**Tile Wallpaper**  
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**Surnames and first names of students, grades:** Cristiana Aanei (student, 10th grade), Ilinca Axinte (student, 10th grade), Elena Iulia Chiruță (student, 10th grade), Bianca Gabriela Luca (student, 10th grade), Diana Andreea Morariu (student, 10th grade), Sorana Scarlat (student, 10th grade).

**School:** “Costache Negruzzi” National High School Iași, Romania

**Teacher:** Ph. D. Ioana Cătălina Anton, “Costache Negruzzi” National High School Iași, Romania

**Researcher and his university:** Ph. D. Iulian Stoleriu, University “Alexandru Ioan Cuza” of Iași, Romania

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**The problem (Abstract)**

A rectangular wall of size $2 \times 10$ is to be covered with tiles. The number of tiles is unlimited and each tile can be rotated according to your preference before mounting. In how many ways can we cover the entire wall using only $2 \times 1$ coloured tiles? What if we have access to both types of tiles, $1 \times 1$ and $2 \times 1$?
For the first part of our problem, we only have 2 x 1 tiles available, therefore we will analyse two cases: the first one, in which the tiles are uncoloured, and the second one, in which the tiles are coloured.

**Case I: Uncoloured tiles**

We are looking for a way of placing the uncoloured tiles. To be able to find a recurrent formula, we consider a 2 x n wall, then we particularize for n = 10.

We denote by $a_n$ the number of ways we can cover the 2 x n wall with 2 x 1 uncoloured tiles.

Firstly we count the number of ways we can cover the last column. We see the last column can be covered in only one way, as displayed below.

The other ($n - 1$) columns will be covered in $a_{n-1}$ ways.

We then count the number of ways we can cover the last two columns. We see that the last two columns can be covered in two ways (as displayed below), but one of these two cases was considered before, when the last column was covered (the second one). So, there is only one way remaining, the first one.
The other \((n-2)\) columns will be covered in \(a_{n-2}\) ways.

If we consider the covering of the last three or more columns, we will not discover new ways that were not considered before. We arrive at the following recurrence relation:

\[
(1) \quad a_n = a_{n-1} + a_{n-2}, \text{ for all } n \geq 3.
\]

The first two terms of the sequence \(a_n\) are:

\[
a_1 = 1; \quad a_2 = 2.
\]

Note that relation (1) is the recurrence for the Fibonacci numbers.

**Case II: coloured tiles**

We denote by \(b_n\) the number of ways we can cover the \(2 \times n\) wall with \(2 \times 1\) coloured tiles. We firstly count the number of ways we can cover the last column. We see the last column can be covered in two ways, as displayed below.

The other \((n-1)\) columns will be covered in \(b_{n-1}\) ways.

We then count the number of ways we can cover the last two columns. We see that the last two columns can be covered in eight ways (as displayed below). However, four of these eight cases (with the tiles placed vertically) were considered before, when the last...
column was covered, and we disregard them. So, there are only **four ways** remaining, the ones marked

with green in the figure below (with the tiles placed horizontally).

The other \((n - 2)\) columns will be covered in \(b_{n-2}\) ways.

If we consider the covering of the last three or more columns, we will not discover new ways that were not considered before. We arrive at the following recurrence relation:

\[
(2) \quad b_n = 2b_{n-1} + 4b_{n-2}, \quad (\forall) \ n \geq 3.
\]

Also note that we can arrive at recurrence (2) in a different way, as follows. As each square in a domino can be painted in two colours, we get that:

\[
b_n = 2^n \cdot a_n, \quad (\forall) \ n \geq 1.
\]

Then, we multiply recurrence (1) by \(2^n\), we arrive at:

\[
b_n = 2^n \cdot (a_{n-1} + a_{n-2}), \quad (\forall) \ n \geq 3,
b_n = 2^n \cdot a_{n-1} + 2^n \cdot a_{n-2}, \quad (\forall) \ n \geq 3.
\]

We want to determine a recurrence relation for the \((b_n)\).

Because: \(b_{n-1} = 2^{n-1} \cdot a_{n-1}\) and \(b_{n-2} = 2^{n-2} \cdot a_{n-2}\), then

\[
b_n = 2 \cdot 2^{n-1} \cdot a_{n-1} + 4 \cdot 2^{n-2} \cdot a_{n-2}, \quad (\forall) \ n \geq 3,
b_n = 2b_{n-1} + 4b_{n-2}, \quad (\forall) \ n \geq 3.
\]

The first two terms of the sequence \(b_n\) are:

\[
b_1 = 2; \ b_2 = 8.
\]

We then have successively:

\[
b_3 = 2b_2 + 4b_1 = 2 \cdot 8 + 4 \cdot 2 = 24;
\]

\[
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quarter
Thus, there are 91136 ways of covering the 2 × n wall with coloured dominos.

We can also determine this number by finding the general formula of the recurrence (2).

We will proceed as follows.

The characteristic equation attached to recurrence $b_n = 2b_{n-1} + 4b_{n-2}$ is:

$$a^2 = 2a + 4.$$ 

Seeing that $b_n$ second order recurrence relation, it will take the following form:

$$b_n = C_1 \alpha_1^n + C_2 \alpha_2^n$$

where $\alpha_1$ and $\alpha_2$ represent the solutions for the equation:

$$\alpha_1 = 1 + \sqrt{5},$$

$$\alpha_2 = 1 - \sqrt{5}.$$ 

We determine $C_1$ and $C_2$:

For $n = 1$, $b_1 = C_1 \cdot \alpha_1 + C_2 \cdot \alpha_2$ then: $2 = C_1(1 + \sqrt{5}) + C_2(1 - \sqrt{5})$;

For $n = 2$, $b_2 = C_1 \cdot \alpha_1^2 + C_2 \cdot \alpha_2^2$, then: $8 = C_1(1 + 2\sqrt{5} + 5) + C_2(1 - 2\sqrt{5} + 5)$.

We form the following system:

$$\begin{align*}
2 &= C_1(1 + \sqrt{5}) + C_2(1 - \sqrt{5}), \\
8 &= C_1(6 + 2\sqrt{5}) + C_2(6 - 2\sqrt{5}) /: 2 \\
2 &= C_1(1 + \sqrt{5}) + C_2(1 - \sqrt{5}), \\
4 &= C_1(3 + \sqrt{5}) + C_2(3 - \sqrt{5}).
\end{align*}$$

Which we will solve using the reduction method:

$$\begin{align*}
C_1(1 + \sqrt{5}) + C_2(1 - \sqrt{5}) &= 2 / (3 - \sqrt{5}), \\
C_1(3 + \sqrt{5}) + C_2(3 - \sqrt{5}) &= 4 / (1 - \sqrt{5}).
\end{align*}$$

\[C_1 = \frac{\sqrt{5} + 5}{10},\]

\[C_2 = \frac{-\sqrt{5} + 5}{10}.\]

We obtain: $b_n = \frac{\sqrt{5} + 5}{10} (1 + \sqrt{5})^n + \frac{-\sqrt{5} + 5}{10} (1 - \sqrt{5})^n$

\[b_n = \frac{\sqrt{5}}{10} (1 + \sqrt{5})^{n+1} + \frac{-\sqrt{5}}{10} (1 - \sqrt{5})^{n+1}\]
The general term formula

To help calculate $b_{10}$ we shall use the Newton binomial:

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + b^n.$$  

Now we can find out the number of combinations needed:

$$b_{10} = \frac{\sqrt{5}}{10} (1 + \sqrt{5})^{11} - \frac{\sqrt{5}}{10} (1 - \sqrt{5})^{11},$$

$$b_{10} = \frac{\sqrt{5}}{10} [(1 + \sqrt{5})^{11} - (1 - \sqrt{5})^{11}],$$

$$b_{10} = \frac{\sqrt{5}}{10} (2 \binom{11}{1} \cdot \sqrt{5} + 2 \binom{11}{3} \cdot \sqrt{5}^3 + \ldots + 2\sqrt{5}^{11}),$$

$$b_{10} = 91136.$$  

After calculations we get that the number of combinations of 2 x 1 colored tiles on a 2 x 10 wall is 91136.

For the second part of our problem both types of tiles are available to us, therefore we will work with 2 cases again, one for uncoloured tiles and one for coloured tiles.

**Case I: Uncoloured tiles**

We consider a 2 x $n$ wall, then we will particularize for $n = 10$. We denote by $c_n$ the number of ways we can cover the 2 x 10 wall with both types of uncolored tiles.

We count the number of ways we can cover the last column, same as we did in the first part of the problem. The following 2 cases are observed:
The other \((n - 1)\) columns will be covered in \(c_{n-1}\) ways.

We can see that the last two columns can be covered in six ways (as displayed below). However, three of the ways (the ones marked with the X) were considered when we covered the last column. Therefore, there will be only three ways of covering the columns (the ones with the tiles placed horizontally).

The remaining \((n - 2)\) columns can be covered in \(c_{n-2}\) ways.

For covering the last 3 columns, we have only 2 new ways:

For each of the last \(k\) columns \((k \geq 4)\), there are exactly 2 ways to cover the extra column: a new way and also its opposite:

\[
2c_{k-4} + 2c_{k-5} + \cdots + 2c_2 + c.
\]

We now write the following relation:

\[
c_n = 2c_{n-1} + 3c_{n-2} + 2c_{n-3} + 2c_{n-4} + \cdots + 2c_2 + c_1.
\]

But, \(c_{n+1} = 2c_n + 3c_{n-1} + 2c_{n-2} + 2c_{n-3} + \cdots + 2c_2 + c_1\).

We subtract the two relations, remaining with:
\[ c_{n+1} - c_n = 2c_n + c_{n-1} - c_{n-1}. \]

We add \( c_n \) on both sides, getting to the relation written below:

\[ c_{n+1} = 3c_n + c_{n-1} - c_{n-2}, \quad (\forall) \ n \geq 3. \]

The first three terms of the \( c_n \) sequence are:

\[ c_1 = 2 \quad c_2 = 7 \quad c_3 = 22. \]

\[ c_{10} = 78243 \text{ ways of covering the } 2 \times 10 \text{ wall with } 2 \times 1 \text{ and } 1 \times 1 \text{ uncoloured tiles.} \]

In conclusion, there are 78243 ways of covering the 2 x 10 wall with both 2 x 1 and 1 x 1 uncoloured dominos.

**Case II: coloured tiles**

For each way of placing the uncoloured tiles, we have to identify how many different situations can appear if they get coloured in the specified colours.

For the last column in the 2 x \( n \) wall, we have already discovered two ways of placing the uncoloured tiles. If we color them, three ways will now be valid, since we can rotate the 2 x 1 dominos.

The other \((n - 1)\) columns can be covered in \( d_{n-1} \) ways.
For the last two columns, there are 3 new ways of placing the uncoloured tiles, and 6 new ways of placing the coloured tiles. In this case the only valid versions will be the ones in which the 2 x 1 tiles are placed horizontally, because the cases in which the tiles are placed vertically have already been counted before, when we covered the last column, for example:

The other \((n - 2)\) columns can be covered in \(d_{n-2}\) ways.

![Diagram of tile placements](image1)

For the last three columns, there are 2 new ways of placing the uncoloured tiles and 8 new ways of placing the coloured tiles:

The other \((n - 3)\) columns can be covered in \(d_{n-3}\) ways:

![Diagram of tile placements](image2)

Thus, we write the following mathematical relation:

\[
d_n = 3 d_{n-1} + 6 d_{n-2} + 8 d_{n-3} + 8 d_{n-4} + \ldots + 8 d_2 + d_1.
\]
For $(n + 1)$ columns:

$$d_{n+1} = 3d_n + 6d_{n-1} + 8d_{n-2} + 8d_{n-3} + ... + 8d_2 + d_1.$$  

If we subtract the two relations we get:

$$d_{n+1} = 4d_n + 3d_{n-1} + 2d_{n-2}, \ n \geq 3.$$  

The terms of the $d_n$ sequence are:

$$d_1 = 3;$$
$$d_2 = 17;$$
$$d_3 = 83;$$
$$d_4 = 4d_3 + 3d_2 + 2d_1 = 4 \cdot 83 + 3 \cdot 17 + 2 \cdot 3 = 389;$$
$$d_5 = 4d_4 + 3d_3 + 2d_2 = 1556 + 249 + 34 = 1839;$$
$$\vdots$$
$$\vdots$$
$$d_{10} = 4d_9 + 3d_8 + 2d_7 = 366538 + 581847 + 82102 = 4329257.$$  

$d_{10} = 4.329.257$ ways of covering the 2 x 10 wall with both types of coloured tiles.

**Conclusion**

Considering the multitude of possibilities, we realized from the beginning that we must be thorough in our calculations. First of all, we considered two main cases: the usage of uncoloured and coloured tiles. Then we realized that we have to discover a general formula to apply to our case. Along the way, the problem became more and more complex. But it was easier when we applied the formula to the cases, finally managing to find out the situations and fulfill our goal.
References

- https://www.mathsisfun.com/combinatorics/combinations-permutations.html
- https://www.whitman.edu/mathematics/cgt_online/book/section03.01.html
- https://www.cuemath.com/data/combinations/