Optimal Route

year 2022-2023

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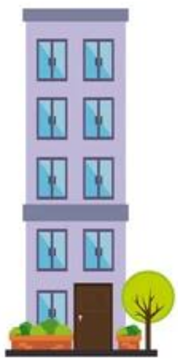
Abstract:

Our research seeks to determine points M and N on the running track h, keeping the running distance d, in which $AM = BN$, as well as the minimum distance that Tom travels to get from one point to another.





The problem

On his way back home, Tom wants to do jogging for d kilometers on the running track. The running track is a straight line (see the attached figure).
Find the location of points M and N on the running track so that $AM = BN$.
What is the location of the point M on the running track where Tom should start jogging, so that the total distance from work to home is minimal?
(You can consider various positions of points A and B in the plane).

Notation:

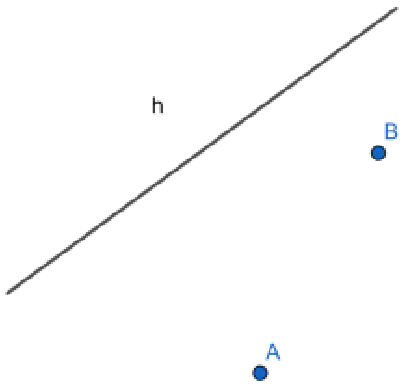
- ✓ h is the running track
- ✓ A and B are the two fixed points
- ✓ A represents Tom's job place
- ✓ B represents Tom's home
- ✓ d is the running distance

Task 1:

We shall consider different positions of points A and B in the plane.

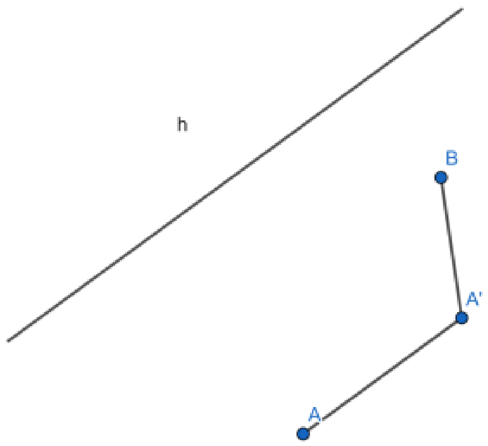
We have to find M and N, points on the running track h so that $AM = BN$.

Case 1: The points A and B are on the same side of the line h.

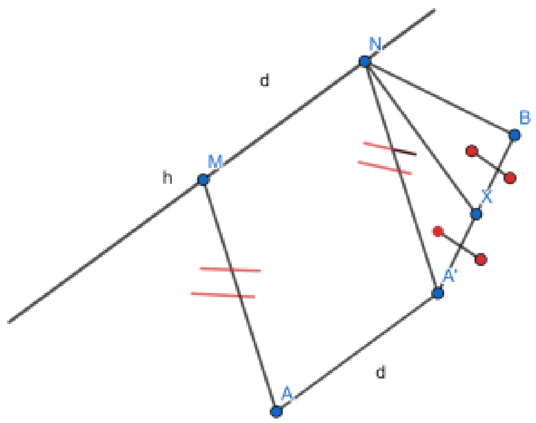


Steps in the construction:

1. Consider the point A' such that $AA' \parallel h$ and the length of AA' is d;
2. Next, we draw the segment $A'B$;



3. Let N be the intersection point of the segment $A'B'$ with the line h.
4. We now consider the point M on h so that $MN = d$ or $AM \parallel A'N$.
5. Consider the point M on the line d such that $MN = d$ (or $AM \parallel BN$).



We now prove that the point M and N are such that $AM = BN$. Firstly, we shall prove that $BN = A'N$. We see that the triangles $\triangle A'XN$ and $\triangle BXN$ are congruent as:

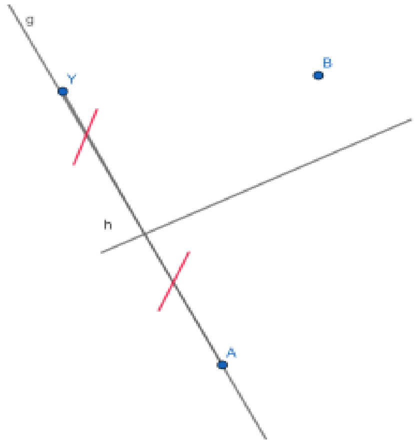
- $\angle X = 90°$
- $A'X = BX$ (point X is the middle of $A'B$)
- $XN = XN$ (common side)

From the congruence $\triangle A'XN \equiv \triangle BXN$ (the cathetus-cathetus case), we get that the other sides are also congruent, thus $BN = A'N$. (1)

Another way of proving this is as follows: the point N is situated on the perpendicular bisector of $A'B$, and thus it has the propriety that it is equidistant from the endpoints of the segment, implying that $BN = A'N$.
From $AA' \parallel MN$ and $AA' = MN = d$, we get that $AA'NM$ is a parallelogram, therefore $AM = A'N$. (2)

From relations (1) and (2), we get that $AM = A'N = BN$, thus $AM = BN$ and $MN = d$.
In conclusion, we have proved the chosen points M and N have the desired properties.

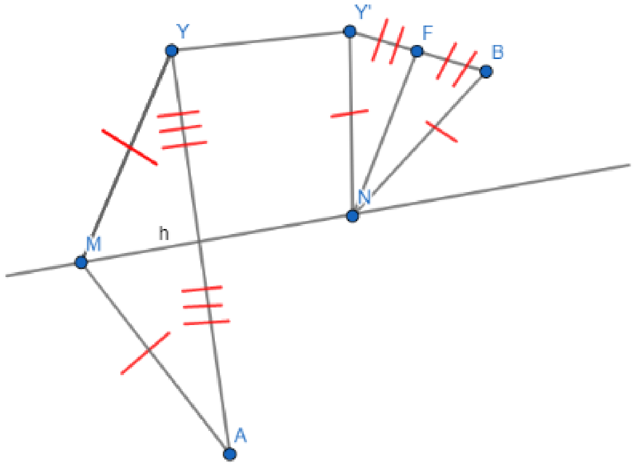
Case 2: The points A and B are on opposite sides of the line h.



In this case, we consider Y to be the symmetrical point of A with respect to the line h. Thus, h is the perpendicular bisector of the segment AY.

We see that he points B and Y are both on the same side of the line h.

Following the same steps as in the previous case, we consider the point Y' such that $YY' = d$ and $YY' \parallel h$. Then, we construct the perpendicular bisector of the segment BY'.
Let N be the intersection of this perpendicular bisector with the line h. Then, the quadrilateral $MYY'N$ is a parallelogram. This implies that $YY' = MN = d$ and $MY = NY'$.



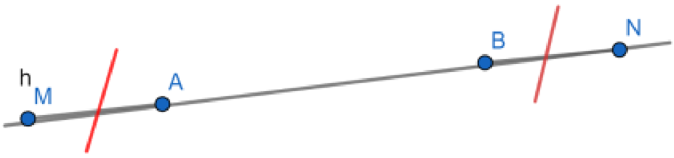
But $MY = AM$ (as M is on the perpendicular bisector of the segment AY), and $BN = NY'$ (as N is on the perpendicular bisector of the segment BY').
In conclusion, we have that $AM = MY = NY' = BN$, thus the constructed points M and N have the desired properties.

Case 3: Both points A and B are on the line h.

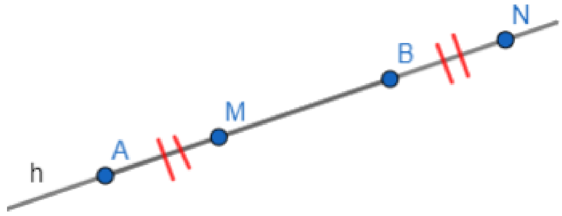
1. If $AB = d$, then Tom runs from home to work. Here, $M = A$ and $N = B$.

In this case, $A = M$ and $B = N$.

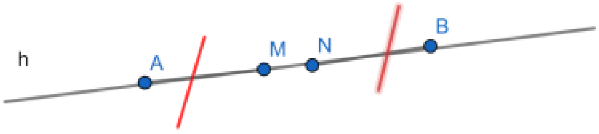
2. If $AB = a < d$, then it is easy to find two points M and N on the line h, lying outside the segment AB, such that $AM = BN = \frac{d-a}{2}$ and $MN = d$. Thus, Tom can walk to point M on the opposite direction to the point B, from there he starts running the full distance of $MN = d$, and then he can walk from point N to point B, as shown in the figure below.



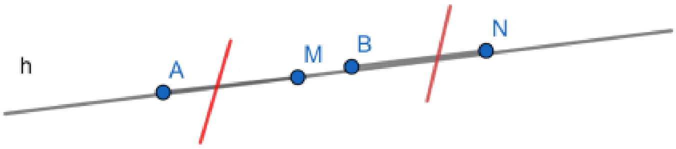
Note that, in this case, Tom can also choose only one of the points M and N inside the segment AB, such that $AM = BN$ and $MN = d$, as shown in the figure below.



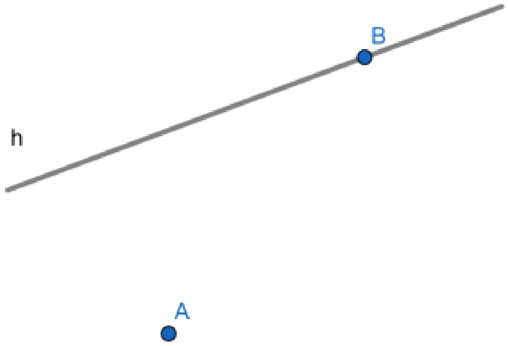
3. If $AB = a > d$, then the points M and N can be located on the segment AB, such that $AM = \frac{a-d}{2}$ and $NB = \frac{a-d}{2}$.



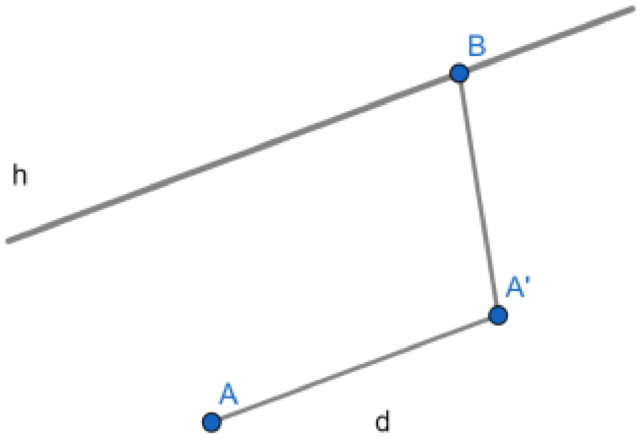
It is also possible to choose only one of the points M and N inside the segment AB, such that $AM = BN$ and $MN = d$, as shown in the figure below.



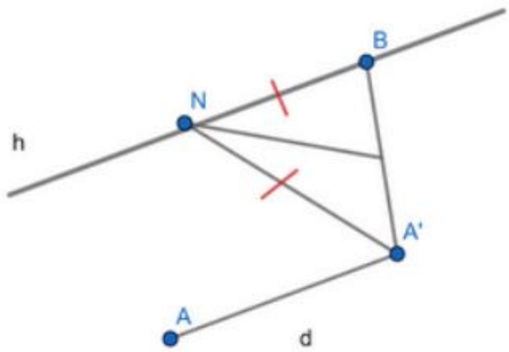
Case 4: Only one of the points A and B are on the line h.
We consider only B on the line h (but we can use the same ideas when only A is on the line h)



We start by drawing AA', so that $AA' = d$ and $AA' \parallel h$.

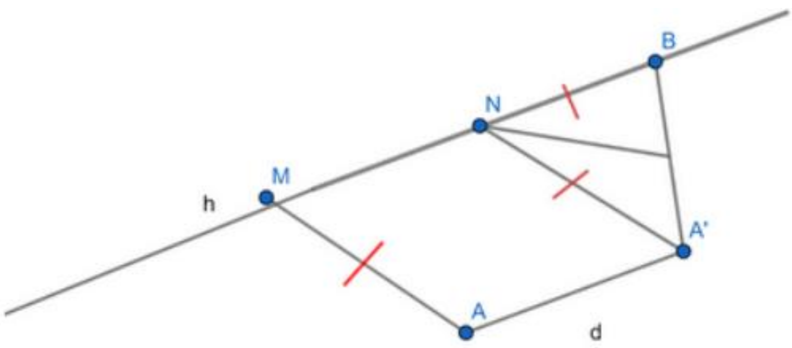


Then, we draw $A'B$, and then we draw its perpendicular bisector. We get:



The intersection of the perpendicular bisector with h is the point N and $A'N = BN$, because N is a point on the perpendicular bisector of the segment and it is equidistant from A' and B.

We draw AM so that $AM \parallel AN$. We see that $AMNA'$ is a parallelogram, as $AM \parallel A'N$ and $AA' \parallel MN$.



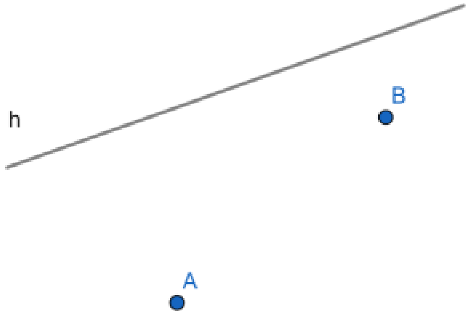
As $BN = A'N$ and $A'N = AM$ (because $AMNA'$ is a parallelogram) we get that $AM = BN$. Thus, the points M and N have the desired properties.

Task 2:

We have to find the location of the point M on the line h such that the total distance $AM + MN + NB$ is minimum.

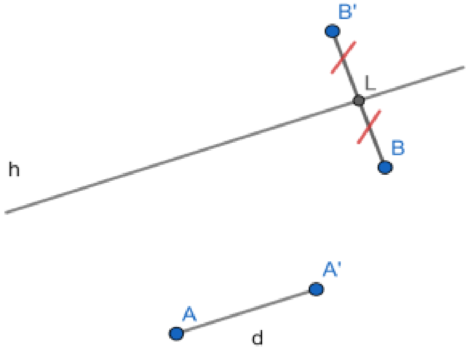
Firstly, we note that $MN = d$ is fixed, so it remains to find the point M such that the sum $AM + NB$ is minimum.

Case 1: The points A and B are on the same side of the line h.

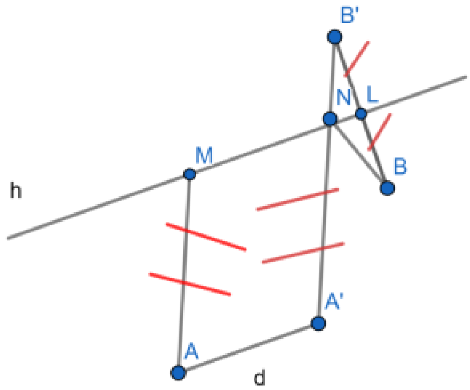


Steps in the construction:

1. We consider the point A' such that $AA' = d$ and $AA' \parallel h$.
2. We consider B' the symmetrical point of B with respect to the line h. Therefore, the line h is the perpendicular bisector of BB'.



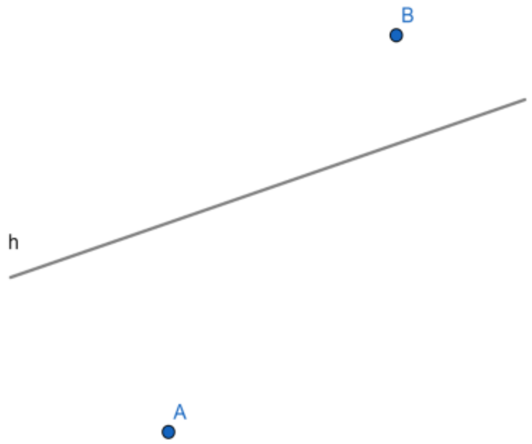
3. Let N be the intersection point of the segment $A'B'$ with the line h.
4. We now consider the point M on h so that $MN = d$ or $AM \parallel A'N$.



We now prove that the point M is the point that we are looking for.

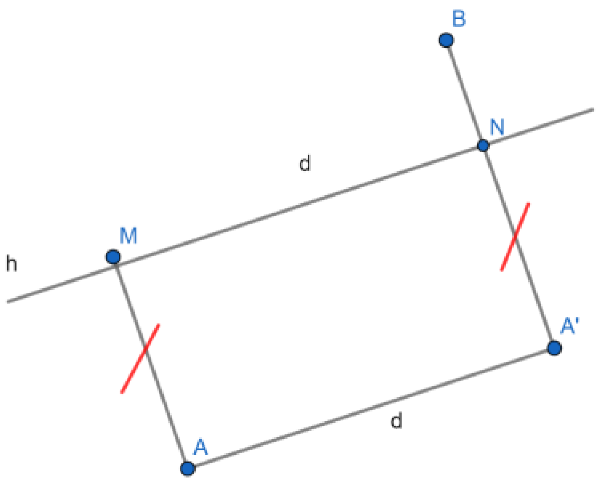
We see that $AA'NM$ is a parallelogram, thus $AM = A'N$. Because N is a point on the perpendicular bisector of BB', we get that $BN = B'N$. Therefore, $AM + BN = A'N + NB' = A'B'$.
If A and B are fixed points, then A' and B' are also fixed points. As the shortest distance between two given points is the straight segment between them, $A'B'$, we conclude that M is the desired point that we are searching for.

Case 2: The points A and B are on opposite sides of the line h.



Steps in the construction:

1. We consider the point A' such that $AA' = d = MN$ and $AA' \parallel h$.
2. Let N be the intersection point of the segment $A'B$ with the line h.
3. We consider the point M on the line h such that $MN = d$ or $AM \parallel A'N$.



We now prove that the point M is the point that we are looking for.

Because $AA' = d$ and $AA' \parallel h$, the quadrilateral $AA'NM$ will be a parallelogram. Thus, $AM = A'N$. Then, $AM + BN = A'N + BN = A'B = minimum$, as the points A', N and B are collinear.

Case 3: Both points A and B are on the line h.

1. $AB = d$

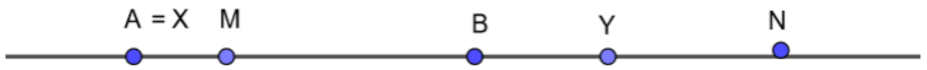
Then $A = M$ and $B = N$, and the shortest distance is $AM + MN + NB = 0 + d + 0 = d$, which is shorter than the other configuration where we have $M \in AB$ and $N \notin AB$.

2. $AB < d$.

For example, the minimum distance $AM + d + NB$ can be obtained when $M = A$ or $N = B$. If $AB = a$ and $MN = d$, then the shortest distance is $d + d - a = 2d - a$.

We can also consider both of the points M, N outside the segment AB, and the shortest distance will be $MN + AM + BN = d + d - a = 2d - a$, as $AM + BN = d - a$, and we get the same result (Tom returns to B).

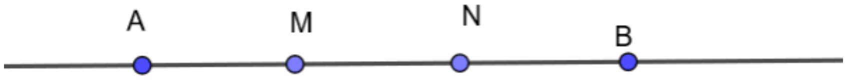
If we consider one of the points M and N between A and B, and one outside the segment AB, the distance will $AM + BN + d > d - a + d = 2d - a$. Indeed, we shall prove that $AM + BN > d - a$.



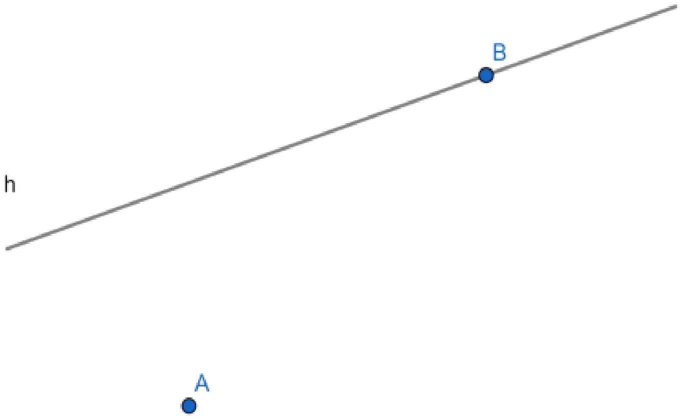
We consider $A = X$ and $Y \in (BN)$ (if $X = A$, $M \in (AB)$ and $XY = MN$ and $MN > AB$, $Y \in (BN)$) so that $XY = MN = d$. $BY = MN - AB = d - a$, Y ∈ (BN) so $BN > BY$ so $AM + BN > BY = d - a$.

3. $AB > d$.

We choose M and N so that both M and $N \in (AB)$. Then, the shortest distance $AM + MN + BN$ will be equal to AB.

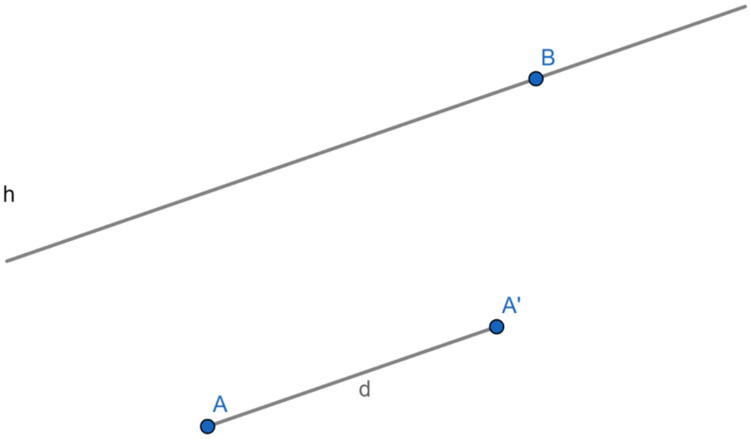


Case 4: Only one of the points A and B are on the line h.
We consider only B on the line h (but we can use the same ideas when only A is on the line h).
We follow almost the same steps as in **Case 2.**



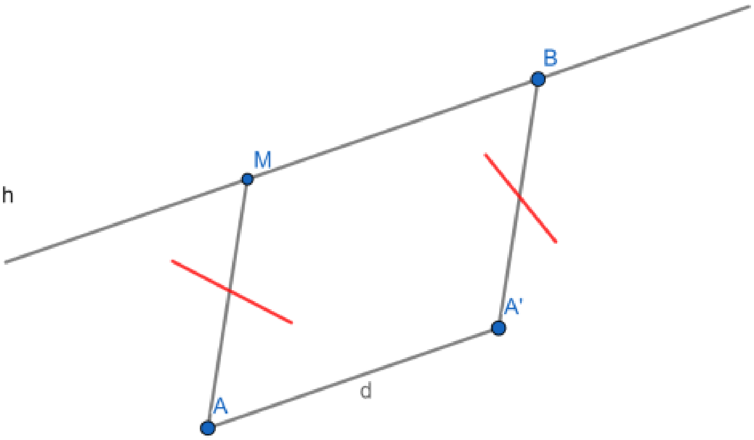
Steps in the construction:

1. We build AA' so that $AA' = d$ and $AA' \parallel h$



2. We draw $A'B$ and $AM \parallel A'B$, (in this case $B = N$ both points B and N are on the line to minimize the distance $AM + d + BN, BN = 0$).

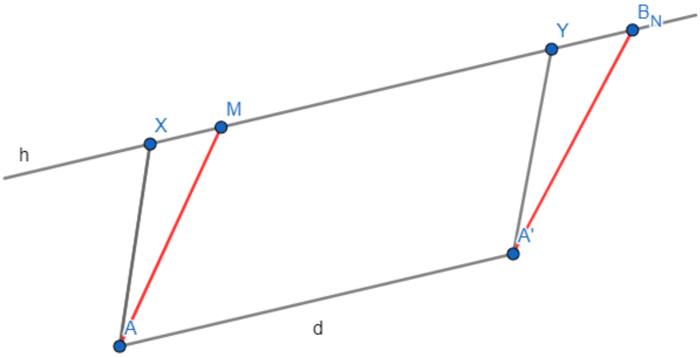
We see that $AA'BM$ is a parallelogram, as: $AM \parallel A'B$ and $AA' \parallel MB$, whence $AM = A'B = A'N$.



As $MB = d$, and $MN = d$, the shortest distance is when $B = N$.

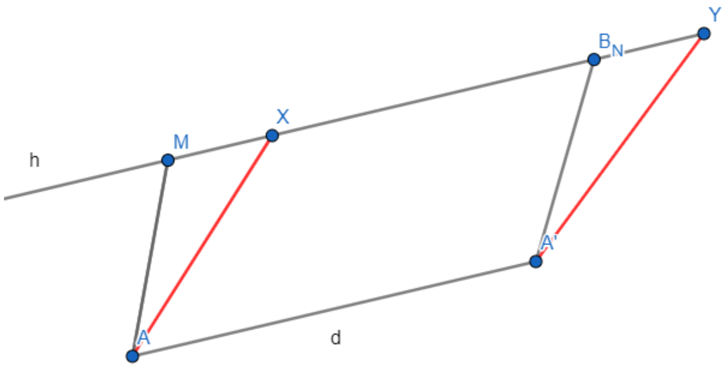
We prove that when M and N are in the positions we considered, the distance is minimum.

a) We consider X and Y ($XY = d$) as alternative position for points M and N, but we keep M and $B = N$ on the figure with $X \notin (MN)$ and $Y \in (MN)$, $M \in (XY)$, $X \neq M$ and $Y \neq N$.



We have $AA'YX$ parallelogram since $AA' = XY = d$ and $AA' \parallel XY$ and $AMBA'$ is also a parallelogram, from the previous construction.
We have to prove that $AX + XY + YB > AM + MN + BN$.
$MN = XY = d$ and $BN = 0$. We have to prove that $AX + YB > AM$, but because $AMBA'$ is a parallelogram $AM = AB$.
But $AX = A'Y$ since $AXYA'$ is a parallelogram we now have to prove that $A'Y + YB > A'B$ which results from the triangular inequality.

b) We consider X and Y ($XY = d$) as alternative position for points M and N, but we keep M and $B = N$ on the figure with $X \in (MN)$ and $Y \notin (MN)$, $M \notin (XY)$, $X \neq M$ and $Y \neq N$.



We have $AA'YX$ parallelogram since $AA' = XY = d$ and $AA' \parallel XY$ and $AMBA'$ is also a parallelogram, from the previous construction.
We have to prove that $AX + XY + YB > AM + MN + BN$. Here, we also added YB because Tom will return to B (his house).
$MN = XY = d$ and $BN = 0$. We have to prove that $AX + YB > A'B$ but because $AMBA'$ is a parallelogram $AM = A'B$.
But $AX = A'Y$ since $AXYA'$ is a parallelogram we now have to prove that $A'Y + YB > A'B$ which results from the triangular inequality.

Conclusion

For the first task, we had to find the position of points, M and N so that $AM = BN$. At first we tried to use circle arcs, but we realized that we weren't respecting d, the fixed running distance, so we used the propriety of the points on the perpendicular bisector of a segment (points on the perpendicular bisector of a segment are equidistant from the endpoints of the segment).

For the second task, we had to minimize $AM + MN + BN$, since MN is fixed, we had to minimize $AM + BN$, we built the symmetrical of B to h to find the minimum distance.

Also, for the part where one of the points (A or B) is on the running track h, we used the triangular inequality to show that only a certainly position of the points is possible.

References

- **Upper School** geometry
- **GeoGebra** notes
- Gazeta Matematică