Calculation of a polygon's area

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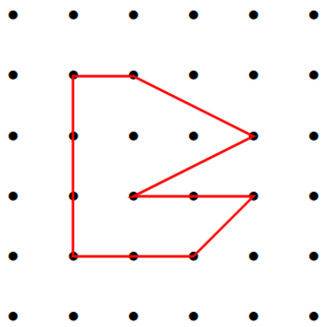
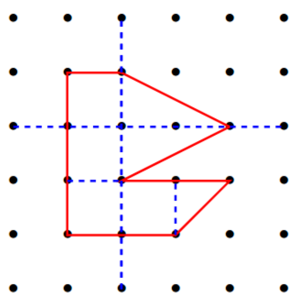
I° Introduction of the topic

Let's consider a grid of equidistant dots on a plan. We draw a polygon on the grid whose tops are some dots of the grid.



Is it possible to calculate the polygon's area on the basis of the dots which are inside and on the edge of the polygon?

II° Annoucement of guesses and results



The first step to solve the problem is by dividing the grid in squares of equal areas. Then, by adding the numbers of squares and triangle-rectangles which compose the area of the polygon, we can find it area. That theory works but it needs to know first the numbers of squares, half-squares and every other triangle-rectangles' shapes which compose the polygon. That's not an efficient way to solve the problem, especially if we work with big polygons.

To solve the problem for any-given polygon, we can use the Pick's theorem:

For any-given polygon of which tops are on the knocks of a grid formed by squares, we have:

S(P), the area of any-given polygon

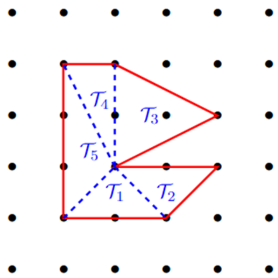
i = the numbers of dots of the grid strictly interior at the polygon

b = the numbers of dots of the grid on the sides of the polygon

$S(P) = i + b/2 - 1$

With the Pick's theorem, we found a solution for every convex polygon we have, but we haven't found yet the particularities of the concave polygons for which the theorem works or not.

A way of using the theorem after determining the area of in initial polygon is by cutting-out the area in triangle shapes, using the theorem to know the area of each of them and then, adding or subtracting another triangle shape.



So, we want to know if a polygon is convex or concave to use the Pick's theorem in consequences. A convex polygon is a polygon with the angle lower than 180°, and a concave polygon is a polygon with at least one of the angle greater than 180°. Then, we just have to calculate if every angles of a polygon are inferior to 180° in a trine direction.

Another way to find that is by using a comparison between two angles' values and two lengths thanks to one reference's top named the top A or the top with the smallest coordinates. If the values follow the ascending of the tops, the polygon is convex.

III° Demonstrations and explanations

A) The Pick's theorem

<u>Proposition:</u>

- If P is a polygon of which the tops are dots of the grid, we write S(P) the area of the polygon, i(P) the number of dots of the grid that are inside the polygon and not on the outline and b(P) the number of dots of the grid on the sides of the polygon, we get:

$S(P) = i(P) + 1/2b(P) - 1$

We divide P in any-given triangles.

If P can be divided in any-given triangles, the sum of the areas of the triangles is equal to the area of P.

- If $S(P')$ is the area of any-given polygon, $S(P)$ is the area of a polygon we already have and $S(T)$ the area of a triangle, we want to prove that:

$S(P') = S(P)+S(T)$

We admit:

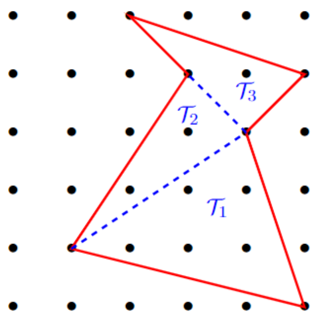
$P' = P \cup T$

<u>Demonstration:</u>

- We want to demonstrate that:

$S(P) = \sum T$

We take on example, a convex polygon divided in three triangles areas:



On one side:

$S(T1) = i(T1) + 1/2b(T1) -1$

$= 5 + 3/2 - 1$

$= 5.5$

$S(T2) = i(T2) + 1/2b(T2) -1$

$= 2 + 3/2 - 1$

$= 2.5$

$S(T3) = i(T3) + 1/2b(T3) -1$

$= 1+ 4/2 -1$

$= 2$

$S(T1) + S(T2) + S(T3) = 5.5 + 2.5 + 2$

$= 10$

On another side:

$S(P) = i(P) + ½(P) -1$

$= 8 + 6/2 -1$

$= 10$

We get $S(P) = S(T1) + S(T2) + S(T3)$.

- We want to demonstrate that:

$S(P') = S(P) + S(T)$

On a side:

Dots inside of P' are the dots n on the border between P and T added to the dots inside of P and T:

$i(P') = i(P) + i(T) + n$

On another side:

Dots on the outline of P' are the dots n on the border between P and T and the two tops repeat two times subtracted from the dots on the outlines of P and T:

$b(P') = b(P) + b(T) - 2n - 2$

We get:

$$\begin{aligned} S(P') &= i(P') + b(P')/2 - 1 \\ &= i(P) + i(T) + n + \tfrac{1}{2}[b(P) + b(T) - 2n - 2] - 1 \\ &= [i(P) + \tfrac{1}{2} b(P) - 1] + i(T) + \tfrac{1}{2} b(T) - 1 \\ &= S(P) + S(T) \end{aligned}$$

The area of P' is equal to the sum of the areas of P and T.

<u>Example:</u>

We take on example the polygon initially given:

$$\begin{aligned} S(T1) = S(T2) = S(T4) &= 0 + 4/2 - 1 \\ &= 1 \end{aligned}$$

$S(T3) = 1 + 4/2 - 1$

$= 2$

$S(T5) = 0 + 5/2 - 1$

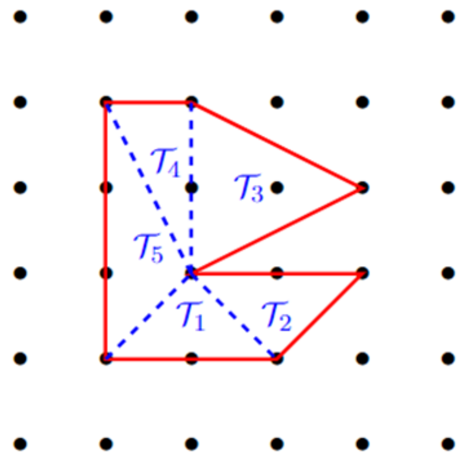
$= 1.5$

$S(P) = S(T1) + S(T2) + S(T3) + S(T4) + S(T5)$

$= 1 + 1 + 2 + 1 + 1.5$

$= 6.5$

We find again $S(P) = 6.5$



B) The theory of angles

Proposition:

- If we take A (0 ; 0) as the first top of a polygon and the top with the smaller coordinates, x as any-given top belonging to the abscissa axe, and B and C two tops of the polygon with B going ahead of C and a trine direction,

We compare the degree of angles xAB° and xAC° and we get this inequality:

$xAB^\circ < xAC^\circ$

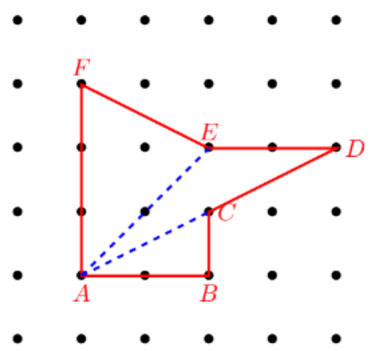
So, the polygon is convex if every angle follows this inequality.

- If two degrees have the same value, we compare the length of (AB) and (AC) and we get this inequality:

$AB < AC$

Example:

We take a polygon where the tops A, C and D are aligned.



- We compare all the angles:

$xAB° = 0°$; $xAC° = 30° = xAD°$; $xAE° = 45°$; $xAF° = 90°$ so

$xAB° < xAC° = xAD° < xAE° < xAF°$

- We get $xAC° = xAD°$. We compare the length AC and AD:

AC (2 ; 1) ; AD (4 ; 2)

$AC^2 = 2^2 + 1^2$ $\qquad AC = \sqrt{AC^2}$

$= 4 + 1$ $\qquad = \sqrt{5}$

$= 5$

$AD^2 = 4^2 + 2^2$ $\qquad AD = \sqrt{AD^2}$

$= 16 + 4$ $\qquad = \sqrt{20}$

$\sqrt{5} < \sqrt{20}$ so $AC < AD$

The polygon follows the two inequalities so it is convex.

<u>Proposition:</u>

We place a polygon in a trine direction (from axe x to axe y)

- If we have the polygon which every top is inferior to 180° in a trine direction, the polygon is convex.
- If we have the polygon which only one top is superior to 180° in a trin direction, the polygon is concave.

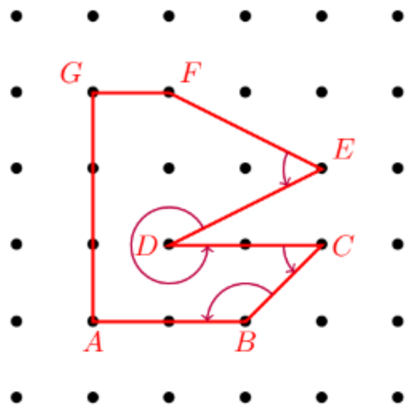
<u>Example:</u>

We take on example the polygon initially given:

$CBA° < 180°$

$DCB° < 180°$

$EDC° > 180°$



FED° < 180°

So the concave part is on the EDC angle.

The polygon is concave.

IV° Conclusion

To find the area of any-given polygon, we determine first if it's a concave or a convex polygon thanks to the theory of angle.

With the Pick's Theorem we can find the area of any-given convex polygon.

Next, we compare the areas of a convex polygon and the same polygon with a concave part. By subtracting the area of the concave part at the area of the convex polygon, we get the area of any-given concave polygon.

For polygons with a polygonal hole, a proposed solution is subtracting the area around the hole from the complete area or using the same method that we have with a polygon without a hole.

Also, we must demonstrate if the Pick's theorem works for every triangle and determine the particularities of the concave polygons for which the theorem works.