1.7 Question 7: How many ways are there to get from a point A to a point B going through a diagonal CD?

This is a generalization of the Question 3. In fact we could consider a diagonal like the street that form the right triangle, with the diagonal as hypotenuse. To calculate the possibilities to go from A to B we can use the formula:

$$n_{ACDB} = n_{AC} \cdot n_{DB}$$

because every path that go from A to C can be continued by every every path that start from the end of the diagon





1.2 Minimal path with weighted edges

So far, a constant length has been assumed for the various segments building up the grid. We shall now attempt to generalize this by allowing each edge to possess a specific length, resulting in weighted edges. Unlike the previous case, here different paths will have in general different lengths, so the goal here becomes to find the shortest path.

1.2.1 Dijkstra's algorithm

In this setting, Dijkstra's algorithm is a procedure that finds the minimum distance between one specific point, called starting *node*, and all the other points in the map. The algorithm presented here, despite its similarity to the original one, has undergone slight modifications, which, however, do not affect the original concept: avoiding the recursion, it finds the shortest path in a graph, which in this particular case is the grid made up of streets perpendicular to each other, whose weight can model some notion of distance (length of the street, time required to go through, cost of the travel, ...).

The map is a grid (i.e. a lattice graph) where every node has these attributes:

- *name*: we used a integer in an ascending order

- *distance from the start*
- *previous node* (of the shortest path from the start)

Every node is linked to others by streets, each having its own weight (chosen randomly between 1 and 9, just to make it easier for us to check the process). The streets that are beyond the borders of the map have a infinite value, since they must not be analyzed. Also once the streets are analyzed their value becomes infinite so that they do not get explored again, avoiding dead-locks.

The starting node is chosen at the beginning (we take 1 in the example). A list keeps track of the current estimated minimal distance between the starting node and each other node; this list is always kept ordered with respect to the value of the distance (1). At first all nodes have infinite distance, apart from starting node which have distance 0 from itself. Hence the starting node is always the first to be analyzed.

The following example illustrates how the algorithm works.

1.2.2 Dijkstra's working example

The city that we are going to consider is depicted in Figure 1.3: it is a square 3×3 grid, where the white dots represent the nodes, numbered in ascending order, and the blue segments denote the streets with their weights.

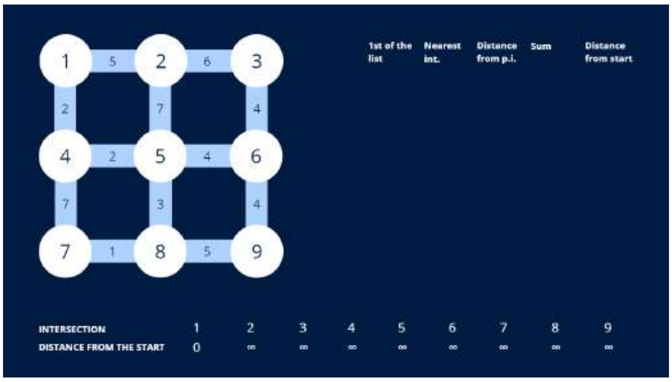


Figure 1.3: Initial configuration

The algorithm starts analyzing the first node of the list (the starting node "1" in the example). Then it compares all the available roads connecting the point "1" to the adjacent nodes (see Figure 1.4, in order to select the shortest one. In this case, the shortest one is the road linked to the point "4", which will be the next one to be analyzed.

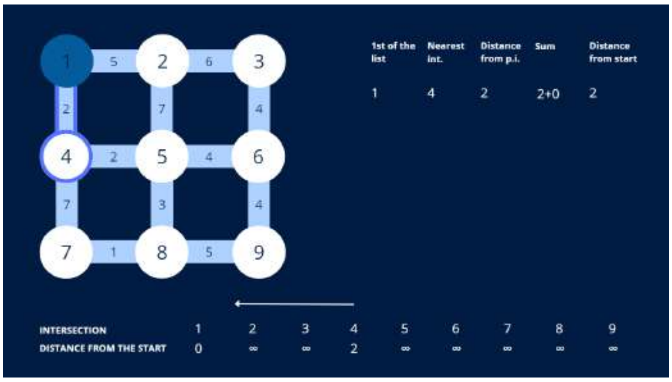
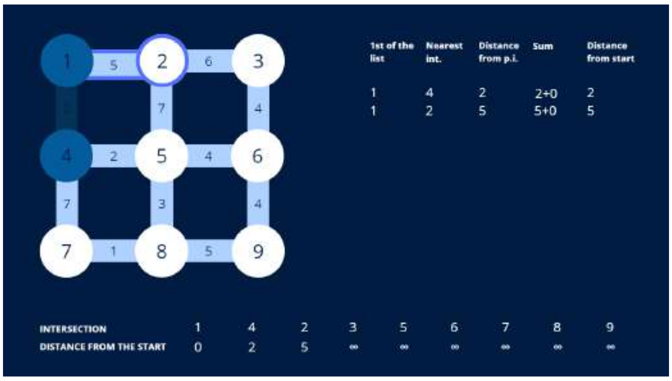


Figure 1.4: Comparing nearest-neighbour nodes

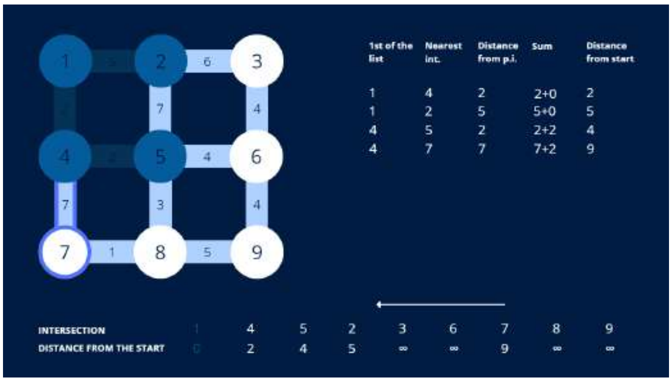
Now the distance from the start of the point "4" equals the sum of the distance from the beginning of the previous point, and the weight of the road that links the previous to the current one.

If this distance is less than the one already memorized, it will be updated to the new one (in case this point has never been analysed, the saved distance is infinite so the new one is always less). In this case, the infinite distance updates to (0+2=2). Then also the list will be updated and “4” will be placed right after the point "1".

Since the point “1” is still the first one in the list, the algorithm analyses the road 1-2 and the point “2” like it did for the road 1-4 and the point “4”.

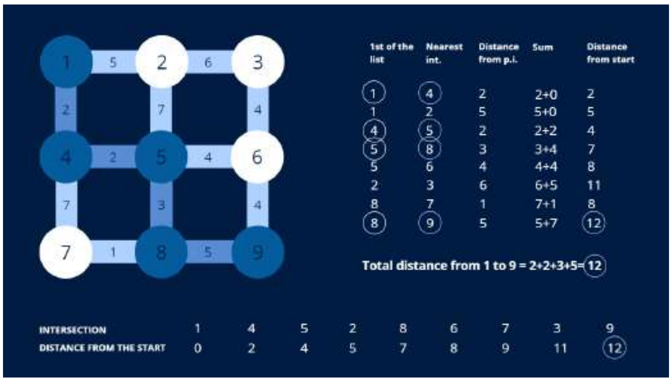


So the algorithm finds that the distance between the start and the point “2” is (0+5=) 5, and the list is updated. Now, the algorithm returns back to the first point of the list, which is still “1”, but it will not find any available road to be analyzed, so it ignores the first one and moves on to the second one in the list, “4”.



This same procedure now applies to all the other points remaining in the grid, until the list of points and distances from the start is completed and all the roads have been analyzed. At the end the values memorized in the list represent the minimal distances from the starting node to the node.

Once the algorithm has already analyzed all the roads, we consider the arrival node. Since all the points have been analyzed and have not only a shortest distance from the start but also a previous point, the algorithm can follow the steps from the arrival back to the starting point and reconstruct the shortest path.



In this example, choosing "1" as the starting point and "9" as the arrival, the shortest path is the one going through the nodes "1", "4", "5", "8", "9", whose length is 12. We can notice that the path going through "6" instead of "8" has the same length; in a case like this the algorithm will only memorize the first, since the path changes only when a strictly shorter distance is found.

Chapter 2

Urban planning issues

2.1 The metric space $\mathbb{Z}^2$ with Manhattan distance

Definition 1. *The **plane** is defined as the metric space $(\mathbb{Z}^2, d)$; its elements $(x, y) \in \mathbb{Z}^2$ are called **points**. The **distance function** d between two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ is defined as*

$$d(A,B) = |\Delta x| + |\Delta y| = |x_A - x_B| + |y_A - y_B| \tag{2.1}$$

This distance function measures the length of the shortest path between the two points A and B, when only the vertical or horizontal streets of the grid are allowed (for this reason it is called *Manhattan distance* or taxicab geometry).

Theorem 1. *The distance function satisfies the following properties:*

I) $d(A,B) > 0 \iff A \neq B$

II) $d(A,B) = 0 \iff A = B$

III) $d(A,B) = d(B,A)$

IV) $d(A,B) \leqslant d(A,C) + d(C,B)$

Proof. We are going to prove each point of the theorem. I, II) By eq. 2.1 the function is computed as the sum of non-negative quantities so $d(A,B) \geqslant 0$ and in particular $d(A,B) = 0 \iff x_A = x_B$ and $y_A = y_B$, therefore the first two properties are satisfied.

III)

$$d(A,B) = |x_A - x_B| + |y_A - y_B| = |x_B - x_A| + |y_B - y_A| = d(B,A)$$

for the definition of absolute value $|x| = |-x|$ therefore the third property is satisfied.

IV)

$$|x_A - x_B| = |x_A - x_C + x_C - x_B| \leqslant |x_A - x_C| + |x_C - x_B|$$

$$|y_A - y_B| = |y_A - y_C + y_C - y_B| \leqslant |y_A - y_C| + |y_C - y_B|$$

Adding member-wise the two inequalities we get

$$|x_A - x_B| + |y_A - y_B| \leqslant |x_A - x_C| + |y_A - y_C| + |x_C - x_B| + |y_C - y_B|$$

and therefore

$$d(A,B) \leqslant d(A,C) + d(C,B).$$

Then the fourth property is satisfied. □

Definition 2. *A **straight line** is defined as the non-empty set of the intersection points between the plane and the euclidean straight line.*

In order to have points, the angular coefficient and the constant term must be rational. Hence the equation of the straight-line can be written as

$$y = \frac{p}{q}x + \frac{s}{t} \qquad x, y, p, q, s, t \in \mathbb{Z} \quad q \neq 0 \text{ and } t \neq 0$$
$$qty = ptx + sq$$

Let $a = pt,\ b = -qt,\ c = sq \in \mathbb{Z}$, then the straight-line can be represented by the equation

$$ax + by + c = 0 \qquad a, b, c \in \mathbb{Z}$$

Moreover, by Bézout's Identity we get that $-c$ must be a multiple of the $GCD(a, b)$.
Infact, denoted with d this GCD, we have $a = dh$, and $b = dk$ for some integers $h, k \in \mathbb{Z}$. Hence

$$-c = ax + by = dhx + dky = d(hx + ky)$$

The previous constraint is trivially satisfied when we assume without loss of generality that a and b are coprime.

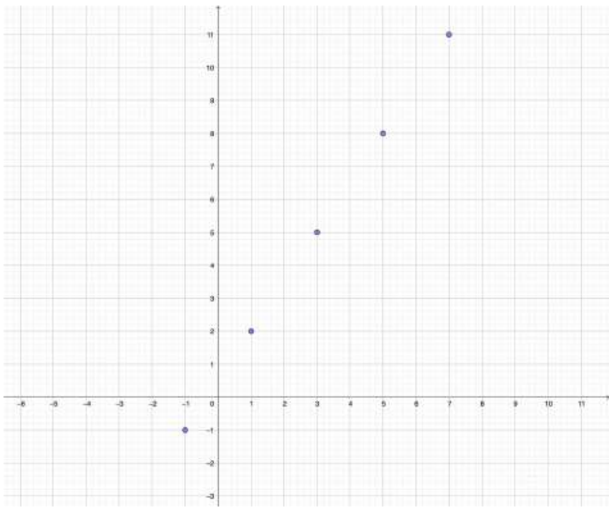


Figure 2.1: Example of a straight-line in $\mathbb{Z}^2$

2.2 Point-line distance

Definition 3. *The **point-line distance** is defined as the minimum distance between the point P and the points of the straight-line r.*

In order to establish the value of the distance point line, as there is no closed forumula for it, we have developed two algorithms that are based on the same idea but differ in the procedure.

2.2.1 Point-line distance first algorithm

We provide first an example to illustrate the procedure which will be generalized afterwards. Given a point $P = (6,1)$ and a straight line $r : y = \dfrac{3}{2}x + \dfrac{1}{2}$,

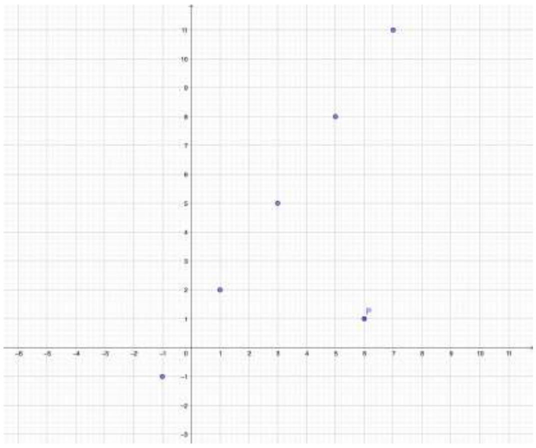


Figure 2.2: Graph of the straight-line $r(x)$ (isolated points) and P

Let's now define the function that expresses the distance of each point of the straight-line from P:

$$d(x) = |x-6| + \left|\frac{3}{2}x - \frac{1}{2}\right| \tag{2.2}$$

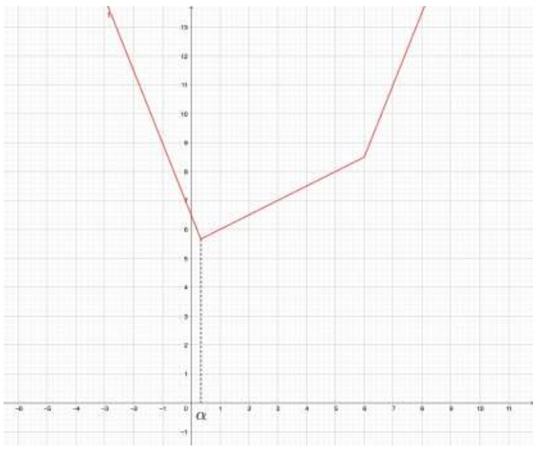


Figure 2.3: Graph of $f(x)$ the point-line distance function

As we can see from the graph, this function presents a minimum in α, however the point with this abscissa could not belong to the straight-line. In fact, in this case $\alpha = \dfrac{1}{3}$ so it cannot belong to the straight-line, but it will certainly lies between two points that belongs to r. In order to find these two points we consider the integer part of α and we subtract a unit iteratively until its image on the straight line is an integer as well.

$$r(\lfloor\alpha\rfloor - k) \quad \text{with } k \in \mathbb{N}$$

In this case $r(\lfloor\alpha\rfloor - 1) = -1$ and consequently we define the point $A = (-1,-1)$.

From the equation of the straight-line we have Δx (2) and we define the other point $B = (\lfloor\alpha\rfloor - 1 + \Delta x, r(\lfloor\alpha\rfloor - 1 + \Delta x)) \Rightarrow B = (1,2)$.

So the value of the point-line distance, recalling the distance point-point 2.1, is

$$d_{P,r} = \min(d(P,A), d(P,B)) = \min(9,6) = 6 \tag{2.3}$$

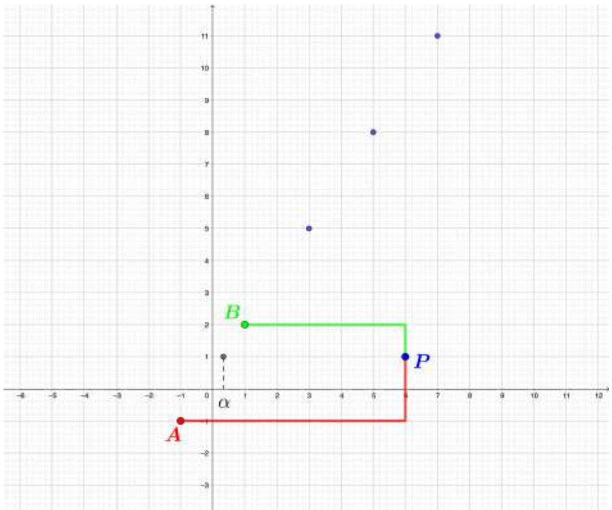


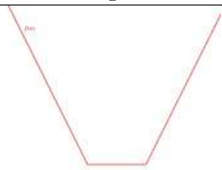
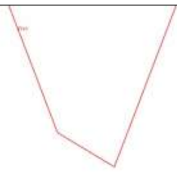
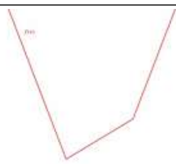
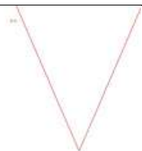
Figure 2.4: Distances $d(P, A)$ and $d(P, B)$

Generalization of the algorithm

Given a point $P = (x_P, y_P)$ and a straight-line $r : ax + by + c = 0$ we consider the function that expresses the distance of each point of the straight-line from P,

$$d(x) = |x - x_P| + \left| -\frac{a}{b}x - \frac{c}{b} - y_P \right| \tag{2.4}$$

This function can present 4 different graphs:

Point position	Coefficient	Graph
	$\left\|-\frac{a}{b}\right\| = 1$	
$y_P > r(x_P)$ $y_P < r(x_P)$	$-1 < -\frac{a}{b} < 0$ or $-\frac{a}{b} > 1$ $-\frac{a}{b} < -1$ or $0 < -\frac{a}{b} < 1$	
$y_P > r(x_P)$ $y_P < r(x_P)$	$-\frac{a}{b} < -1$ or $0 < -\frac{a}{b} < 1$ $-1 < -\frac{a}{b} < 0$ or $-\frac{a}{b} > 1$	
$y_P = r(x_P)$		

Each of these graphs presents exactly one minimum that we call α, apart from the first case that we analyse in the end.

Since the point with abscissa α may not belong to the straight-line in $\mathbb{Z}^2$, if either α or its image $r(\alpha)$ is not an integer, let's look for the two points that belongs to r that lie next to α. In order to do that consider the integer part of α and its image on $r(x)$, if this one is not an integer we iteratively subtract a unit until its image is an integer.

$$r(\lfloor\alpha\rfloor - k) \qquad k \in \mathbb{N}$$

We then define the points

$$A = (\lfloor\alpha\rfloor - k, r(\lfloor\alpha\rfloor - k)), \qquad B = (\lfloor\alpha\rfloor - k + \Delta x, r(\lfloor\alpha\rfloor - k) + \Delta x)$$

As we can see from each graph, every other point would be at a greater distance from P and hence not useful to calculate the distance point-line, which is

$$d_{P,r} = \min(d(P,A), d(P,B))$$

In the first case, in which we have multiple minima, we consider all the integer minima and if none of them has an integer image on $r(x)$ we apply the same procedure as before on each of the extremes of the horizontal segment of the graph.

2.2.2 Point-line distance second algorithm

Now we are going to present a different approach.

Algorithm structure

Given a point $P = (x_P; y_P)$ and a straight-line $r: ax + by + c = 0$, let's imagine we worked in $\mathbb{R}^2$ and considered the (filled) straight-line: the point of r nearest to P would be the horizontal/vertical projection of P onto r, depending on whether the slope of r is greater/less than 1 in absolute value (the abscissa of this point is the minimum α of the distance function d previously defined). As before, this point in general does not belong to r, but surely lies between two points of r. How to determine these points?

The key idea here is to observe that the points of r enjoy some periodicity: they follow each other at step b for the abscissas, and step a for the ordinates, once we assume without loss of generality[1] $GCD(a,b) = 1$.

We look for the first point after the y-axis, so the first point of the straight-line with non-negative abscissa x_0. In order to do that we consider the equation given by the reminders of the division[2] by the coefficient b of the straight-line equation.

$$\left[a \cdot x + b \cdot y + c\right]_b = [0]_b$$
$$[a]_b \cdot [x]_b + \cancel{[0]_b \cdot [y]_b} + [c]_b = [0]_b$$
$$[x_0]_b = -[c]_b \cdot [a]_b^{-1}$$

We then define the numbers δ and δ' that count by defect/excess[3] respectively how many points of the straight-line lie in between x_0 and α.

$$\delta = \text{int}\left(\frac{\alpha - x_0}{b}\right) \qquad \delta' = \delta + \text{sgn}(\delta)$$

These numbers could be both positive or negative depending on whether x_0 is less or greater than α.

Then the two candidates for the searched nearest point of r are

[1] In general, when $GCD(a,b) = d$ we can always divide by d both sides of the equation $ax + by + c = 0$, obtaining the steps $\frac{a}{d}$ for the ordinates, $\frac{b}{d}$ for the absicssas.

[2] We denote with $[x]_b$ the remainder of the division $x : b$. Its inverse, denoted with $[x]_b^{-1}$, is the number that multiplied by $[x]_b$ gives $[1]_b$, and is computed with the extended Euclidean algorithm.

[3] $\text{int}(x)$, the integer part of x, is equal to greatest integer less than or equal to x, denoted $\lfloor x\rfloor$, if $x \geqslant 0$ and to the least integer greater than or equal to x, denoted $\lceil x\rceil$ if $x < 0$.

$$A = (x_0 + \delta \cdot b; r(x_0 + \delta \cdot b)) \qquad B = (x_0 + \delta' \cdot b; r(x_0 + \delta' \cdot b))$$

and again, the point-line distance is the minimum of the distance between P and A and B:

$$d_{P,r} = \min(\overline{PA}, \overline{PB})$$

An example for the second point-line distance algorithm

Given a point $P = (7; 9)$ and straight-line $r : 3x - 2y + 1 = 0$ we consider the equation given by the reminders of the division by 2:

$$[x]_2 + [1]_2 = [0]_2 \quad \rightarrow \quad [x]_2 = [1]_2 \quad \rightarrow \quad x_0 = 1$$

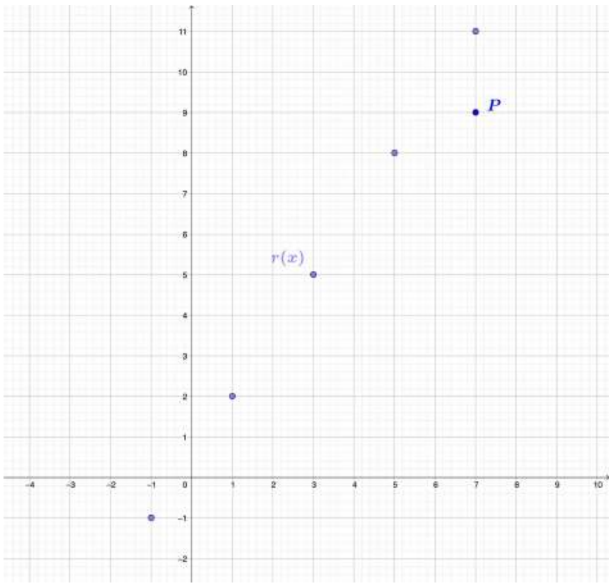


Figure 2.5: Graph of the straight line $r(x)$ (isolated points) and P

The minimum of the distance function $d(x) = |x - 7| + \left|\frac{3x-17}{2}\right|$ being $\alpha = \frac{17}{3}$, we obtain

$$\delta = \text{int}\left(\frac{\frac{17}{3} - 1}{2}\right) = \text{int}\left(\frac{7}{3}\right) = 2 \qquad \delta' = 2 + 1 = 3$$

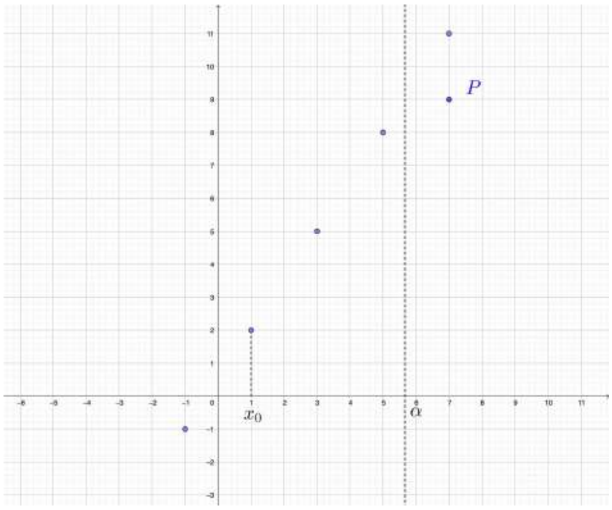


Figure 2.6: Graph of $r(x)$, P and the representation of α and x_0

Therefore we get the two candidate nearest points

$$A = (1+2\cdot 2; r(1+2\cdot 2)) = (5;8) \qquad \text{and} \qquad B = (1+3\cdot 2; r(1+3\cdot 2)) = (7;11)$$

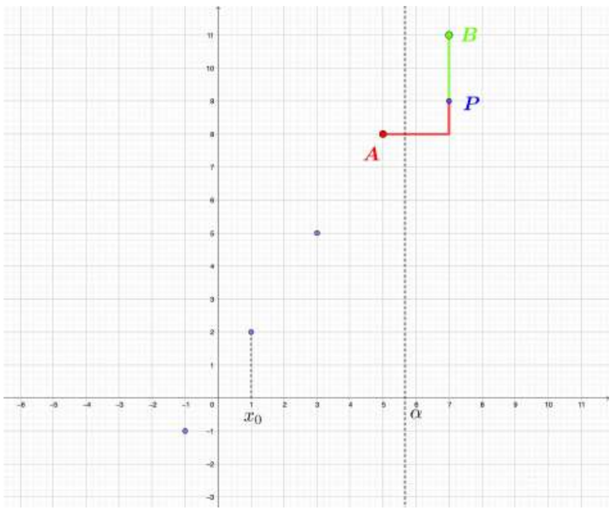


Figure 2.7: Point-point distances between P and A and B, respectively.

Comparing the distances form P of A and B respectively, we get that the minimal is

$$d_{P,r} = \min\big(d(P,A), d(P,B)\big) = \min(3,2) = 2$$

2.3 Common geometric loci given the Manhattan distance

In this section we will show which shapes the classical geometric loci (e.g. circle, ellipse, parabola and hyperbola) take in this metric space, using the definition of Manhattan distance. We will then use these results to tackle some problems related to city planning.

2.3.1 Circumference

According to its definition, the circumference $\mathscr{C}$ is the set of all the points P whose distance from another point C, the *center*, is equal to a parameter $r \in \mathbb{Z}^+$, called *radius*:

$$\mathscr{C} := \left\{ P \in \mathbb{Z}^2 \;\middle|\; |x_P - x_C| + |y_P - y_C| = r \right\}$$

As a consequence of the new definition of distance between two points, we obtain the figure shown in the Figure 2.8, which takes the shape of a rotated square.

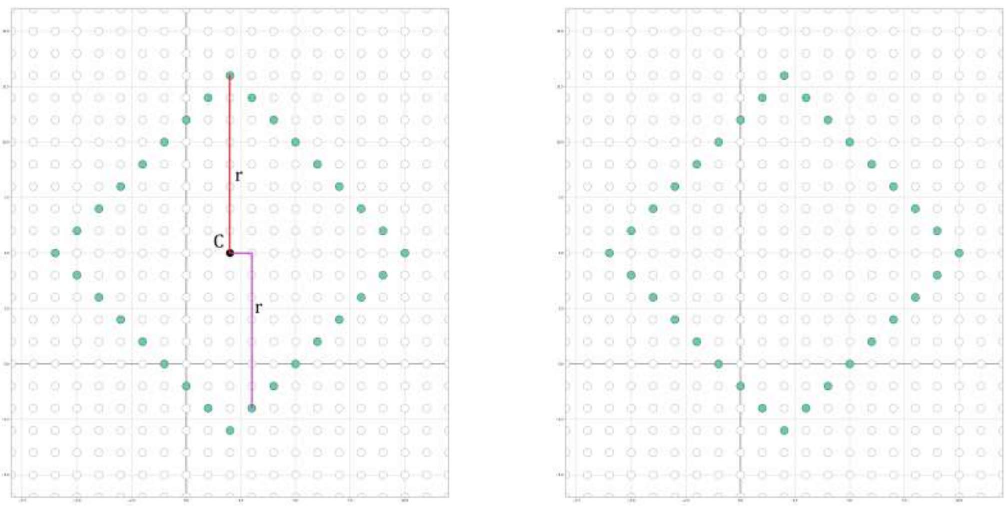


Figure 2.8: Circumference

Length of the circumference

We define the perimeter of $\mathscr{C}$ as the cardinality of the set of distinct points forming the circumference:

$$p_{\mathscr{C}} = \#\left\{P \in \mathbb{Z}^2 \mid P \in \mathscr{C}\right\} = 4r$$

We obtained this relation by projecting the points of the circumference onto the horizontal radius, as shown in the following picture (Figure 2.9):

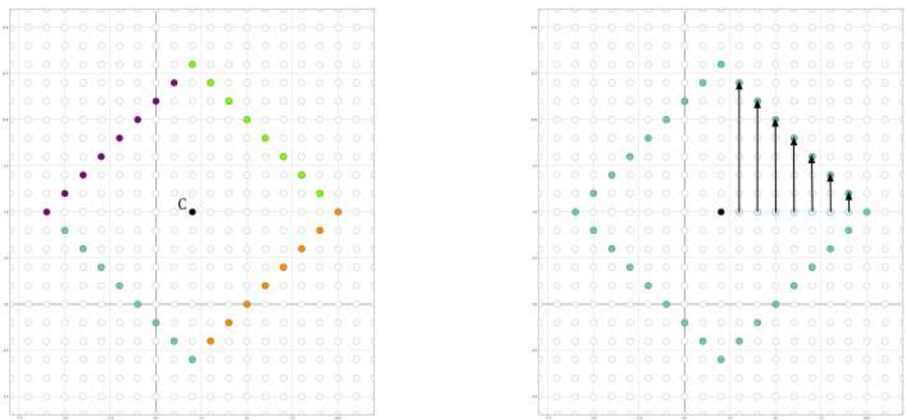


Figure 2.9: Circumference and visual proof of the formula

Furthermore, if we divide the circumference by $2r$, which is the diameter, we obtain 2. This means that, in this metric space, the constant π is equal to 2 (3).

Area of the circle

We define the area enclosed by the circumference as the cardinality of the set of the points whose distance from the centre is less than or equal to the radius (Figure 2.10):

$$\mathscr{A}_{\mathscr{C}} = \#\left\{P \in \mathbb{Z}^2 \mid |x_P - x_C| + |y_P - y_C| \leq r\right\} = 2r(r+1) + 1 \tag{2.5}$$

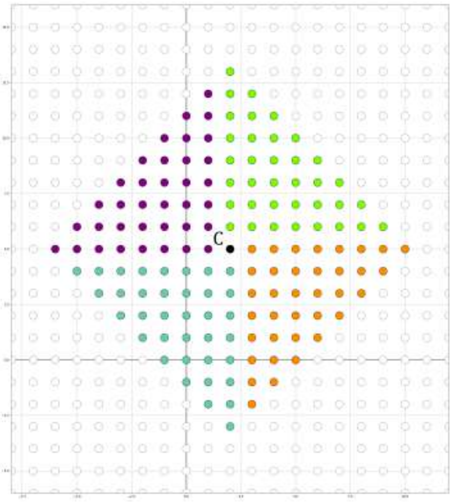


Figure 2.10: Visual proof of the formula

To get this result we can divide the circle into four triangles around a central point. The area of each triangle is given by the sum of the first integers up to the radius, thus

$$\mathscr{A}_{tri} = 1 + 2 + \ldots + r = \frac{r(r+1)}{2}$$
$$\mathscr{A}_{\mathscr{C}} = 4\,\mathscr{A}_{tri} + 1 = 4\,\frac{r(r+1)}{2} + 1$$

which simplifies to eq. (2.5).

2.3.2 Ellipse

We then moved on the study of the ellipse $\mathscr{E}$, which we recall to be the locus of points P whose sum of the distances from two foci F_1 and F_2 is equal to a parameter k:

$$\mathscr{E} := \left\{P \in \mathbb{Z}^2 \mid |x_P - x_{F_1}| + |y_P - y_{F_1}| + |x_P - x_{F_2}| + |y_P - y_{F_2}| = k\right\} \text{(4)}$$

Using our definition of distance, we obtain the following Figure 2.11:

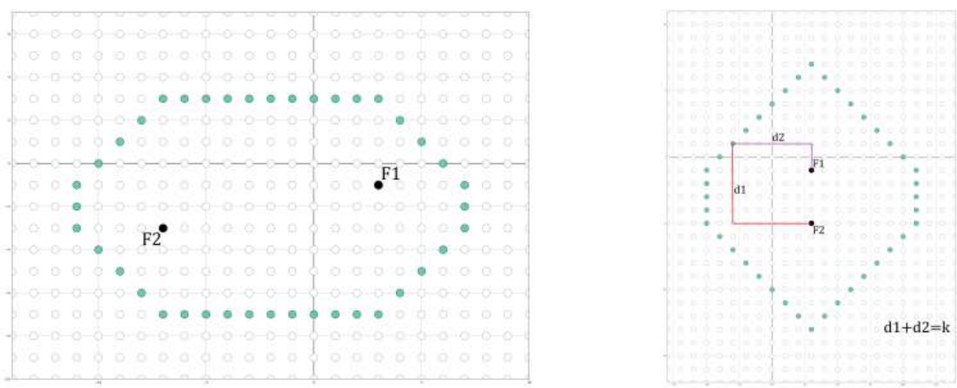


Figure 2.11: Ellipse

As we can see, the pecularity of this geometric locus is that it can assume various shapes, depending on the relative position of its foci.

The perimeter of the ellipse

We define the perimeter as the cardinality of the set of points belonging to $\mathscr{E}$:

$$p_{\mathscr{E}} = \#\{P \mid P \in \mathscr{E}\} = 2k$$

where the last equality is demonstrated by the image below (Figure 2.12):

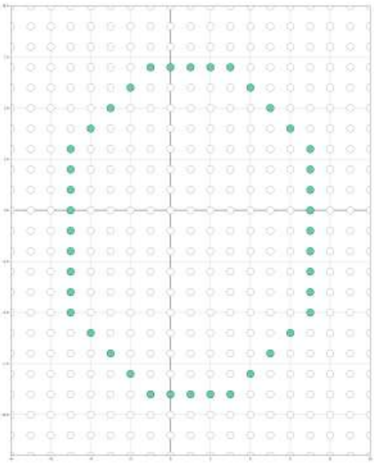


Figure 2.12: Ellipse contour

The area of the ellipse

We define the area enclosed by the ellipse $\mathscr{E}$ as the cardinality of the set of the points whose sum of the distances from the two foci is less than or equal to k.

$$\mathscr{A}_{\mathscr{E}} = \#\{P \in \mathbb{Z}^2 \mid |x_P - x_{F_1}| + |y_P - y_{F_1}| + |x_P - x_{F_2}| + |y_P - y_{F_2}| \leq k\} \quad (5)$$

We can obtain this area by observing that the ellipse is contained in a rectangle. We can see that, if we subtract the area of four triangles at the corners of this rectangle, we are left with the area of the ellipse (Figure 2.13).

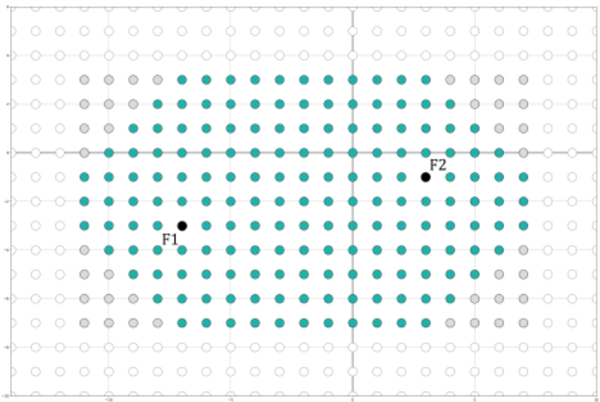


Figure 2.13: Visual proof of the formula

By noting $\Delta x = |x_{F_1} - x_{F_2}|$ and $\Delta y = |y_{F_1} - y_{F_2}|$, we have

$$\mathscr{A}_{rect} = (k - \Delta y + 1)(k - \Delta x + 1)$$

$$\mathscr{A}_{tri} = 1 + 2 + \ldots + \frac{k - \Delta x - \Delta y}{2} = \frac{\left(\frac{k-\Delta x-\Delta y}{2}\right)\left(\frac{k-\Delta x-\Delta y}{2}+1\right)}{2}$$

$$\mathscr{A}_{\mathscr{E}} = \mathscr{A}_{rect} - 4\mathscr{A}_{tri} = (k - \Delta y + 1)(k - \Delta x + 1) - \frac{1}{2}(k - \Delta x - \Delta y)(k - \Delta x - \Delta y + 2)$$

Limiting case

We can see that when k is equal to the distance between the two foci, we obtain for the ellipse $\mathscr{E}$ a set of points enclosed by a rectangle (Figure 2.14). This happens because for all the points between the foci the sum of their distances from them is precisly equal to the distance between the two foci itself.

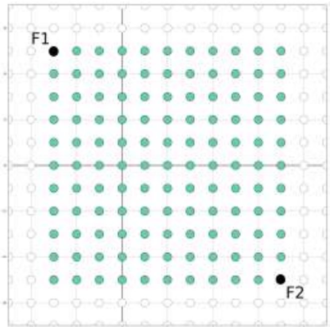


Figure 2.14: Illustration of the limiting case

2.3.3 Hyperbola

Moving on to the hyperbola $\mathscr{H}$, we define it as the set of points whose difference of distances from two foci F_1 and F_2 is equal to a parameter k:

$$\mathscr{H} := \left\{P \in \mathbb{Z}^2 \mid \left| |x_P - x_{F_1}| + |y_P - y_{F_1}| - |x_P - x_{F_2}| - |y_P - y_{F_2}| \right| = k \right\} \quad (6)$$

We thus obtain Figure 2.15:

Figure 2.15: Hyperbola

Limiting case

In the particular case when k is equal to the distance between the two foci, the points that satisfy the obtained equation lie on two portions of the plane, as shown in the following picture (Figure 2.16):

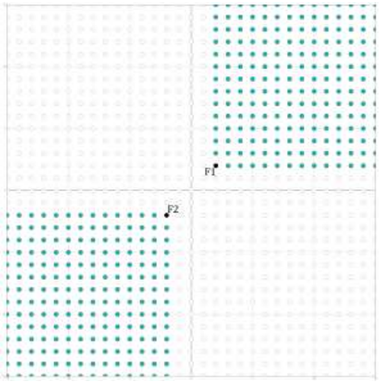


Figure 2.16: Illustration of the limiting case

2.3.4 Parabola

Finally we define the parabola $\mathscr{P}$ as the set of points whose distance from a straight line d, the *directix*, is equal to their distance from another point F, which is the *focus*.

Defining the parabolas whose directrix is parallel to either of the axis was no particular challenge. It was much harder to work with parabolas whose directrix had a certain slope, since we do not have a closed formula for the point-line distance. For this reason, we developed an algorithm to ease our work. We obtained the following results (Figure 2.17):

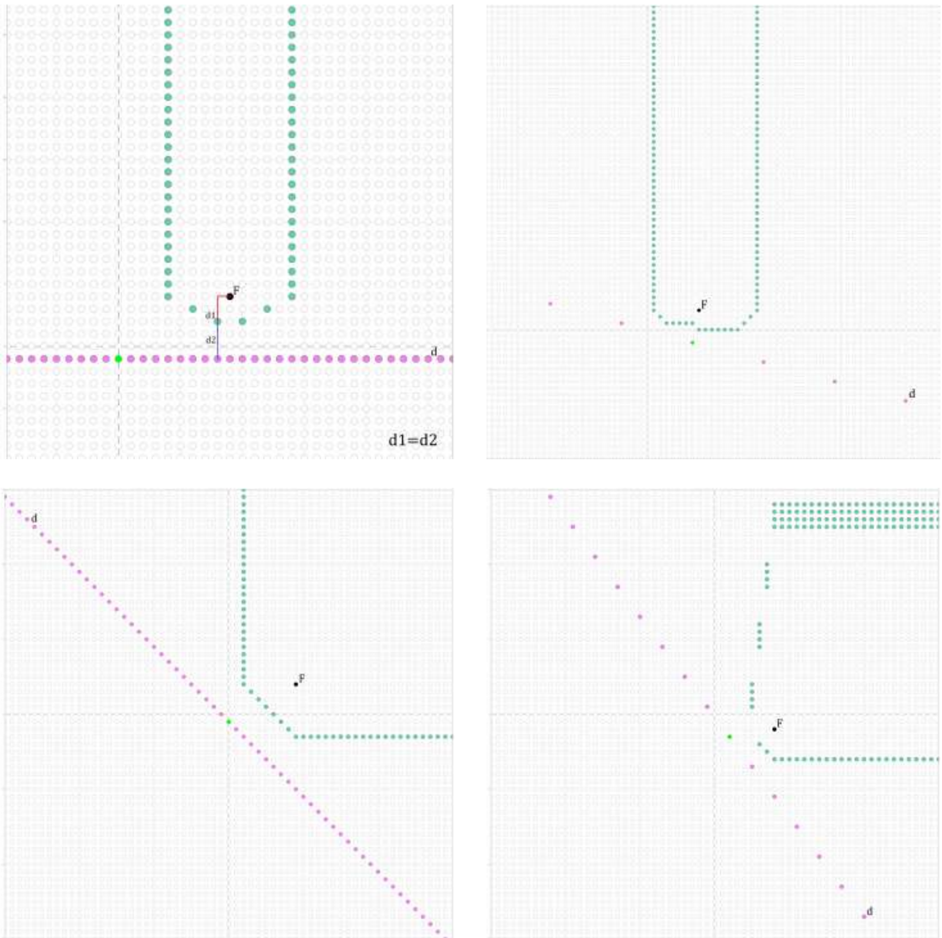


Figure 2.17: Examples of parabolas

2.4 Analysis of problem related to simple city planning situations

We then used the results to tackle problems related to urban planning and find optimal solutions to the organization of the city. In particular we have answered the following questions (illustrated in Fig. 2.18):

Question 1 Some children are participating in a treasure hunt and they know that clues are all placed within a certain distance from the starting point. Where should they look for them?

Question 2 If a university student has a side-job, where should he rent an apartment such that the sum of its distances from the university and the workplace does not exceed a given distance?

Question 3 A couple lives in the same apartment, but the two of them work in different places. Where would be the best place to rent an apartment?

Question 4 Someone lives in an apartment which is equally distant from a park and a series of metro stations along a straight line. Where is his house located?

(a) (b) (c) (d)

Figure 2.18: Examples to questions related to urban planning situations

The answers to these questions are areas enclosed by, respectively, a circumference, an ellipse, a hyperbole and a parabola which were previously discussed in the article.

1. If all the clues are placed within a certain distance from the starting point (which will be the center), their distribution will look like that depicted in Figure 2.18a.
2. If we consider his apartment and the university as 2 foci, we can deduce that, if the sum of the distances does not exceed a given length, we obtain an area enclosed by an ellipse (Figure 2.18b)
3. The optimal solution is to find an area where the distances from the two workplaces do not differ by much, which is outlined by an hyperbole (Figure 2.18c)
4. We can easily deduce that the metro line is the directrix of a parabola, whose focus is represented by the park (Figure 2.18d).

2.5 Other questions related to urban planning and their relation to the concept of axis

Let's consider the rectangular city with dimensions $l \times h$ with $l, h \in \mathbb{Z}^+$. In this section we are going to discuss the following questions:

Question 1 Given two points of interest $A(x_A; y_A)$, $B(x_B; y_B) \in \mathbb{Z}^2$ of the city, where can we find a house in the city at the same distance from the two points?

Question 2 Given a positive integer $n \in \mathbb{Z}^+$ of hospitals, how can we divide the city in areas so that each area has only one hospital and the distance between every point of the area and the respective hospital is less than that of any other hospital?

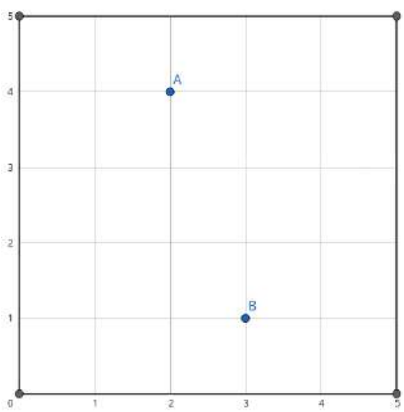
Question 3 Given a $l \times l$ city and a maximum distance $r \in \mathbb{Z}^+$, what is the minimum number N of hospitals that cover the city so that the distance from every point to the nearest hospital is less or equal to r?

2.5.1 Living at the same distance from two points of interest

Given two points of interest $A(x_A; y_A), B(x_B; y_B) \in \mathbb{Z}^2$ of the city, where can we find a house in the city at the same from the two points?

The set of the houses

Let's consider a 5×5 city and two points $A(2;4)$ and $B(3;1)$:



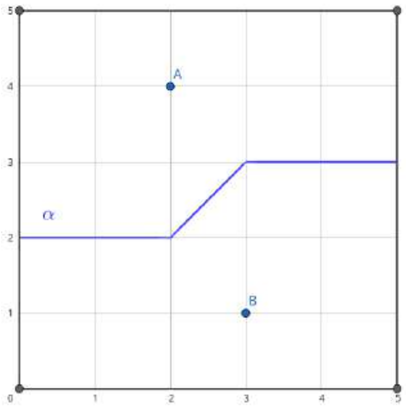
To find a house in the city which has the same distance between the two points we have to find the locus $\mathscr{L}$ of all the points of the plane $\mathbb{Z}^2$ that have the same distance from A and B. To do that we can simply equal the distance of a generic point $P(x;y)$ from A to its distance from B and find the integer solutions of the equation that we'll find:

$$d(P,A) = d(P,B) \rightarrow |x - x_A| + |y - y_A| = |x - x_B| + |y - y_B|$$

Which corresponds to the axis α of the points A and B:

Definition 4. *Given two points $A = (x_A, y_A), B = (x_B, y_B) \in \mathbb{Z}^2$ the* ***axis*** *α of A and B is the locus of the points of the euclidian plane $\mathbb{R}^2$ which have the same distance from A and B:*

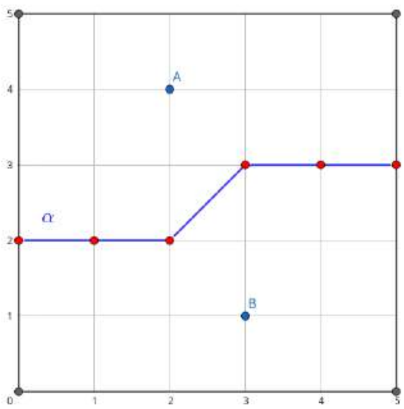
$$\alpha := \{\, (x, y) \in \mathbb{R}^2 \mid |x - x_A| + |y - y_A| = |x - x_B| + |y - y_B| \,\}$$

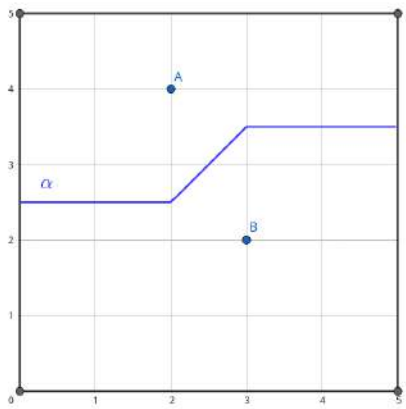


The set $\mathscr{L}$ of the points of the city which have the same distance from the two points of interest is given by the intersection of the solution set α of the equation of the axis and the plane $\mathbb{Z}^2$:

$$\mathscr{L} = \alpha \cap \mathbb{Z}^2$$

If $\mathscr{L}$ is a non empty set, we define it as the *integer axis* of A and B, while if it is an empty set, we define the axis α as an *imaginary axis*.



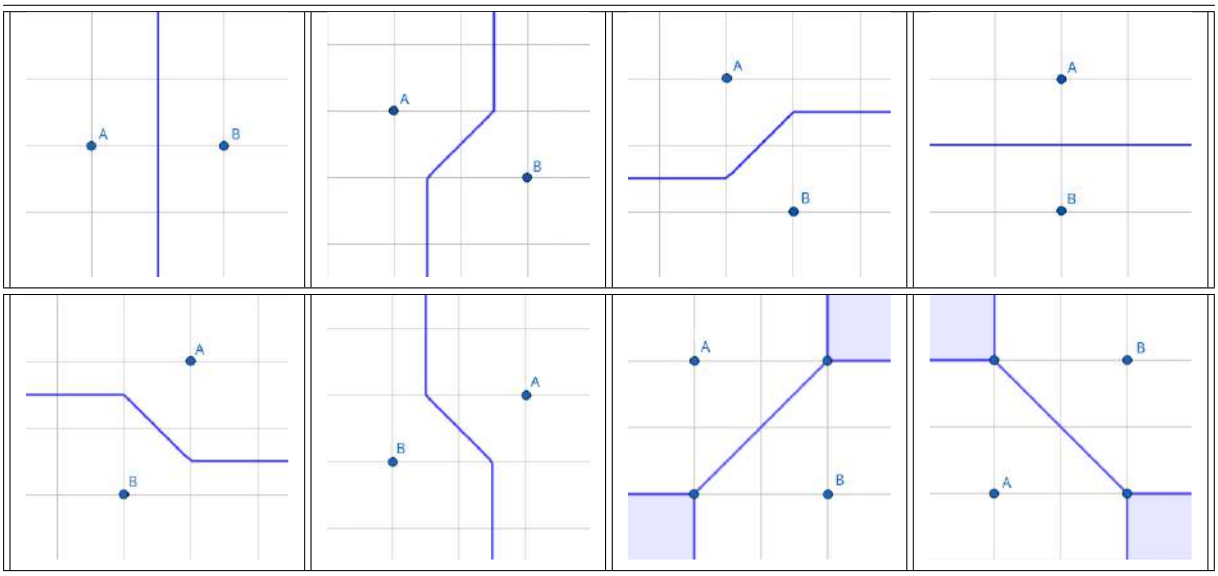


2.5.2 The different possible forms of the axis

Let's consider the equation of the axis α of the points $A(x_A; y_A)$ and $B(x_B; y_B) \in \mathbb{R}^2$

$$\alpha : |x - x_A| + |y - y_A| = |x - x_B| + |y - y_B|$$

All the different forms of the axis are summarized in the following table:



We can see that the form that the axis assumes depends on the difference between the abscissas $\Delta x = x_A - x_B$ and the difference between the ordinates $\Delta y = y_A - y_B$.

An axis can be made of (7)

- one horizontal/vertical straight line if $\Delta x = 0$ or if $\Delta y = 0$;
- one diagonal segment and two half-lines if $\Delta x \neq \Delta y$ and $\Delta x \neq 0$ and $\Delta y \neq 0$;
- one diagonal segment and two parts of the plane if $\Delta x = \Delta y \neq 0$

2.5.3 Splitting the city

Given a positive integer $n \in \mathbb{N}_0$ of hospitals, how can we divide the city in areas so that each area has only one hospital and the distance between each point of the area and the respective hospital is less than that to any other hospital?

2.5.4 The $n = 2$ case

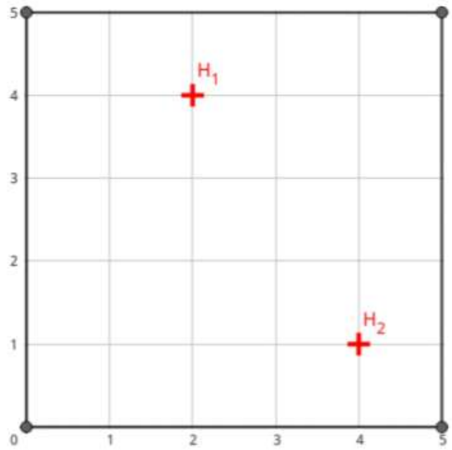
Let's consider the specific case where we only have two hospitals in a 5×5 city, like the following: where the two hospitals H_1 and H_2 have coordinates $(2;4)$ and $(4;1)$.

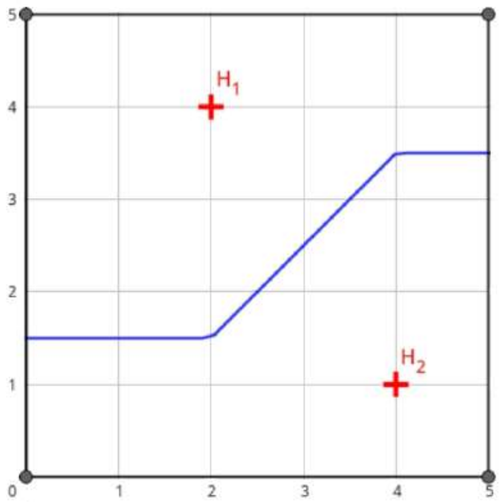
To draw a division of the city we have to find all the points of the euclidean plane which have the same distance from the two hospitals, since that all the other points will have a lower distance from one of the two hospitals and will be part of one of the areas.

To do that, we can equal the distance from one hospital of a generic point $P(x;y)$ with the distance of the point from the other hospital. The equation we will find corresponds to the equation of the axis α of the points H_1, H_2.

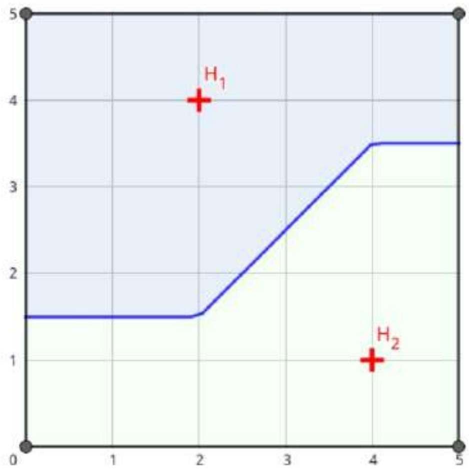
$$d(P,H_1) = d(P,H_2) \rightarrow |x - x_{H_1}| + |y - y_{H_1}| = |x - x_{H_2}| + |y - y_{H_2}| \tag{2.6}$$

If we sketch the real solutions of the previous equation (2.6) on the Cartesian coordinate system of the city we get the following curve:



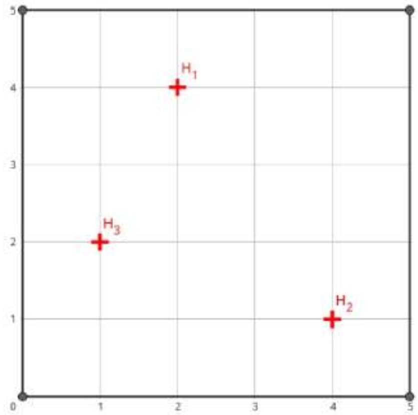


By drawing the points which have the same distance from the two hospitals, we have drawn the division of the city and we can divide the city it in two areas limited by the axis itself.



2.5.5 The case where $n = 3$

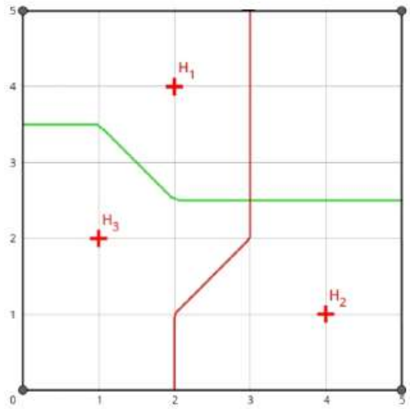
Let's now consider the same city, with a third hospital H_3 on the point (1;2):



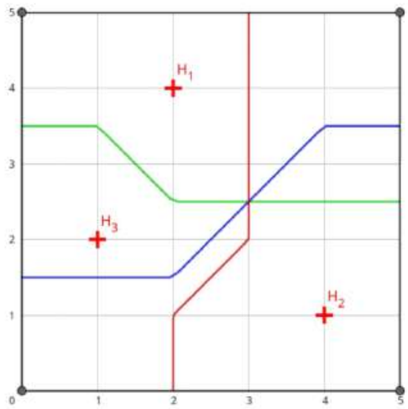
To divide the city this time we have to add to the previous axis, the ones of the points H_3, H_1 and H_3, H_2. To find the two new axes we have to build the two following equations:

$$d(P, H_3) = d(P, H_1) \to |x - x_{H_3}| + |y - y_{H_3}| = |x - x_{H_1}| + |y - y_{H_1}|$$
$$d(P, H_3) = d(P, H_2) \to |x - x_{H_3}| + |y - y_{H_3}| = |x - x_{H_2}| + |y - y_{H_2}|$$

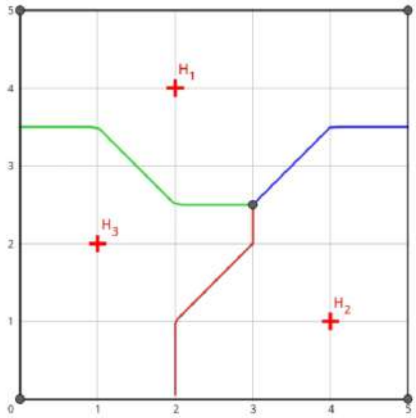
If we sketch the real solutions of these two equations, we get the two following axes:



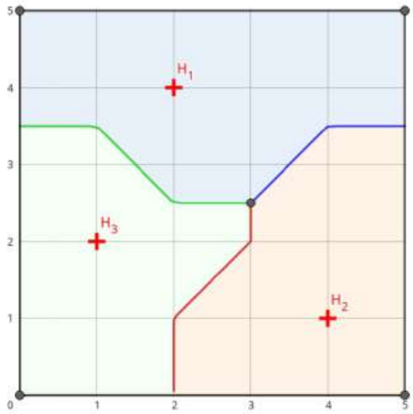
And if we add these axes to the previous one we get



Since there are parts of the axes of the points, for example H_1, H_2, that get in the areas of the other hospital, in this case H_3, we have to remove those parts. To do that we have to remove all the points $P(x_P; y_P)$ of the axis of the points H_1, H_2 for which $d(P, H_3) < d(P, H_1) = d(P, H_2)$.



Now we can divide the city into three areas limited by the three axes.



2.5.6 The generic case with $n \in \mathbb{Z}^+$

After analyzing the specific cases where $n = 2$ and $n = 3$, we can now consider the generic case of a number $n \in \mathbb{Z}^+$ of hospitals. To split the city we have to follow these steps:

- First we need to find all the possible couples of hospitals, without repeating them.
- After that we have to draw the axes of the points given by the couples of hospitals. The number S of axes that we will have to draw is given by the following relation:

$$S = \frac{n(n-1)}{2}$$

- Last we have to remove the parts of the axes which invade the areas of other hospitals. In order to do that we have to remove all the points $P(x_P; y_P)$ of each axis whose distance from the nearest hospital H_i is less than the distance of one of the points H_j, H_k that generate the axis.

$$d(P, H_j) = d(P, H_k) < d(P, H_i)$$

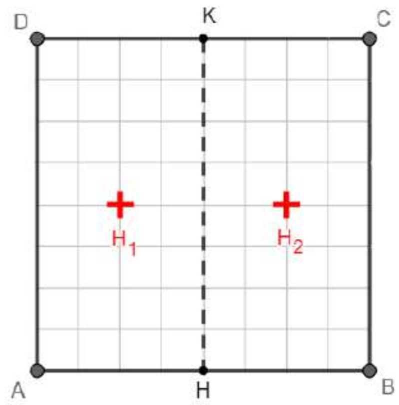
2.6 Minimum number of hospitals to covering a given city

Given a $l \times l$ city and a maximum distance $r \in \mathbb{Z}^+$, what is the minimum number N of hospitals so that the distance of every point from the nearest hospital is less or equal to r?

To answer this question we can find the number of hospitals that we need to cover the area of the city with circumferences with radius r and center in each hospital. We decided to analyze separately the cases where $r < \frac{3}{4}l$ and $r \geqslant \frac{3}{4}l$. We can easily prove that if $r \geqslant l$ then $N = 1$ (8), so we can just study the cases where $r < \frac{3}{4}l$ and $\frac{3}{4}l \leqslant r < l$.

2.6.1 The case where $\frac{3}{4}l \leqslant r < l$

Let's first consider the specific cases where $\frac{3}{4}l \leqslant r < l$.



As we can see, if we split the city into two rectangles with sizes l and $\frac{l}{2}$ and put two hospitals in the intersections of the diagonals of each rectangle, the distance of the hospitals from each vertex of the rectangle is $\frac{3}{4}l$ (9).

$$d(H_1, D) = d(H_2, C) = \frac{3}{4}l$$

Since for $r = \frac{3}{4}l$ then $N = 2$ and if $r < l$ then $N > 1$, if $\frac{3}{4}l \leqslant r < l$ the number N of hospitals is 2.

2.6.2 The case where $r < \frac{3}{4}l$

Let's now consider the case where $r < \frac{3}{4}l$. This time to cover the city we have chosen a pattern that will be repeated to cover each city. Considering an unlimited city, the pattern we chose to cover the area of the city is shown in Fig. 2.19.

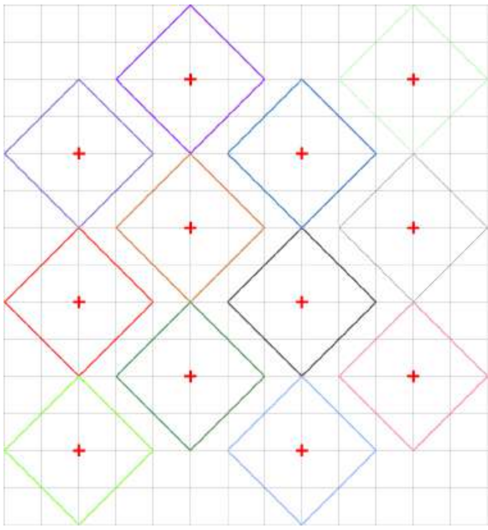


Figure 2.19: The area is covered following the pattern shown in this example.

Definition 5. *A **column of hospitals** is a set of hospitals which have the same abscissas.*
*Given a $l \times l$ city and a maximum distance r, with $l, r \in \mathbb{Z}^+$ a column of hospitals is an **even/odd column** iff the the common abscissa of the hospitals x_H is a non negative even/odd integer multiple of $r+1$:*

$$x_H = k(r+1), \qquad \textit{with } k \in \mathbb{Z}^+ \textit{ even/odd}$$

Let's consider the case of an $l \times l$ city (Fig. 2.20), like the following:

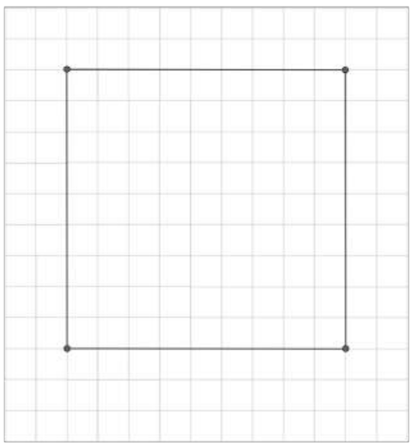


Figure 2.20: $l \times l$ city

First we have to calculate the number of hospitals of each column of the city (10). As we can see, it is given by the ceiling quotient between the length l of the city and two times the diameter r of the circumferences if the column is odd and by the ceiling quotient between $l+r$ and $2r$ if the column is even:

$$H_O = \left\lceil \frac{l}{2r} \right\rceil \tag{2.7}$$

$$H_E = \left\lceil \frac{l+r}{2r} \right\rceil \tag{2.8}$$

After that we have to find the number of columns needed to cover all the city (see Fig. 2.21).

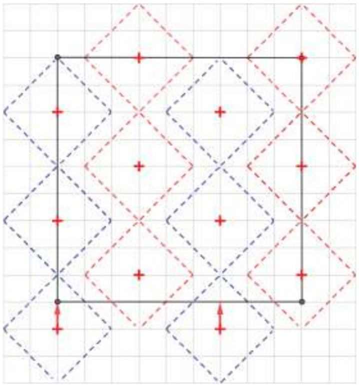


Figure 2.21: Arranging columns to cover the city.
Red arrows show where to move the hospitals (represented by a red cross).

We can see that the number of odd columns is given by the ceiling quotient between $l+r+1$ and $2r+2$ while the number of even columns is given by the ceiling quotient between l and $2r+2$:

$$C_O = \left\lceil \frac{l+r+1}{2r+2} \right\rceil \tag{2.9}$$

$$C_E = \left\lceil \frac{l}{2r+2} \right\rceil \tag{2.10}$$

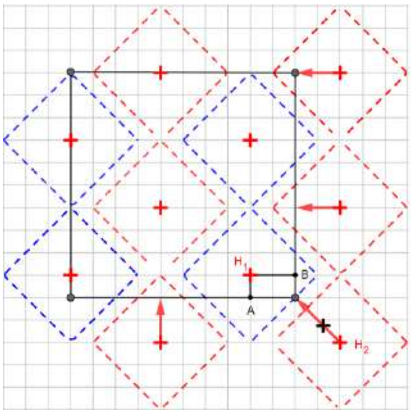
The total number of columns is now given by the sum of the odd and even number of columns (11).

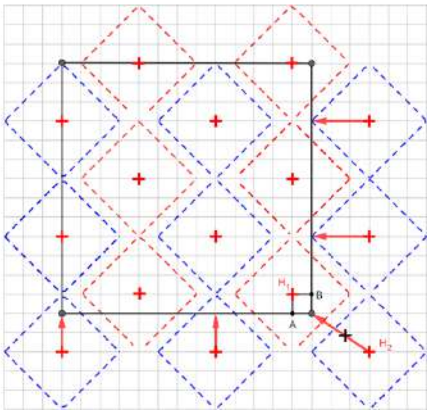
$$C = C_O + C_E = \left\lceil \frac{l+r+1}{2r+2} \right\rceil + \left\lceil \frac{l}{2r+2} \right\rceil \tag{2.11}$$

Since we have found the number of hospitals of each column and the total number of columns as a function of the length l of the city and the maximum distance r, we can now calculate the total number of hospitals needed to cover the city by summing the number of hospitals of the even columns times the number of even columns and the number of hospitals of the odd columns times the number of odd columns:

$$N_H = H_O \cdot C_O + H_E \cdot C_E$$

To find a more general relation, although, we have to consider the cases where one hospital can be omitted, like in the following examples:



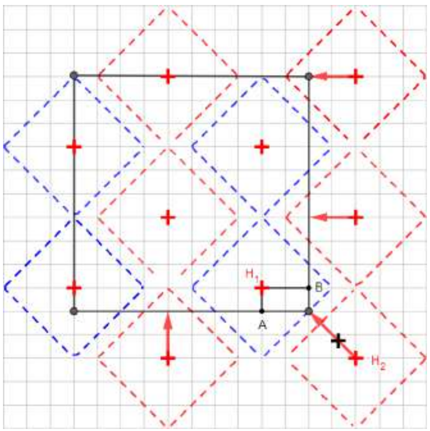


We can consider separately the cases where the last column is odd or even and so the cases where C is odd or even. In both the situations, the last hospital H_2 can be removed if the sum of the distances $d(H_1, A)$ and $d(H_1, B)$ between the hospital H_1 of the second last column and the sides of the city is less than or equal to the maximum distance r (12):

$$d(H_1, A) + d(H_1, B) \leqslant r \rightarrow N = N_H - 1$$

***C* even**

Let's first consider the case where C is even.



As we can see, the distances $d(H_1, A)$ and $d(H_1, B)$ are given by the following relations:

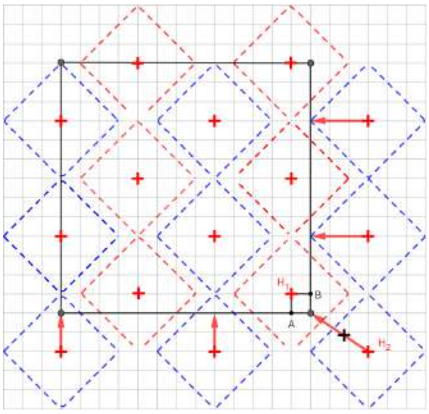
$$d(H_1, A) = l - \big((H_O - 1)2r + r\big) = l - r(2H_O - 1)$$
$$d(H_1, B) = l - \big((C_O - 1)(2r + 2)\big) = l - 2(r + 1)(C_O - 1)$$

and the relation that describes the number of hospitals will be the following:

$$N = \begin{cases} H_O C_O + H_E C_E, & \text{if } d(H_1, A) + d(H_1, B) > r \\ H_O C_O + H_E C_E - 1, & \text{if } d(H_1, A) + d(H_1, B) \leqslant r \end{cases}$$

***C* odd**

Let's now consider the case where C is odd.



In this situation the distances $d(H_1, A)$ and $d(H_1, B)$ are given by the following formulas:

$$d(H_1, A) = l - \big((H_E - 1)2r\big) = l - 2r(H_E - 1)$$

$$d(H_1, B) = l - \big((C_E - 1)(2r + 2) + 1\big) = l - (r + 1)(2C_E - 1)$$

The relation that describes the number of hospitals will be the same as the previous case.

$$N = \begin{cases} H_O \cdot C_O + H_E \cdot C_E, & \text{if } d(H_1, A) + d(H_1, B) > r \\ H_O \cdot C_O + H_E \cdot C_E - 1, & \text{if } d(H_1, A) + d(H_1, B) \leqslant r \end{cases}$$

2.6.3 Generic case with $r \in \mathbb{Z}^+$

We can now consider the generic case of an integer distance $r \in \mathbb{Z}^+$. The function that describes the number of hospitals N needed to cover the city is given by the following equations: (13)

$$N(r, l) = \begin{cases} 1, & \text{if } r \geqslant l \\ 2, & \text{if } \frac{3}{4} l \leqslant r < l \\ \left\lceil \frac{l}{2r} \right\rceil \left\lceil \frac{l + r + 1}{2r + 2} \right\rceil + \left\lceil \frac{l}{2r} + \frac{1}{2} \right\rceil \left\lceil \frac{l}{2r + 2} \right\rceil, & \text{if } r < \frac{3}{4} l \text{ and } k > r \\ \left\lceil \frac{l}{2r} \right\rceil \left\lceil \frac{l + r + 1}{2r + 2} \right\rceil + \left\lceil \frac{l}{2r} + \frac{1}{2} \right\rceil \left\lceil \frac{l}{2r + 2} \right\rceil - 1, & \text{if } r < \frac{3}{4} l \text{ and } k \leqslant r \end{cases} \tag{2.12}$$

Where k is equal to

- $2l - r\left(2\left\lceil \frac{l}{2r} \right\rceil - 1\right) - 2(r + 1)\left(\left\lceil \frac{l + r + 1}{2r + 2} \right\rceil - 1\right)$ if C is even;
- $2l - 2r\left(\left\lceil \frac{l}{2r} + \frac{1}{2} \right\rceil - 1\right) - (r + 1)\left(2\left\lceil \frac{l}{2r + 2} \right\rceil - 1\right)$ if C is odd.

with $C = \left\lceil \frac{l + r + 1}{2r + 2} \right\rceil + \left\lceil \frac{l}{2r + 2} \right\rceil$.

In the case where $r < \frac{3}{4} l$ the actual minimum number N^* of hospitals needed to cover the city is not in general equal to the number N we just computed: in fact the pattern of hospitals we heuristically chose might not be the best possible way to cover the city. Despite that, N^* surely lies between our N as upper bound and a lower bound N_{inf} given by the quotient between the area of the city and the area of the circumference of each hospital.

$$\frac{\mathscr{A}_{city}}{\mathscr{A}_{hospital}} = N_{inf} < N^* \leq N$$

Chapter 3

The grid as a metric space

So far we have considered only the intersections of the streets as points of interest. What happens if we admit as of interest also any other point of the street?

3.1 Definitions

Definition 6. *The plane is defined as* $(\mathbb{N} \times \mathbb{R}^+) \cup (\mathbb{R}^+ \times \mathbb{N})$*; its elements are called points. The distance function between* $A = (x_A, y_A)$ *and* $B = (x_B, y_B)$ *is defined as:* (14)

- *If* $\lfloor y_A \rfloor \neq \lfloor y_B \rfloor$ *and* $\lfloor x_A \rfloor \neq \lfloor x_B \rfloor$*, or* $\lfloor y_A \rfloor = \lfloor y_B \rfloor$ *and* $\lfloor x_A \rfloor = \lfloor x_B \rfloor$*, then*

$$d(A, B) = |\Delta x| + |\Delta y| = |x_A - x_B| + |y_A - y_B|$$

- *If only* $\lfloor y_A \rfloor = \lfloor y_B \rfloor$ *then*
  - *If* $\{y_A\} + \{y_B\} \leqslant 1$ *then* $d(A, B) = |x_A - x_B| + |y_A - 2\lfloor y_B \rfloor + y_B|$
  - *If* $\{y_A\} + \{y_B\} > 1$ *then* $d(A, B) = |x_A - x_B| + |y_A - 2\lceil y_B \rceil + y_B|$
- *If only* $\lfloor x_A \rfloor = \lfloor x_B \rfloor$ *then*
  - *If* $\{x_A\} + \{x_B\} \leqslant 1$ *then* $d(A, B) = |x_A - 2\lfloor x_B \rfloor + x_B| + |y_A - y_B|$
  - *If* $\{x_A\} + \{x_B\} > 1$ *then* $d(A, B) = |x_A - 2\lceil x_B \rceil + x_B| + |y_A - y_B|$

Although we want to capture the same idea as before, i.e. that the distance still coincides with the length of the shortest path between A and B, the definition of the distance function gets more complicated here. The problem is when the two points lie between two integers that differs by a unit in either the abscissas or the ordinates: in this case it's necessary to determine which one of the two paths is the shortest. We are going to show the case where two points have the same integer part in the ordinates, but the reasoning is the same for the abscissas.

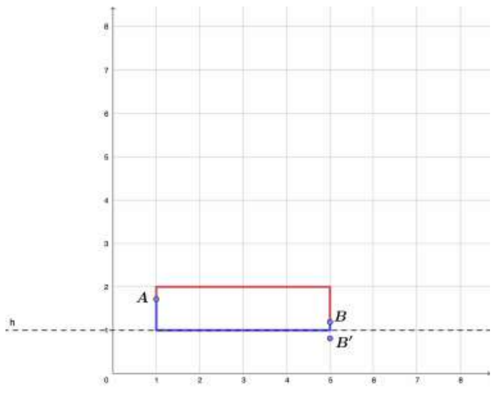


Figure 3.1: Possible minimal paths between points with same integer part of ordinates.

Since the horizontal segments of both paths are congruent, the length only depends on the sum of the lengths of the vertical segments which are respectively represented by the sum of the fractional part of the ordinates (for the blue one) and the sum of 1 minus the fractional part of the ordinates (for the red one).

$$\{y_A\} + \{y_B\} \lesseqgtr 1 - \{y_A\} + 1 - \{y_B\}$$

$$\{y_A\} + \{y_B\} \lesseqgtr 1$$

Once determined the shortest path it has the same length as the distance between one point and the symmetric of the other one with respect to the line $y = \lfloor y_A \rfloor$ if the lower path is the shortest (blue one) or $y = \lceil y_A \rceil$ if the upper path is the shortest (red one).

This function is a well-behaved metric function, in that it enjoys these four properties:

I) $d(A,B) > 0 \Leftrightarrow A \neq B$

II) $d(A,B) = 0 \Leftrightarrow A = B$

III) $d(A,B) = d(B,A)$

IV) $d(A,B) \leqslant d(A,C) + d(C,B)$

3.2 Differences with $\mathbb{Z}^2$

One of the implications of this new definition is the representation of some geometric loci.

3.2.1 Circumferences in $(\mathbb{N} \times \mathbb{R}^+) \cup (\mathbb{R}^+ \times \mathbb{N})$

The circumference, which is still defined as the locus of all and only the points equidistant from a point called center, can now present a real number as both radius and one coordinate of the center.

Real coordinate for the center O

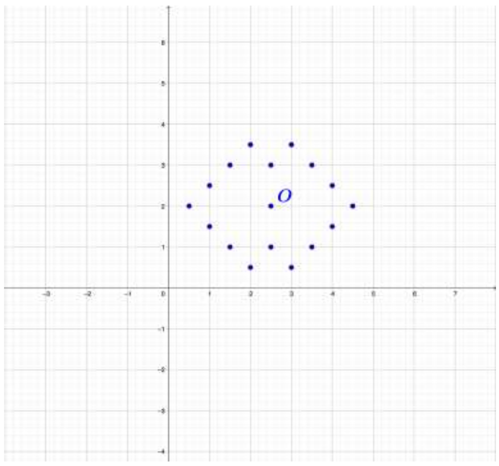


Figure 3.2: circumference with centre $O\left(\frac{5}{2}, 2\right)$ and radius $r = 2$

Real radius r

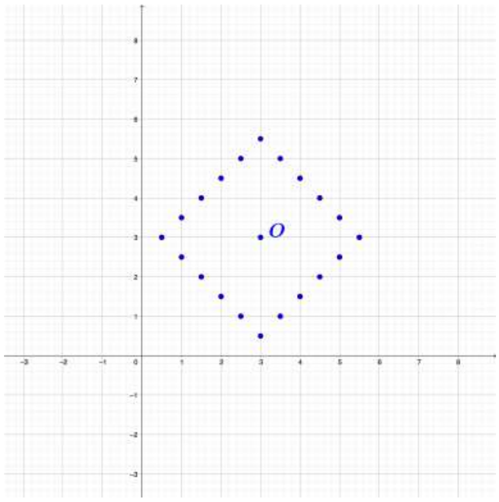


Figure 3.3: circumference with centre $O(3,3)$ and radius $r = \dfrac{5}{2}$

Editing Notes

(1) What is most important in Dijkstra's algorithm is to select inductively a list of nodes for which it is certain that the current estimated distance is its minimum distance to the starting node in the whole map. In the present version these are the first nodes, up to the one being examined.
At step k, the algorithm will have identified nodes 1 to $k-1$ as such "good" nodes and updated the distances of their neighbors, the list of nodes being ordered by non-decreasing distance. Then node k may be added to good nodes, because we cannot exit the group of nodes 1 to $k-1$ by a shorter path than its current distance. Now, as the routes from nodes 1 to $k-1$ have already been tested, we just need to update the distances for the neighbors of node k (also entering k as previous node when a shorter distance is found) and to reorder the list of nodes.

(2) Δx is the difference between the abscissas of two successive points of the line. It is equal to b if a and b are coprime; here $\Delta x = 2$.

(3) However, π cannot be replaced by 2 in other geometry formulas. For example, the area of the disk calculated below is not equal to $2r^2$.

(4) Of course, k has to be larger than the distance between the foci F_1 and F_2 – for the degenerate case $k = d(F_1, F_2)$, see the limiting case below. But there is also a constraint of parity: indeed, $|x_P - x_{F_1}| + |x_P - x_{F_2}|$ has the same parity as $x_{F_2} - x_{F_1}$ and $|y_P - y_{F_1}| + |y_P - y_{F_2}|$ has the same parity as $y_{F_2} - y_{F_1}$, so k must have same parity as $|x_{F_2} - x_{F_1} + |y_{F_2} - y_{F_1}| = d(F_1, F2)$. Otherwise, the "ellipse" is empty (and the formula given for the perimeter is not valid).

(5) Here, k may not have the same parity as the distance between F_1 and F_2, but in this case there is no point P with $d(P, F_1) + d(P, F_2) = k$ and k must be replaced by $k-1$ in the calculation below and in the result (the number of points beyond a focus to the nearest edges is actually the integer part of $\big(k - d(F_1, F_2)\big)/2$).

(6) Same remark as (4) on the parity of k.

(7) We can add that the "axis" is empty if $d(A, B) = |\Delta x| + |\Delta y|$ is odd. Indeed, $|x - x_B| - |x - x_A|$ always has the same parity as Δx and $|y - y_B| - |y - y_A|$ always has the same parity as Δy, so $d(P, B) - d(P, A)$ has the same parity as $d(A, B) = |\Delta_x| + |\Delta_y|$.

(8) Since l is the length of an edge and not the number of integer points on an edge (see the example of a 5×5 city above), for any integer point P there exists a vertex of the square whose distance from P is $\geqslant 2\lceil l/2 \rceil$, that is $l+1$ in case when l is odd, and then we need a radius $l+1$ to cover the square with a single circumference.

(9) It is not always possible to place hospitals at the centers of the rectangles, whose coordinates are not necessarily integers. However, by choosing the closest points, it can be shown that this distance is $\leq \lceil 3l/4 \rceil$.

(10) The columns are arranged as shown in Figure 2.21, where the abscissa of the left-hand edge of the square is equal to 1, so that the first column is odd.

(11) This sum is equal to $\lceil l/(r+1) \rceil + 1$.

(12) This is only for the case where H_1 is is above the lower edge of the square. If H_1 is below or on the edge, then H_2 is above it and cannot be removed (while the following calculation yields the opposite). This latter case occurs when the column of H_1 is odd ($\lceil l/(r+1) \rceil$ odd) and the remainder of the division of l by $2r$ lies between 1 and r, and when the column is even and the remainder of the division of l by $2r$ is zero or lies between r and $2r-1$.

(13) Actual conditions are slightly different from those given below, according to notes 8 and 12.

(14) Explanations are given after the definition, in the following text.
$\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of a real number x.