Optimal Route
year 2022-2023

Surnames and first names of students, grades: Danu Evelina-Teodora (7th grade), Fedorenciuc Lara-Andreea (7th grade), Lău Ruxandra-Sofia (7th grade), Tudose Eric-Mihnea-Constantin (7th grade)
School: "Costache Negruzzi" National College of Iași, Romania
Teacher: Ph.D. Ioana Cătălina Anton, „Costache Negruzzi” National High School of Iași, Romania
Researcher and his university: Ph.D Iulian Stoleriu, University „Alexandru Ioan Cuza” of Iași, Romania
Keywords: minimum distance, running, Tom

Abstract:
Our research seeks to determine points $M$ and $N$ on the running track $h$, keeping the running distance $d$, in which $AM = BN$, as well as the minimum distance that Tom travels to get from one point to another.

The problem
On his way back home, Tom wants to do jogging for $d$ kilometers on the running track. The running track is a straight line (see the attached figure).
Find the location of points $M$ and $N$ on the running track so that $AM = BN$.
What is the location of the point $M$ on the running track where Tom should start jogging, so that the total distance from work to home is minimal?
(You can consider various positions of points $A$ and $B$ in the plane).
Notation:
✔️ \( h \) is the running track
✔️ \( A \) and \( B \) are the two fixed points
✔️ \( A \) represents Tom’s job place
✔️ \( B \) represents Tom’s home
✔️ \( d \) is the running distance

**Task 1:**
We shall consider different positions of points \( A \) and \( B \) in the plane.

We have to find \( M \) and \( N \), points on the running track \( h \) so that \( AM = BN \).

**Case 1:** The points \( A \) and \( B \) are on the same side of the line \( h \).

**Steps in the construction:**

1. Consider the point \( A' \) such that \( AA' \parallel h \) and the length of \( AA' \) is \( d \);
2. Next, we draw the segment \( A'B \);
3. Let \( N \) be the intersection point of the segment \( A'B \) with the line \( h \).
4. We now consider the point \( M \) on \( h \) so that \( MN = d \) or \( AM \parallel A'N \).
5. Consider the point \( M \) on the line \( d \) such that \( MN = d \) (or \( AM \parallel BN \)).
We now prove that the point $M$ and $N$ are such that $AM = BN$. Firstly, we shall prove that $BN = A'N$. We see that the triangles $\triangle A'XN$ and $\triangle BNX$ are congruent as:
- $\angle X = 90^\circ$
- $A'X = BX$ (point $X$ is the middle of $A'B$)
- $XN = XN$ (common side)

From the congruence $\triangle A'XN \equiv \triangle BNX$ (the cathetus-cathetus case), we get that the other sides are also congruent, thus $BN = A'N$. (1)

Another way of proving this is as follows: the point $N$ is situated on the perpendicular bisector of $A'B$, and thus it has the property that it is equidistant from the endpoints of the segment, implying that $BN = A'N$.

From $AA' \parallel MN$ and $AA' = MN = d$, we get that $AA'NM$ is a parallelogram, therefore $AM = A'N$. (2)

From relations (1) and (2), we get that $AM = A'N = BN$, thus $AM = BN$ and $MN = d$.

In conclusion, we have proved the chosen points $M$ and $N$ have the desired properties.

**Case 2:** The points $A$ and $B$ are on opposite sides of the line $h$.

In this case, we consider $Y$ to be the symmetrical point of $A$ with respect to the line $h$. Thus, $h$ is the perpendicular bisector of the segment $AY$.

We see that the points $B$ and $Y$ are both on the same side of the line $h$.

Colegiul Național „Costache Negruzzi” Iași
Following the same steps as in the previous case, we consider the point \( Y' \) such that \( YY' = d \) and \( YY' \parallel h \). Then, we construct the perpendicular bisector of the segment \( BY' \).

Let \( N \) be the intersection of this perpendicular bisector with the line \( h \). Then, the quadrilateral \( MYY'N \) is a parallelogram. This implies that \( YY' = MN = d \) and \( MY = NY' \).

But \( MY = AM \) (as \( M \) is on the perpendicular bisector of the segment \( AY \)), and \( BN = NY' \) (as \( N \) is on the perpendicular bisector of the segment \( BY' \)).

In conclusion, we have that \( AM = MY = NY' = BN \), thus the constructed points \( M \) and \( N \) have the desired properties.

**Case 3:** Both points \( A \) and \( B \) are on the line \( h \).

1. If \( AB = d \), then Tom runs from home to work. Here, \( M = A \) and \( N = B \).

   ![Diagram 1](image1)

   In this case, \( A = M \) and \( B = N \).

2. If \( AB = a < d \), then it is easy to find two points \( M \) and \( N \) on the line \( h \), lying outside the segment \( AB \), such that \( AM = BN = \frac{d-a}{2} \) and \( MN = d \). Thus, Tom can walk to point \( M \) on the opposite direction to the point \( B \), from there he starts running the full distance of \( MN = d \), and then he can walk from point \( N \) to point \( B \), as shown in the figure below.

   ![Diagram 2](image2)

   Note that, in this case, Tom can also choose only one of the points \( M \) and \( N \) inside
the segment $AB$, such that $AM = BN$ and $MN = d$, as shown in the figure below.

3. If $AB = a > d$, then the points $M$ and $N$ can be located on the segment $AB$, such that $AM = \frac{a-d}{2}$ and $NB = \frac{a-d}{2}$.

It is also possible to choose only one of the points $M$ and $N$ inside the segment $AB$, such that $AM = BN$ and $MN = d$, as shown in the figure below.

**Case 4:** Only one of the points $A$ and $B$ are on the line $h$.
We consider only $B$ on the line $h$ (but we can use the same ideas when only $A$ is on the line $h$)

We start by drawing $AA'$, so that $AA' = d$ and $AA' \parallel h$. 
Then, we draw $A'B'$, and then we draw its perpendicular bisector. We get:

![Diagram showing perpendicular bisector and parallelogram]

The intersection of the perpendicular bisector with $h$ is the point $N$ and $A'N = BN$, because $N$ is a point on the perpendicular bisector of the segment and it is equidistant from $A'$ and $B$.

We draw $AM$ so that $AM \parallel AN$. We see that $AMNA'$ is a parallelogram, as $AM \parallel A'N$ and $AA' \parallel MN$.

As $BN = A'N$ and $A'N = AM$ (because $AMNA'$ is a parallelogram) we get that $AM = BN$. Thus, the points $M$ and $N$ have the desired properties.

**Task 2:**
We have to find the location of the point $M$ on the line $h$ such that the total distance $AM + MN + NB$ is minimum.

Colegiul Național „Costache Negruzzi” Iași
Firstly, we note that $MN = d$ is fixed, so it remains to find the point $M$ such that the sum $AM + NB$ is minimum.

**Case 1:** The points $A$ and $B$ are on the same side of the line $h$.

**Steps in the construction:**

1. We consider the point $A'$ such that $AA' = d$ and $AA' \parallel h$.
2. We consider $B'$ the symmetrical point of $B$ with respect to the line $h$. Therefore, the line $h$ is the perpendicular bisector of $BB'$.
3. Let $N$ be the intersection point of the segment $A'B'$ with the line $h$.
4. We now consider the point $M$ on $h$ so that $MN = d$ or $AM \parallel A'N$.

We now prove that the point $M$ is the point that we are looking for.
We see that $AA'NM$ is a parallelogram, thus $AM = A'N$. Because $N$ is a point on the perpendicular bisector of $BB'$, we get that $BN = B'N$. Therefore, $AM + BN = A'N + NB' = A'B'$.

If $A$ and $B$ are fixed points, then $A'$ and $B'$ are also fixed points. As the shortest distance between two given points is the straight segment between them, $A'B'$, we conclude that $M$ is the desired point that we are searching for.

**Case 2:** The points $A$ and $B$ are on opposite sides of the line $h$.

**Steps in the construction:**

1. We consider the point $A'$ such that $AA' = d = MN$ and $AA' \parallel h$.
2. Let $N$ be the intersection point of the segment $A'B$ with the line $h$.
3. We consider the point $M$ on the line $h$ such that $MN = d$ or $AM \parallel A'N$.

We now prove that the point $M$ is the point that we are looking for.

Because $AA' = d$ and $AA' \parallel h$, the quadrilateral $AA'NM$ will be a parallelogram. Thus, $AM = A'N$. Then, $AM + BN = A'N + BN = A'B = minimum$, as the points $A', N$ and $B$ are collinear.

Colegiul Național „Costache Negruzzi” Iași
Case 3: Both points $A$ and $B$ are on the line $h$.

1. $AB = d$

![Diagram](image1)

Then $A = M$ and $B = N$, and the shortest distance is $AM + MN + NB = 0 + d + 0 = d$, which is shorter than the other configuration where we have $M \in AB$ and $N \notin AB$.

2. $AB < d$.

For example, the minimum distance $AM + d + NB$ can be obtained when $M = A$ or $N = B$. If $AB = a$ and $MN = d$, then the shortest distance is $d + d - a = 2d - a$.

![Diagram](image2)

We can also consider both of the points $M, N$ outside the segment $AB$, and the shortest distance will be $MN + AM + BN = d + d - a = 2d - a$, as $AM + BN = d - a$, and we get the same result (Tom returns to $B$).

![Diagram](image3)

If we consider one of the points $M$ and $N$ between $A$ and $B$, and one outside the segment $AB$, the distance will $AM + BN + d > d - a + d = 2d - a$. Indeed, we shall prove that $AM + BN > d - a$.

![Diagram](image4)

We consider $A = X$ and $Y \in (BN)$ (if $X = A$, $M \in (AB)$ and $XY = MN$ and $MN > AB$, $Y \in (BN)$) so that $XY = MN = d$. $BY = MN - AB = d - a$, $Y \in (BN)$ so $BN > BY$ so $AM + BN > BY = d - a$.

3. $AB > d$.

We choose $M$ and $N$ so that both $M$ and $N \in (AB)$. Then, the shortest distance $AM + MN + BN$ will be equal to $AB$.

![Diagram](image5)
**Case 4:** Only one of the points $A$ and $B$ are on the line $h$.
We consider only $B$ on the line $h$ (but we can use the same ideas when only $A$ is on the line $h$).
We follow almost the same steps as in **Case 2**.

**Steps in the construction:**
1. We build $AA'$ so that $AA' = d$ and $AA' \parallel h$.
2. We draw $A'B$ and $AM \parallel A'B$, (in this case $B = N$ both points $B$ and $N$ are on the line to minimize the distance $AM + d + BN, BN = 0$).
We see that $AA'BM$ is a parallelogram, as: $AM \parallel A'B$ and $AA' \parallel MB$, whence $AM = A'B = A'N$.

As $MB = d$, and $MN = d$, the shortest distance is when $B = N$.

We prove that when $M$ and $N$ are in the positions we considered, the distance is minimum.
a) We consider $X$ and $Y$ ($XY = d$) as alternative position for points $M$ and $N$, but we keep $M$ and $B = N$ on the figure with $X \notin (MN)$ and $Y \in (MN)$, $M \in (XY)$, $X \neq M$ and $Y \neq N$.

We have $AA'YX$ parallelogram since $AA' = XY = d$ and $AA' \parallel XY$ and $AMBA'$ is also a parallelogram, from the previous construction.

We have to prove that $AX + XY + YB > AM + MN + BN$. $MN = XY = d$ and $BN = 0$. We have to prove that $AX + YB > AM$, but because $AMBA'$ is a parallelogram $AM = AB$.

But $AX = A'Y$ since $AXYA'$ is a parallelogram we now have to prove that $A'Y + YB > A'B$ which results from the triangular inequality.

b) We consider $X$ and $Y$ ($XY = d$) as alternative position for points $M$ and $N$, but we keep $M$ and $B = N$ on the figure with $X \in (MN)$ and $Y \notin (MN)$, $M \notin (XY)$, $X \neq M$ and $Y \neq N$.

We have $AA'YX$ parallelogram since $AA' = XY = d$ and $AA' \parallel XY$ and $AMBA'$ is also a parallelogram, from the previous construction.

We have to prove that $AX + XY + YB > AM + MN + BN$. Here, we also added $YB$ because Tom will return to $B$ (his house).

$MN = XY = d$ and $BN = 0$. We have to prove that $AX + YB > A'B$ but because $AMBA'$ is a parallelogram $AM = A'B$.

But $AX = A'Y$ since $AXYA'$ is a parallelogram we now have to prove that $A'Y + YB > A'B$ which results from the triangular inequality.

**Conclusion**
For the first task, we had to find the position of points, $M$ and $N$ so that $AM = BN$. At first we tried to use circle arcs, but we realized that we weren’t respecting $d$, the fixed running distance, so we used the propriety of the points on the perpendicular bisector of a segment (points on the perpendicular bisector of a segment are equidistant from the endpoints of the segment).

For the second task, we had to minimize $AM + MN + BN$, since $MN$ is fixed, we had to minimize $AM + BN$, we built the symmetrical of $B$ to $h$ to find the minimum distance.

Also, for the part where one of the points ($A$ or $B$) is on the running track $h$, we used the triangular inequality to show that only a certainly position of the points is possible.

References

- Upper School geometry
- GeoGebra notes
- Gazeta Matematică