SophistiCat analysis

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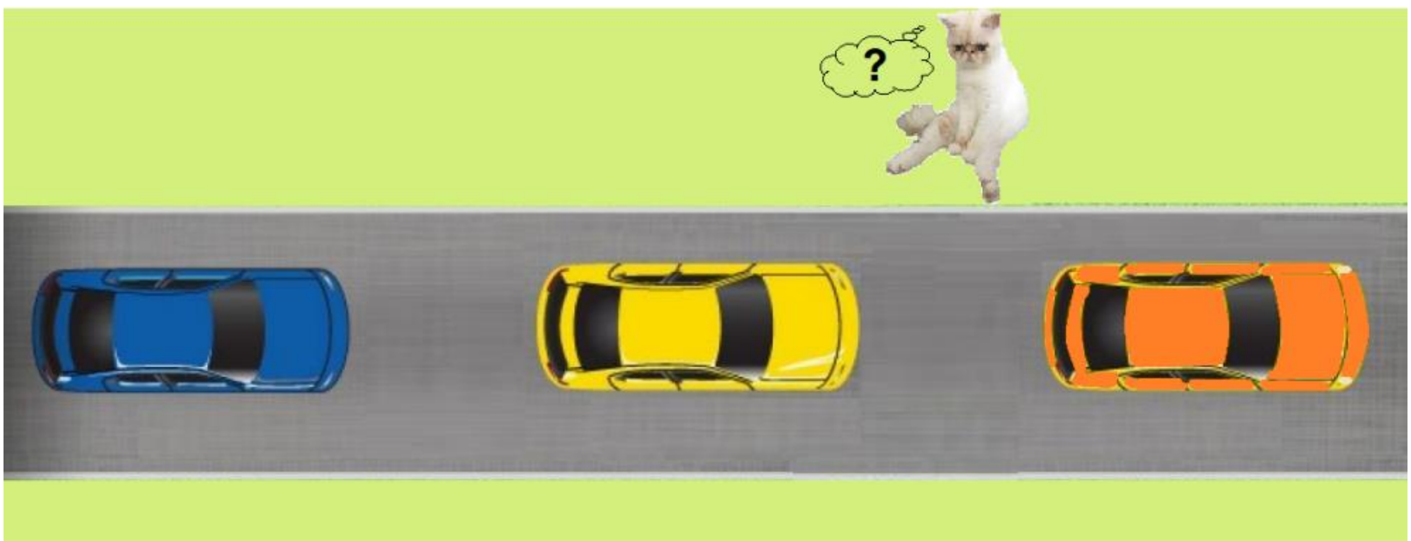
Abstract

Our research deals with calculating the speed of a cat on a road which is full of cars, so that she won’t get hit by them. Given the measure of the cars and the road, we must find the minimum speed of the cat and also the time she needs to cross the road safely.

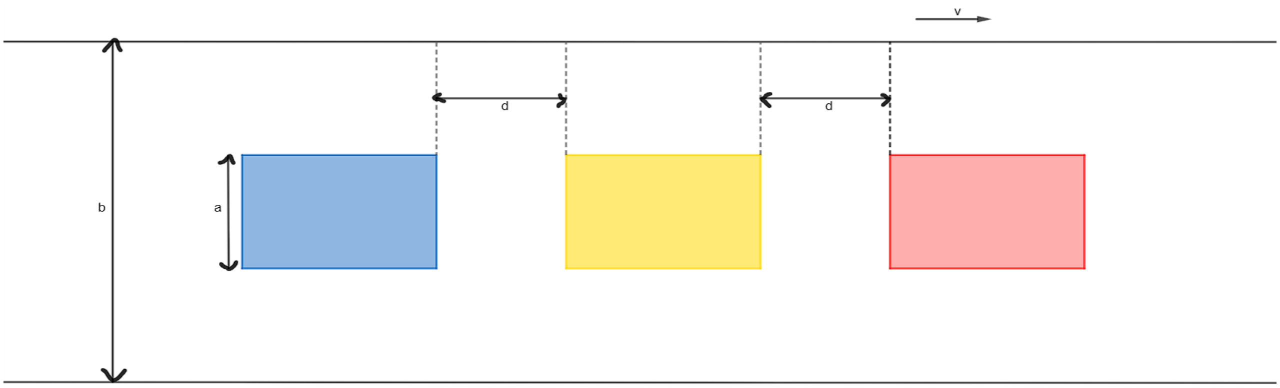


The problem

Misty is a sophisticat that was away on some business, and now she is in a big rush to get home. In order to get back to the loved ones, she has to cross a road of breadth b. Unfortunately, the road is full of cars and she sees no big gaps between them. However, after a quick but thorough analysis, she has noticed that all cars run in one stream, and each car has roughly the same width a. Moreover, all cars are following each other in a single line, at the same speed (say, v), keeping the same distance (say, d) among them.

She is planning to run in a straight line, at a constant speed (say, s), from one side of the road to the other. With this thought in mind, she has calculated the right angle (say, θ) that this straight line will make with the direction of traffic.

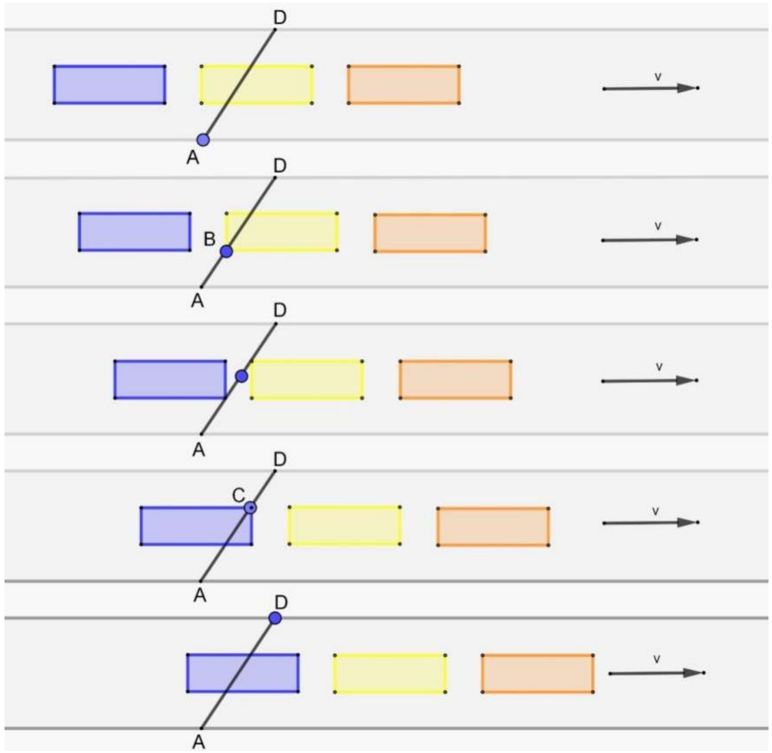
However, being extra cautious and willing to keep her precious fur untouched, she would be very thankful if you could help her. . She wishes to double check with you her minimum constant speed s that she has to run. Also, calculate the crossing time corresponding to s and θ.



We thought of AD as the cat's trajectory. Thus, we considered the very first case, in which the cars are static. Because they don't move at all (they might be waiting for the traffic light to turn green), the fastest way for Misty to go from point A to point D is in a straight line, so the angle θ will be equal to 90° (1). Therefore, the distance she will have to run is equal to the width of the road, so b, and the time she will have is equal to the time when the traffic light is red. It's speed will be calculated by the formula $speed = \frac{distance}{variation\ of\ time}$. So, $s = \frac{b}{t}$. We have to find the minimal speed of our cat so the time in which she can cross the road must be maximal (if the speed increases, the time available for Misty to reach the other side of the road decreases).

The second case is represented by a usual image of the road, so when the cars are moving. The three rectangles represent the cars that have a constant speed; our Misty must pass to the other side of the road between them. Assume AD is the trajectory (straight line) of Misty.

The easiest way for the cat would be to cross the road as in the following image (otherwise, if it crossed perpendicularly it would be hit by the car). Assume Misty starts from point A and ends in the point D:



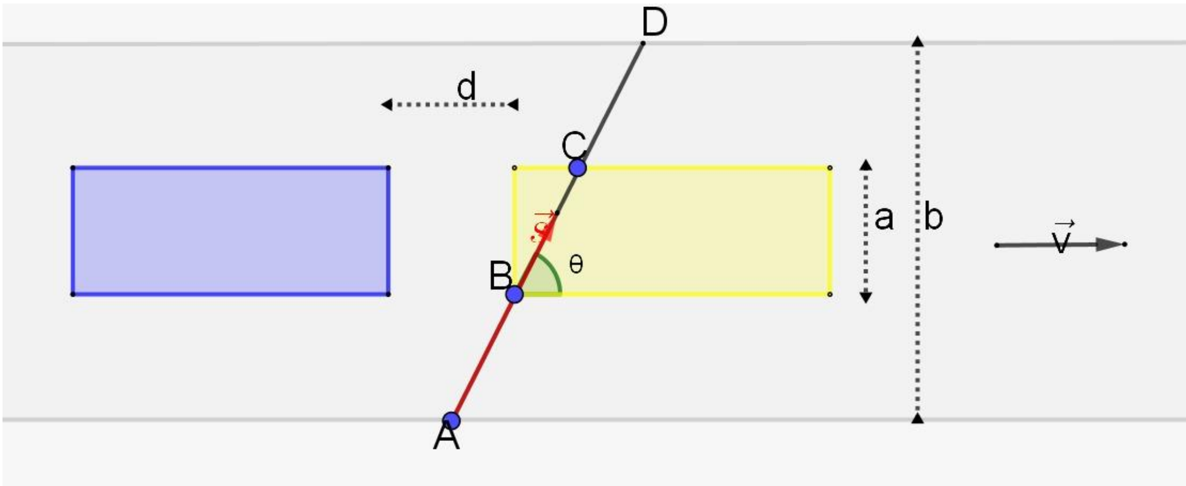
We tried to include in this graph the positions that the cat will have, while crossing the street. Therefore, Misty is represented by the blue dot and its route is starting from point A and ending at point B. In the second image, the cat avoids (at the limit) hitting the back corner of the right side of the yellow car (the one in front of it) and in the third picture it also avoids any contact with the back corner of the yellow car, which is overtaking the cat. In the fourth image, the cat avoids hitting the front corner of the blue car, being in front of it. And in the last image, the cat reaches point D, thus crossing the street successfully.

<u>**The notations given in the problem:**</u>

a - the width of the car
b - the breadth of the road
d **-** the distance between 2 cars
v - the speed of the cars
s - cat's speed
θ - the angle that the cat's trajectory will make with the direction of traffic

<u>**The notations we made:**</u>

A - the point where the cat leaves
B - the place where the cat should be, so as to avoid (at the limit) a collision with the yellow car
C **-** the place where the cat should be, so as to avoid (at the limit) a collision with the blue car
D **-** the place where the cat will arrive



Firstly, we considered a Cartesian coordinate system with the origin B as in the previous image. The speed of the cat ($\vec{s}$) has the direction BC, its horizontal component is equal to $s \cdot \cos\theta$. Therefore, the horizontal component of the relative speed of the cat (related to the blue car) is

$$\boxed{s \cdot \cos\theta - v}$$

Misty passes the danger zone BC in time t_{BC} if

$$d + (s \cdot \cos\theta - v) \cdot t_{BC} \geq 0,$$

where $(s \cdot cos\theta - v) \cdot t_{BC}$ represents the relative distance of the cat to the blue car, traveled in time t_{BC}.

The width a of the car will be equal to $s \cdot t_{BC} \cdot \sin\theta$. Therefore,

$$t_{BC} = \frac{a}{s \cdot \sin\theta}.$$

We will thus replace, obtaining the following relation $(s \cdot \cos\theta - v)\frac{a}{s \cdot \sin\theta} + d \geq 0$.

So, we deduce that

$$s \cdot a \cdot \cos\theta - v \cdot a + d \cdot s \cdot \sin\theta \geq 0$$

$$s \cdot (a \cdot \cos\theta + d \cdot \sin\theta) \geq v \cdot a$$

$$s \geq \frac{a \cdot v}{a \cdot \cos\theta + d \cdot \sin\theta}.$$

where θ falls within the range $\left(0, \frac{\pi}{2}\right)$.

We want to calculate the minimum value of s, the speed of the cat, as the angle θ ranges between 0 and $\frac{\pi}{2}$. We observe that, in order to obtain this, we have to find the maximum value of the following expression

$$E(\theta) = a \cdot \cos\theta + d \cdot \sin\theta, \ \text{with } \theta \in \left(0, \frac{\pi}{2}\right).$$

We take the forced common factor $\sqrt{a^2+d^2}$ in the above expression and get

$$E(\theta) = \sqrt{a^2+d^2} \cdot \left(\frac{a}{\sqrt{a^2+d^2}} \cdot cos\theta + \frac{d}{\sqrt{a^2+d^2}} \cdot sin\theta\right).$$

As

$$\left(\frac{a}{\sqrt{a^2+d^2}}\right)^2 + \left(\frac{d}{\sqrt{a^2+d^2}}\right)^2 = \frac{a^2+d^2}{a^2+d^2} = 1,$$

there is an unique angle $\gamma \epsilon \left(0, \frac{\pi}{2}\right)$ such that $\cos\gamma = \frac{a}{\sqrt{a^2+d^2}}$ and $\sin\gamma = \frac{d}{\sqrt{a^2+d^2}}$.

Therefore, $\tan\gamma = \frac{d}{a}$ and thus

$$\boxed{\gamma = \arctan\left(\frac{d}{a}\right).}$$

Hence, $E(\theta) = \sqrt{a^2+d^2} \cdot (\cos\theta \cdot \cos\gamma + \sin\theta \cdot \sin\gamma)$, and the expression of $E(\theta)$ becomes

$$E(\theta) = \sqrt{a^2+d^2} \cdot \cos(\gamma - \theta)$$

Then, the maximum of $E(\theta)$ occurs when $\cos(\gamma - \theta) = 1$. As both, $\theta, \gamma \in \left(0, \frac{\pi}{2}\right)$, this means that $\theta = \gamma = \arctan\left(\frac{d}{a}\right)$. In conclusion, the minimum constant speed of the cat that guarantees a safe crossing of the road must be

$$\boxed{s_{min} = \frac{a \cdot v}{\sqrt{a^2+d^2}}}$$

and the right angle needed is $\theta^* = \arctan\left(\frac{d}{a}\right)$ (2).

The corresponding crossing time T of the entire road, from point A to point D, is

$$T = \frac{AD}{s} = \frac{\frac{b}{\sin\theta}}{\frac{a \cdot v}{d \cdot \sin\theta + a \cdot \cos\theta}} = \frac{b \cdot (d \cdot \sin\theta + a \cdot \cos\theta)}{a \cdot v \cdot \sin\theta} = \frac{b \cdot (d + a \cdot \cot\theta)}{a \cdot v} = \frac{b}{v} \cdot \left(\cot\theta + \frac{d}{a}\right).$$

If the cat runs at minimum velocity, then $\theta = \arctan\left(\frac{d}{a}\right)$, and thus, $cot\ \theta = \frac{a}{d}$.

Therefore,

$$T = \frac{b}{v}\left(cot\ \theta + \frac{d}{a}\right) = \frac{b}{v}\left(\frac{a}{d} + \frac{d}{a}\right)$$

In conclusion, if Misty runs at minimum velocity, the crossing time of the entire road will be

$$\boxed{T^* = \frac{b}{v}\left(\frac{a}{d} + \frac{d}{a}\right).}$$

The subject could be more relevant to us if we take into consideration two particular cases:

I. **Misty is crossing a town street**

In this case, the speed of the cars is limited to $v = 50$ km/h ≈ 14 m/s. We assume that the width of cars is $a = 2$ m, the distance between two following cars is $d = 5$ m. From the formula for s_{min}, we obtain:

$$s_{min} = \frac{2\cdot 50}{\sqrt{2^2+5^2}} = \frac{100}{\sqrt{29}} \approx 18.58 \text{ km/h} = 310 \text{ m/min} \approx 5.20 \text{ m/s}.$$

As the average speed of a running cat is 40 km/h (Wikipedia), which means that Misty needs 9 seconds for each 100 meters, we conclude that our minimum speed from above is realistic.

If the breadth of the road is $b = 7$ m and Misty runs at minimum velocity, we obtain that the crossing time of the entire road is

$$T^* \approx \frac{7}{14} \cdot \left(\frac{2}{5} + \frac{5}{2}\right) \approx 1.45\ s$$

and the angle corresponding to the minimum velocity is (see Figure 1)

$$\theta^* = \arctan\left(\frac{5}{2}\right) \approx 1.19 \text{ rad} \ \text{ (about } 68°\text{)}.$$

We consider the speed of the cat as a function of θ (3),

$$s(\theta) = \frac{a \cdot v}{a \cdot \cos\theta + d \cdot \sin\theta} = \frac{2 \cdot 14}{2 \cdot \cos\theta + 5 \cdot \sin\theta}$$

Its graph is displayed in Figure 1.

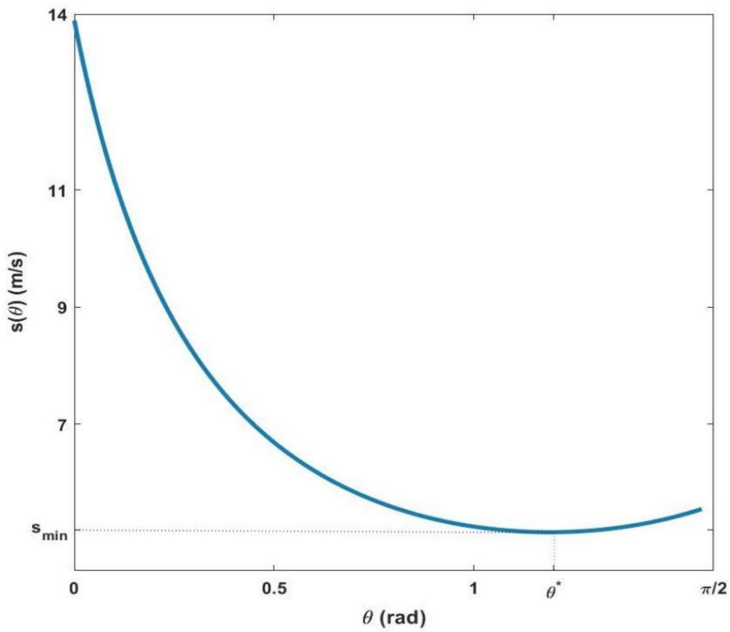


Figure 1. Cat's velocity s as a function of θ. Here, $v = 50\ km/h$, $a = 2\ m$, $b = 7\ m$, $d = 5\ m$.

We also considered the cases where the constant distance between following cars are $d = 10$ m and $d = 30$ m (to avoid rush hour, for example).

For $d = 10$ m, then the minimum cat velocity is $s_{min} = 2.72$ m/s, the crossing time of the road at minimum velocity is $T^* = 2.6$ s, and the angle corresponding to the minimum velocity is $\theta^* \approx 78°$.

For $d = 30$ m, then the minimum cat velocity is $s_{min} = 0.93$ m/s, the crossing time of the road at minimum velocity is $T^* = 7.5$ s, and the angle corresponding to the minimum velocity is $\theta^* \approx 86°$.

This means that, if the cars are further apart, Misty can cross the road almost perpendicular to the direction of traffic, at smaller velocities and the corresponding crossing time is larger, as Misty will be no longer in a big rush.

If we consider the time needed to cross the street as a function of θ, $T(\theta) = \frac{b}{v}\left(\cot\theta + \frac{d}{a}\right)$ (4), then we obtain the graph displayed in Figure 2.

We observe that, the larger the angle θ is, the lower time Misty spends on crossing the street.

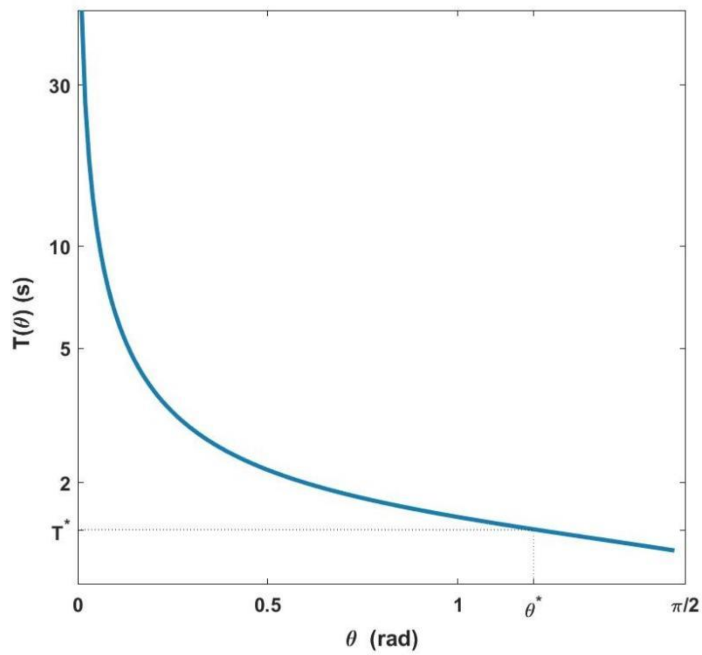


Figure 2. Crossing time T as a function of θ. Here, v = 50km/h, a = 2 m, b = 7 m, d = 5 m.

II. **Misty is crossing a motorway**

The speed of the cars on a motorway is $v = 130$ km/h ≈36 m/s. We assume that the width of all cars is $a = 2$ m, distance between following cars is $d = 5$ m (5) and the breadth of road is $b = 7$ m.

From the above formula, we obtain that the minimum velocity of the cat must be $s_{min} = 13.38$ m/s (see Figure 3). This is realistic only if Misty is really fast by nature (we can assume that Misty is trained for speed). Also, we also get the crossing time of the road at this minimum velocity is $T^* = 0.56$ s and the optimal angle is $\theta^* \approx 1.19$ rad (about 68°) (see Figure 4).

If we now consider that the distance between cars is higher, say $d = 10$ m, we obtain that:

$$s_{min} = 7.06 \text{ m/s}, \; T \approx 1 \text{ s and } \theta^* \approx 78°.$$

If we consider the speed of Misty as a function of θ, then

$$s(\theta) = \frac{72}{2\cos\theta + 5\sin\theta} .$$

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Its graph is displayed in Figure 3 below.

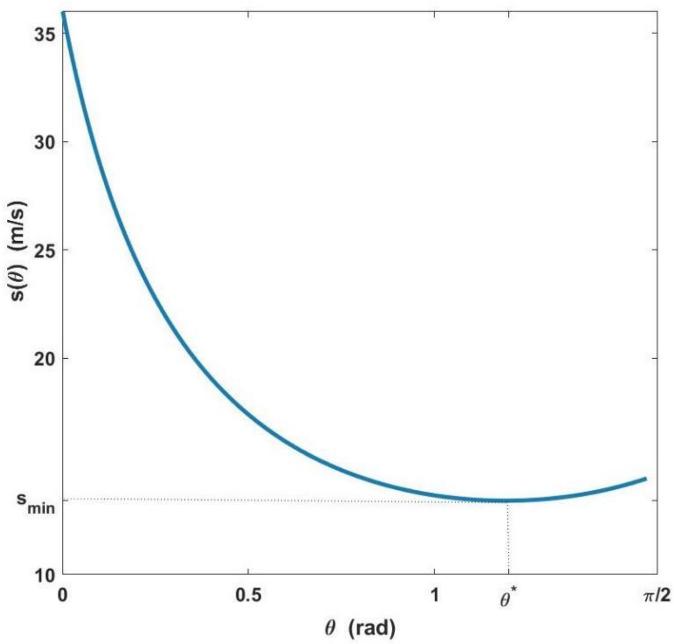

Figure 3. Cat's velocity s *as a function of* θ*. Here,* $v = 130\ km/h$, $a = 2\ m$, $b = 7\ m$, $d = 5\ m$.

The cat's crossing time as a function of θ is displayed in Figure 4.

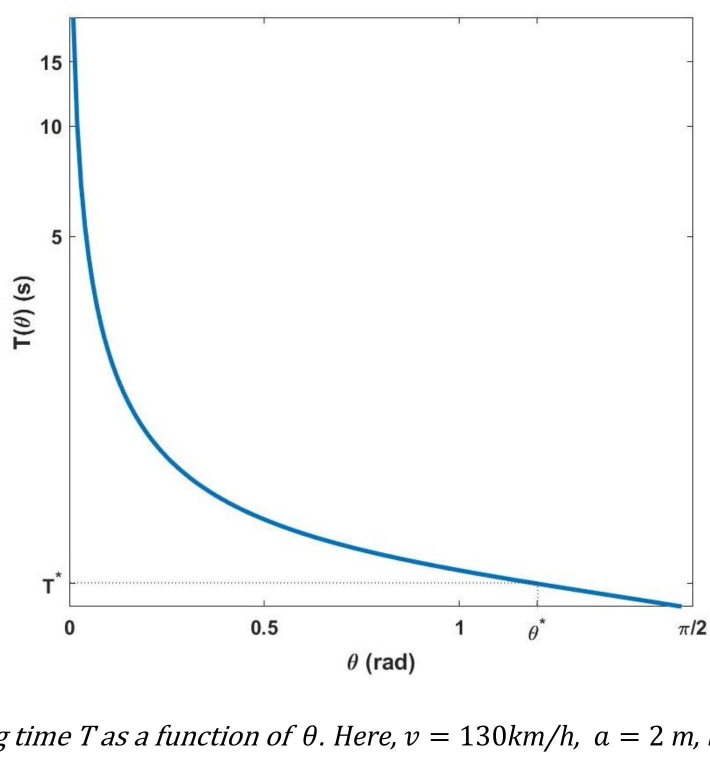

Figure 4. Crossing time T *as a function of* θ*. Here,* $v = 130km/h$, $a = 2\ m$, $b = 7\ m$, $d = 5\ m$.

Conclusion

In conclusion, in order for the cat to successfully cross the street, it must have a route determined by the trigonometric functions of a unique angle θ. The speed and the time in which it travels the street are determined by the measurements of the road and the cars, compared to the angle made by the trajectory of the cat on the road. In order to calculate the cat's minimal speed, we have to consider the maximal value of the angle θ. Through some numerical calculations, we come to the conclusion that both the cat's speed and the time of crossing the street can be written in the form of trigonometric equations according to θ so that our mathematical cat has a route as safe as possible.

In the case when the cars are static, the cat can cross the street with any speed and any trajectory angle which falls within the range $(0, \pi)$, so it can both run, in case of haste, and go for a promenade around the city.

Editing Notes

(1) Here, the point D is the closest point to A on the other side of the road, so that AD is perpendicular to the road.

(2) We may add that Misty does not need a calculator to find the best direction: this angle θ is such that BC is perpendicular to the line joining B to the front left corner of the following car.
Indeed, if we note E this corner and F the rear left corner of the previous car, then he triangle EFB is right-angled at F, with $EF = d$ and $BF = a$, so that $\tan \widehat{EBF} = d/a$. It follows that $\theta = \widehat{EBF}$ and $\widehat{EBC} = \theta + \widehat{FBC}$ is also a right angle.

(3) More precisely, $s(\theta)$ is the *minimum* speed that allows the cat to cross without being hit for a given angle θ.

(4) Similarly, $T(\theta)$ is the crossing time at the minimum speed not to be hit, for a given angle θ.

(5) This distance is not very realistic. It is generally considered that a car must be at least 2 seconds behind the one preceding it, which corresponds to 72 m at 36 m/s. Even if she fears that cars may be twice that close, Misty can cross quietly!