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REMARKABLE INTEGERS

The Problem. A number is called *remarkable* if there exists a multiple of it that is written (use Base-10) as a string of 9, followed (or not) by a string of 0. For example, 3 is remarkable (justify). What are the remarkable numbers less than 6? In general, what are the remarkable numbers?

Definition. An integer x, other than 0, is called *a remarkable number* if there exists M_x, a multiple of x such that M_x can be written as $\underbrace{99\ldots9}_{m\in N^*}\underbrace{00\ldots0}_{n\in N}$.

Question 1. 3 is a remarkable number? Why?
Answer 1.1. *3 is a remarkable number,* because there exists
$M_3 = 3 \times 3 = 9,$ with $m = 1$ and $n = 0$.

Question 2. Which numbers, smaller than 6, are remarkable?
Remark 2.1. The remarkable numbers that are smaller than 6 can be divided in two categories:
1) positive integers (1,2,3,4,5);
2) negative integers.

Answer 2.1. 1, 2, 3, 4, 5 *are remarkable numbers*, because there exists
$M_1 = 1 \times 9 = 9,$ with $m = 1$ and $n = 0$.
$M_2 = 2 \times 45 = 90,$ with $m = 1$ and $n = 1$.
$M_3 = 3 \times 3 = 9,$ with $m = 1$ and $n = 0$.
$M_4 = 4 \times 225 = 900,$ with $m = 1$ and $n = 2$.
$M_5 = 5 \times 18 = 90,$ with $m = 1$ and $n = 1$.
Note that the specified multiples are not unique.
Answer 2.2. To determine all negative integers that are remarkable numbers, the general form of the remarkable positive numbers will be determined later.

Question 3. In general, which numbers are remarkable?
Answer 3.1−Statement. All positive integers are remarkable numbers.
Proof. We will prove that any positive integer x has a multiple of form $\underbrace{99\ldots9}_{m\in N^*}\underbrace{00\ldots0}_{n\in N}$ with m and n unspecified.

Remark 1. Finding $x \in Z, x \neq 0$ remarkable, for which there exists a multiple with the form
$M_x = \underbrace{99\ldots9}_{m\in N^*}\ \underbrace{00\ldots0}_{n\in N} = 9 \times \underbrace{11\ldots1}_{m\in N^*} \times 2^n \times 5^n$, $m\in N^*$, $n\in N$
is like looking for the divisors of the number $9 \times \underbrace{11\ldots1}_{m\in N^*} \times 2^n \times 5^n$ using the divisibility criteria in Number Theory.
Next, we intend to specify, for a given positive integer x, those numbers m and n from the sought multiple M_x.
Using the Scientific WorkPlace Program for the decomposition into prime factors for $\underbrace{11\ldots1}_{m\in N^*}$, we will study some examples of prime numbers and powers of prime numbers that are remarkable numbers.

Using the remainders theory, we will discover rules for determining the divisors for $\underbrace{11\ldots1}_{m\in N^*}$.

We will present and demonstrate, using number theory, statements about the structure of remarkable numbers:

Statement 1: For all $n_1 \in N$, all $n_2 \in N$ and all $s \in N$, the numbers $2^{n_1} \times 5^{n_2} \times 3^s$ *are remarkable numbers.*

Statement 2: All prime numbers $p \in N^*$, $p > 5$ *are remarkable numbers*

Statement 3: For all $\alpha \in N$ and all prime numbers $p \in N^*$, $p > 5$, the numbers p^α *are remarkable numbers.*

Statement 4: For any $\alpha_1, \alpha_2, \ldots, \alpha_k \in N$, and any prime number $p_1, p_2, \ldots, p_k \in N^*$, greater than 5, the numbers $p_1^{\alpha_1} \times p_2^{\alpha_2} \times \ldots \times p_n^{\alpha_n}$ *are remarkable numbers.*

Statement 5: Any positive integer x is a *remarkable number.*

Answer 3.2.1−Statement 1. For all $n \in N$, the numbers $2^n \times 5^n$ *are remarkable numbers.*

Proof. For all $n \in N$, there exists $M_n = \underbrace{11\ldots1}_{m\in N^*} \times 9 \times 2^n \times 5^n$, which is a multiple of $2^n \times 5^n$.

Out of curiosity, we used the Scientific WorkPlace Program for the decomposition into prime factors for $\underbrace{11\ldots1}_{n\in N^*}$ and we obtain, for $m = \overline{1, 81}$:

$m = 1 : 1 = 1$
$m = 2 : 11 = 11$
$m = 3 : 111 = 3 \times 37$
$m = 4 : 1111 = 11 \times 101$
$m = 5 : 11111 = 41 \times 271$
$m = 6 : 111111 = 3 \times 7 \times 11 \times 13 \times 37$
$m = 7 : 1111111 = 239 \times 4649$
$m = 8 : 11111111 = 11 \times 73 \times 101 \times 137$
$m = 9 : 11111111 = 11 \times 73 \times 101 \times 137$
$m = 10 : 1111111111 = 11 \times 41 \times 271 \times 9091$
$m = 11 : 11111111111 = 21649 \times 513239$
$m = 12 : 111111111111 = 3 \times 7 \times 11 \times 13 \times 37 \times 101 \times 9901$
$m = 13 : 1111111111111 = 53 \times 79 \times 265371653$
$m = 14 : 11111111111111 = 11 \times 239 \times 4649 \times 909091$
$m = 15 : 111111111111111 = 3 \times 31 \times 37 \times 41 \times 271 \times 2906161$
$m = 16 : 1111111111111111 = 11 \times 17 \times 73 \times 101 \times 137 \times 5882353$
$m = 17 : 11111111111111111 = 2071723 \times 5363222357$
$m = 18 : 111111111111111111 = 3^2 \times 7 \times 11 \times 13 \times 19 \times 37 \times 52579 \times 33367$
$m = 19 : 1111111111111111111 = 1111111111111111111$
$m = 20 : 11111111111111111111 = 11 \times 41 \times 101 \times 271 \times 3541 \times 9091 \times 27961$
$m = 21 : 111111111111111111111 = 3 \times 37 \times 43 \times 239 \times 1933 \times 4649 \times 10838689$
$m = 22 : 1111111111111111111111 = 11^2 \times 23 \times 4093 \times 8779 \times 21649 \times 513239$
$m = 23 : 11111111111111111111111 = 11\,111\,111\,111\,111\,111\,111\,111$
$m = 24 : 111111111111111111111111 = 3 \times 7 \times 11 \times 13 \times 37 \times 73 \times 101 \times 137 \times 9901 \times 99\,990\,001$
$m = 25 : 1111111111111111111111111 = 41 \times 271 \times 21\,401 \times 25\,601 \times 182\,521\,213\,001$
$m = 26 : 11111111111111111111111111 = 11 \times 53 \times 79 \times 859 \times 265\,371\,653 \times 1058\,313\,049$
$m = 27 : 111111111111111111111111111 = 3^3 \times 37 \times 757 \times 333667 \times 440334654777631$
$m = 28 : 1111111111111111111111111111 = 11 \times 29 \times 101 \times 239 \times 281 \times 4649 \times 909\,091 \times 121\,499\,449$
$m = 29 : 11111111111111111111111111111 = 3191 \times 16\,763 \times 43\,037 \times 62\,003 \times 77\,843\,839\,397$
$m = 30 : \underbrace{11\ldots1}_{30} = 3 \times 7 \times 11 \times 13 \times 31 \times 37 \times 41 \times 211 \times 241 \times 271 \times 2161 \times 9091 \times 2906\,161$
$m = 31 : \underbrace{11\ldots1}_{31} = 2791 \times 6943\,319 \times 57\,336\,415\,063\,790\,604\,359$
$m = 32 : \underbrace{11\ldots1}_{32} = 11 \times 17 \times 73 \times 101 \times 137 \times 353 \times 449 \times 641 \times 1409 \times 69\,857 \times 5882\,353$

$m = 33:\ \underbrace{11\dots1}_{33} = 3 \times 37 \times 67 \times 21\,649 \times 513\,239 \times 1344\,628\,210\,313\,298\,373$

$m = 34:\ \underbrace{11\dots1}_{34} = 11 \times 103 \times 4013 \times 2071\,723 \times 5363\,222\,357 \times 21\,993\,833\,369$

$m = 35:\ \underbrace{11\dots1}_{35} = 41 \times 71 \times 239 \times 271 \times 4649 \times 123\,551 \times 102\,598\,800\,232\,111\,471$

$m = 36:\ \underbrace{11\dots1}_{36} = 3^2 \times 7 \times 11 \times 13 \times 19 \times 37 \times 101 \times 9901 \times 52579 \times 333667 \times 999999000001$

$m = 37:\ \underbrace{11\dots1}_{37} = 2028\,119 \times 247\,629\,013 \times 2212\,394\,296\,770\,203\,368\,013$

$m = 38:\ \underbrace{11\dots1}_{38} = 11 \times 909\,090\,909\,090\,909\,091 \times 1111\,111\,111\,111\,111\,111$

$m = 39:\ \underbrace{11\dots1}_{39} = 3 \times 37 \times 53 \times 79 \times 265\,371\,653 \times 900\,900\,900\,900\,990\,990\,990\,991$

$m = 40:\ \underbrace{11\dots1}_{40} = 11 \times 41 \times 73 \times 101 \times 137 \times 271 \times 3541 \times 9091 \times 27\,961 \times 1676\,321 \times 5964\,848\,081$

$m = 41:\ \underbrace{11\dots1}_{41} = 83 \times 1231 \times 538\,987 \times 201\,763\,709\,900\,322\,803\,748\,657\,942\,361$

$m = 42:\ \underbrace{11\dots1}_{42} = 3 \times 7^2 \times 11 \times 13 \times 37 \times 43 \times 127 \times 239 \times 1933 \times 2689 \times 4649 \times 459691 \times 909091 \times 10838689$

$m = 43:\ \underbrace{11\dots1}_{43} = 173 \times 1527\,791 \times 1963\,506\,722\,254\,397 \times 2140\,992\,015\,395\,526\,641$

$m = 44:\ \underbrace{11\dots1}_{44} = 11^2 \times 23 \times 89 \times 101 \times 4093 \times 8779 \times 21649 \times 513239 \times 1052788969 \times 1056689261$

$m = 45:\ \underbrace{11\dots1}_{45} = 3^2 \times 31 \times 37 \times 41 \times 271 \times 238681 \times 333667 \times 2906161 \times 4185502530133110721$

$m = 46:\ \underbrace{11\dots1}_{46} = 11 \times 47 \times 139 \times 2531 \times 549\,797\,184\,491\,917 \times 11\,111\,111\,111\,111\,111\,111\,111$

$m = 47:\ \underbrace{11\dots1}_{47} = 35\,121\,409 \times 316\,362\,908\,763\,458\,525\,001\,406\,154\,038\,726\,382\,279$

$m = 48:\ \underbrace{11\dots1}_{48} = 3 \times 7 \times 11 \times 13 \times 17 \times 37 \times 73 \times 101 \times 137 \times 9901 \times 5882\,353 \times$

$\times\, 99\,990\,001 \times 9999\,999\,900\,000\,001$

$m = 49:\ \underbrace{11\dots1}_{49} = 239 \times 4649 \times 505\,885\,997 \times 1976\,730\,144\,598\,190\,963\,568\,023\,014\,679\,333$

$m = 50:\ \underbrace{11\dots1}_{50} = 11 \times 41 \times 251 \times 271 \times 5051 \times 9091 \times 21\,401 \times 25\,601 \times 182\,521\,213\,001 \times$

$\times\, 78\,875\,943\,472\,201$

$m = 51:\ \underbrace{11\dots1}_{51} = 3 \times 37 \times 613 \times 210\,631 \times 2071\,723 \times 52\,986\,961 \times 5363\,222\,357 \times 13\,168\,164\,561\,429\,877$

$m = 52:\ \underbrace{11\dots1}_{52} = 11 \times 53 \times 79 \times 101 \times 521 \times 859 \times 265\,371\,653 \times 1058\,313\,049 \times$

$\times\, 1900\,381\,976\,777\,332\,243\,781$

$m = 53:\ \underbrace{11\dots1}_{53} = 107 \times 1659\,431 \times 1325\,815\,267\,337\,711\,173 \times 47\,198\,858\,799\,491\,425\,660\,200\,071$

$m = 54:\ \underbrace{11\dots1}_{54} = 3^3 \times 7 \times 11 \times 13 \times 19 \times 37 \times 757 \times 52579 \times 333667 \times 70541929 \times 14175966169 \times$

$\times\, 440334654777631$

$m = 55:\ \underbrace{11\dots1}_{55} = 41 \times 271 \times 1321 \times 21\,649 \times 62\,921 \times 513\,239 \times 83\,251\,631 \times$

$\times\, 1300\,635\,692\,678\,058\,358\,830\,121$

$m = 56:\ \underbrace{11\dots1}_{56} = 11 \times 29 \times 73 \times 101 \times 137 \times 239 \times 281 \times 4649 \times 7841 \times 909\,091 \times$

$\times\, 121\,499\,449 \times 127\,522\,001\,020\,150\,503\,761$

$m = 57:\ \underbrace{11\dots1}_{57} = 3 \times 37 \times 21\,319 \times 10\,749\,631 \times 1111\,111\,111\,111\,111\,111 \times$

$\times\, 3931\,123\,022\,305\,129\,377\,976\,519$

$m = 58:\ \underbrace{11\dots1}_{58} = 11 \times 59 \times 3191 \times 16\,763 \times 43\,037 \times 62\,003 \times 77\,843\,839\,397 \times$

$\times\, 154\,083\,204\,930\,662\,557\,781\,201\,849$

$$m = 59:\ \underbrace{11\ldots1}_{59} = 2559\,647\,034\,361 \times 4340\,876\,285\,657\,460\,212\,144\,534\,289\,928\,559\,826\,755\,746\,751$$

$$m = 60:\ \underbrace{11\ldots1}_{60} = 3 \times 7 \times 11 \times 13 \times 31 \times 37 \times 41 \times 61 \times 101 \times 211 \times 241 \times 271 \times 2161 \times 3541 \times$$
$$\times 9091 \times 9901 \times 27\,961 \times 4188\,901 \times 2906\,161 \times 39\,526\,741$$

$$m = 61:\ \underbrace{11\ldots1}_{61} = 733 \times 4637 \times 329\,401 \times 974\,293 \times 1360\,682\,471 \times 106\,007\,173\,861\,643 \times$$
$$\times 7061\,709\,990\,156\,159\,479$$

$$m = 62:\ \underbrace{11\ldots1}_{62} = 11 \times 2791 \times 6943\,319 \times 57\,336\,415\,063\,790\,604\,359 \times$$
$$\times 909\,090\,909\,090\,909\,090\,909\,090\,909\,091$$

$$m = 63:\ \underbrace{11\ldots1}_{63} = 3^2 \times 37 \times 43 \times 239 \times 1933 \times 4649 \times 10837 \times 23311 \times 45613 \times 333667 \times 10838689 \times$$
$$\times 45\,121\,231 \times 1921\,436\,048\,294\,281$$

$$m = 64:\ \underbrace{11\ldots1}_{64} = 11 \times 17 \times 73 \times 101 \times 137 \times 353 \times 449 \times 641 \times 1409 \times 19\,841 \times 69\,857 \times 5882\,353 \times$$
$$\times 6187\,457 \times 976\,193 \times 834\,427\,406\,578\,561$$

$$m = 65:\ \underbrace{11\ldots1}_{65} = 41 \times 53 \times 79 \times 271 \times 265\,371\,653 \times 162\,503\,518\,711 \times$$
$$\times 5538\,396\,997\,364\,024\,056\,286\,510\,640\,780\,600\,481$$

$$m = 66:\ \underbrace{11\ldots1}_{66} = 3 \times 7 \times 11^2 \times 13 \times 23 \times 37 \times 67 \times 4093 \times 8779 \times 21649 \times 599144041 \times 513239 \times$$
$$\times 183\,411\,838\,171 \times 1344\,628\,210\,313\,298\,373$$

$$m = 67:\ \underbrace{11\ldots1}_{67} = 493\,121 \times 79\,863\,595\,778\,924\,342\,083 \times 28\,213\,380\,943\,176\,667\,001\,263\,153\,660\,999\,177\,245\,677$$

$$m = 68:\ \underbrace{11\ldots1}_{68} = 11 \times 101 \times 103 \times 4013 \times 2071\,723 \times 1491\,383\,821 \times 28\,559\,389 \times 21\,993\,833\,369 \times$$
$$\times 5363\,222\,357 \times 2324\,557\,465\,671\,829$$

$$m = 69:\ \underbrace{11\ldots1}_{69} = 3 \times 37 \times 277 \times 203\,864\,078\,068\,831 \times 11\,111\,111\,111\,111\,111\,111\,111 \times$$
$$\times 1595\,352\,086\,329\,224\,644\,348\,978\,893$$

$$m = 70:\ \underbrace{11\ldots1}_{70} = 11 \times 41 \times 71 \times 239 \times 271 \times 4649 \times 9091 \times 123\,551 \times 909\,091 \times 4147\,571 \times$$
$$\times 102\,598\,800\,232\,111\,471 \times 265\,212\,793\,249\,617\,641$$

$m = 71:\ \underbrace{11\ldots1}_{71} =$ the Scientific WorkPlace program does not work.

$$m = 72:\ \underbrace{11\ldots1}_{72} = 3^2 \times 7 \times 11 \times 13 \times 19 \times 37 \times 73 \times 101 \times 137 \times 3169 \times 9901 \times 52579 \times 98641 \times 333667$$
$$\times 99\,990\,001 \times 999\,999\,000\,001 \times 3199\,044\,596\,370\,769$$

$m = 73:\ \underbrace{11\ldots1}_{73} =$ the Scientific WorkPlace program does not work.

$$m = 74:\ \underbrace{11\ldots1}_{74} = 11 \times 7253 \times 2028\,119 \times 247\,629\,013 \times 422\,650\,073\,734\,453 \times 296\,557\,347\,313\,446\,299$$
$$\times 2212\,394\,296\,770\,203\,368\,013$$

$$m = 75:\ \underbrace{11\ldots1}_{75} = 3 \times 31 \times 37 \times 41 \times 151 \times 271 \times 4201 \times 21\,401 \times 25\,601 \times 2906\,161 \times 182\,521\,213\,001$$
$$\times 15\,763\,985\,553\,739\,191\,709\,164\,170\,940\,063\,151$$

$$m = 76:\ \underbrace{11\ldots1}_{76} = 11 \times 101 \times 909\,090\,909\,090\,909\,091 \times 1369\,778\,187\,490\,592\,461 \times$$
$$\times 722\,817\,036\,322\,379\,041 \times 1111\,111\,111\,111\,111\,111$$

$$m = 77:\ \underbrace{11\ldots1}_{77} = 239 \times 4649 \times 5237 \times 21\,649 \times 42\,043 \times 29\,920\,507 \times 513\,239 \times$$
$$\times 136\,614\,668\,576\,002\,329\,371\,496\,447\,555\,915\,740\,910\,181\,043$$

$$m = 78:\underbrace{11\ldots1}_{78} = 3 \times 7\ 13^2 \times 37 \times 53 \times 79 \times 157 \times 859 \times 6397 \times 216451 \times 265371653 \times 1058313049 \times$$
$$\times 388\,847\,808\,493 \times 900\,900\,900\,900\,990\,990\,990\,991$$

$$m = 79:\ \underbrace{11\ldots1}_{79} = 317 \times 6163 \times 10\,271 \times 307\,627 \times 49\,172\,195\,536\,083\,790\,769 \times$$
$$\times 3660\,574\,762\,725\,521\,461\,527\,140\,564\,875\,080\,461\,079\,917$$

$m = 80: \underbrace{11 \ldots 1}_{80} = 11 \times 17 \times 41 \times 73 \times 101 \times 137 \times 271 \times 3541 \times 9091 \times 27961 \times 5882353 \times 1676321$

$\times 5070721 \times 5964848081 \times 19721061166646717498359681$

$m = 81: \underbrace{11 \ldots 1}_{81} = 3^4 \times 37 \times 163 \times 757 \times 9397 \times 333667 \times 2462401 \times 440334654777631 \times$

$\times 676421558270641 \times 130654897808007778425046117$

Heuristic 1. For all $s \in N^*$, the numbers 3^s divides $\underbrace{11 \ldots 1}_{3^s}$.

Proof. Using mathematical induction, we will prove the divisibility statement:

"$P(s): 3^s$ divides $\underbrace{11 \ldots 1}_{3^s}$, for all $s \in N^*$ "

$\underline{P(1)}$ is true, that is the basic step. Indeed,
3^1 divides $111 = 3 \times 37$.
We assume that:

$P(k)$: 3^k divides $\underbrace{11 \ldots 1}_{3^k}$ for all $k \in N^*$. Then, there exists $d \in N^*$ such that $\underbrace{11 \ldots 1}_{3^k} = d \times 3^k$.

Let us prove that P(k+1) is true, that is the induction step:

$P(k+1)$: 3^{k+1} divides $\underbrace{11 \ldots 1}_{3^{k+1}}$. We observe that

$$\underbrace{11 \ldots 1}_{3^{k+1}} = \underbrace{11 \ldots 1}_{3^k}\underbrace{11 \ldots 1}_{3^k}\underbrace{11 \ldots 1}_{3^k}$$

$$\underbrace{11 \ldots 1}_{3^{k+1}} = \underbrace{11 \ldots 1}_{3^k}\underbrace{11 \ldots 1}_{3^k}\underbrace{11 \ldots 1}_{3^k} = d \times 3^k \times 10^{3^k} \times 10^{3^k} + d \times 3^k \times 10^{3^k} + d \times 3^k \times 1$$

$$= d \times 3^k \times \left(10^{3^k} \times 10^{3^k} + 10^{3^k} + 1\right)$$

Using the rule of divisibility by 3, since the sum of digits for

$10^{3^k} \times 10^{3^k} + 10^{3^k} + 1 = \underbrace{100 \ldots 01}_{3^k - 1}\underbrace{00 \ldots 01}_{3^k - 1}$ is 3,

we obtain that there exists $d' \square N^\square$ such that

$10^{3^k} \times 10^{3^k} + 10^{3^k} + 1 = d' \times 3.$

Then there exists $d \times d' \in N^*$ such that $\underbrace{11 \ldots 1}_{3^{k+1}} = d \times d' \times 3^{k+1}$ so, 3^{k+1} divides $\underbrace{11 \ldots 1}_{3^{k+1}}$ for all $k \in N^*$.

By the principle of mathematical induction, the statement is true for all positive integers.

Answer 3.2.2– For all $n_1 \in N$, all $n_2 \in N$ and all $s \in N$ the numbers $2^{n1} \times 5^{n2} \times 3^s$ *are remarkable numbers.*

Proof. Indeed, there exists

$M = 9 \times \underbrace{11 \ldots 1}_{3^s} \times 2^n \times 5^n$, with $m = 3^s$ and $n = max(n_1, n_2)$., so the numbers $2^{n1} \times 5^{n2} \times 3^s$ are remarkable.

Answer 3.2.3. It was verified that the *every prime number greater than 5 and less than* 100, $x \in \{7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97\}$, *is remarkable number*, because x divides $\underbrace{11 \ldots 1}_{m}$, with not unique m, that is

$x = 7$ with $m = 6$, m = 12, m = 18, m = 24, m = 30, m = 36, m = 42 (for $x = 7^2$ too),
$m = 48$, m = 54, m = 60, m = 66, m = 72, m = 78.

$x = 11$ with $m = 2$, m = 4, m = 6, m = 8, m = 10, m = 12, m = 14, m = 16, m = 18, m = 20,
$m = 22$ (for $x = 11^2$ too), m = 24, m = 26, m = 28, m = 30, m = 32, m = 34, m = 36,
$m = 38$, m = 40, m = 42, m = 44 (for $x = 11^2$ too), m = 46, m = 48, m = 50, m = 54,

m = 56, m = 58, m = 60, m = 62, m = 64, m = 66 (for $x=11^2$ too),
m = 68, m = 70, m = 72, m = 74, m = 76, m = 78, m = 80.

x = 13 with m = 6, m = 12, m = 18, m = 24, m = 30, m = 36, m = 42, m = 48, m = 54, m = 60,
m = 66, m = 72, m = 78 (for $x=13^2$ too).

x = 17 with m = 16, m = 32, m = 48, m = 64, m = 80.

x = 19 with m = 18, m = 36, m = 54, m = 72.

x = 23 with m = 22, m = 44, m = 66.

x = 29 with m = 28, m = 56.

x = 31 with m = 15, m = 30, m = 45, m = 60, m = 75.

x = 37 with m = 3, m = 9, m = 12, m = 15, m = 18, m = 24, m = 27, m = 30, m = 33, m = 36,
m = 39, m = 42, m = 45, m = 48, m = 51, m = 54, m = 57, m = 60, m = 63, m = 66, m = 69,
m = 72, m = 75, m = 78, m = 81.

x = 41 with m = 5, m = 10, m = 15, m = 20, m = 25, m = 30, m = 35, m = 40, m = 45, m = 50,
m = 55, m = 60, m = 65, m = 70, m = 75, m = 80.

x = 43 with m = 42, m = 63.

x = 47 with m = 46.

x = 53 with m = 13, m = 26, m = 39, m = 52, m = 65, m = 78.

x = 59 with m = 58.

x = 61 with m = 60.

x = 67 with m = 33, m = 66.

x = 71 with m = 35, m = 70.

x = 73 with m = 8, m = 16, m = 24, m = 32, m = 40, m = 48, m = 56, m = 64, m = 72, m = 80.

x = 79 with m = 13, m = 26, m = 39, m = 52, m = 65, m = 78.

x = 83 with m = 41, m = 82.

x = 89 with m = 44, m = 88.

x = 97 with m = 96

and so on.

Heuristic 2.

$\underbrace{11\ldots1}_{m\in N^*} = 10^{m-1}+10^{m-2}+\ldots+10^2+10^1+1.$

We will analyse a few examples so that we can find a rule to write m depending on prime number $p\in N^*$, $p>5$ such that $p \mid \underbrace{11\ldots1}_{m\in N^*}$

For example, p = 19. When we divide the powers of then by p = 19 we will obtain the following remainders:

$10^0\to1$	$10^5\to3$	$10^{10}\to9$	$10^{15}\to8$
$10^1\to10$	$10^6\to11$	$10^{11}\to14$	$10^{16}\to4$
$10^2\to5$	$10^7\to15$	$10^{12}\to7$	$10^{17}\to2$
$10^3\to12$	$10^8\to17$	$10^{13}\to13$	$10^{18}\to1$
$10^4\to6$	$10^9\to18$	$10^{14}\to16$	$10^{19}\to10\ldots$

and so on. Then

$1+2+\ldots+18 = \frac{18\times19}{2} = 9\times19 = M_{(19)} \Rightarrow m = M_{(18)}$

For example, p = 7. When we divide the powers of then by p = 7 we will obtain the following remainders:

$10^0\to1$	$10^6\to1$	$10^{12}\to1$
$10^1\to3$	$10^7\to3$	$10^{13}\to3$
$10^2\to2$	$10^8\to2$	$10^{14}\to2$
$10^3\to6$	$10^9\to6$	$10^{15}\to6$
$10^4\to4$	$10^{10}\to4$	$10^{16}\to4$
$10^5\to5$	$10^{11}\to5$	$10^{17}\to5\ldots$

and so on. Then

$1 + 2 + \ldots + 6 = \frac{6\times7}{2} = 3 \times 7 = M_7 \Rightarrow m = M_{(6)}$

We repeat this process for other examples.

It can be assumed that, in general, $m = M_{(p-1)}$.

Heuristic 3.

$\underbrace{11\ldots1}_{m\in N^*} = 10^{m-1} + 10^{m-2} + \cdots + 10^2 + 10^1 + 1$.

a) We will analyse a few examples so that we can find a rule to write m depending on prime number $p\in N^*$, $p>5$ such that $p^2 \mid \underbrace{11\ldots1}_{m\in N^*}$

For example, $p^2 = 7^2 = 49$. When we divide the powers of ten by $p^2 = 7^2 = 49$ we will obtain the following remainders:

$10^0\to1$	$10^5\to40$	$10^{10}\to32$	$10^{15}\to6$	$10^{20}\to44$	$10^{25}\to45$	$10^{30}\to36$	$10^{35}\to19$	$10^{40}\to25$
$10^1\to10$	$10^6\to8$	$10^{11}\to26$	$10^{16}\to11$	$10^{21}\to48$	$10^{26}\to9$	$10^{31}\to17$	$10^{36}\to43$	$10^{41}\to5$
$10^2\to2$	$10^7\to31$	$10^{12}\to15$	$10^{17}\to12$	$10^{22}\to39$	$10^{27}\to41$	$10^{32}\to23$	$10^{37}\to38$	$10^{42}\to1$
$10^3\to20$	$10^8\to10$	$10^{13}\to3$	$10^{18}\to22$	$10^{23}\to47$	$10^{28}\to18$	$10^{33}\to34$	$10^{38}\to37$	$10^{43}\to10$
$10^4\to4$	$10^9\to13$	$10^{14}\to30$	$10^{19}\to24$	$10^{24}\to29$	$10^{29}\to33$	$10^{34}\to46$	$10^{39}\to27$	…

and so on. We notice that $p = 7$ remainders are missing.

It can be assumed that, in general, $m = M_{(p^2-p)} = M_{(p(p-1))}$

b) After analysing above examples we observe that it can be assumed:

$\forall\, p, \alpha \in N^*$, p–prime, $p \geq 7$,

$p \mid \underbrace{11\ldots1}_{M_{(p-1)}}$

$p^2 \mid \underbrace{11\ldots1}_{M_{(p^1(p-1))}}$

…

$p^\alpha \mid \underbrace{11\ldots1}_{M_{(p^{\alpha-1}(p-1))}}$

$p_1^{\alpha_1} \times p_2^{\alpha_2} \times p_3^{\alpha_3} \times \ldots\ldots\ldots\ldots \times p_n^{\alpha_n} \mid \underbrace{111\ldots\ldots1111}$

$M(p_1^{\alpha_1-1} \times (p_1-1) \times p_2^{\alpha_2-1} \times (p_2-1) \times \ldots\ldots \times p_n^{\alpha_{n-1}} \times (p_n-1)$

In conclusion, after analysing all the possible cases, we can state that every integer, positive or negative, is a remarkable number because we were able to find a multiple of it written as a string of 9 followed (or not) by a string of 0.