Circle Pavements

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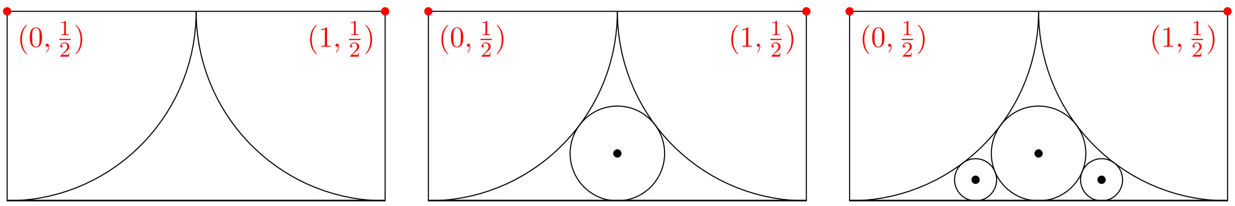
1 Introduction

The problem

We construct an initial area bounded by the Ox axis and circles of center $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$ and of radius 0.5.
With each step we add the largest circles tangent to Ox and to the other circles.
What can be said about the circles' center and radius at each stage?



2 Results

2.1 Unicity

Even if the text of the problem says that with each step we add the biggest circles tangent to three specific points there is actually only one possible option. By associating to every point of tangency a vertex of a triangle, the only circle that passes through all three points is the circumcircle of that triangle.

2.2 Recursive Formulas

We started by trying to find a recursive formula based on the center's coordinates of the last constructed circle, by going around one of the initial circles of the problem and finding out the center's coordinates of every circle tangent to it. To do this, we considered the center's coordinates of the initial circle on the left as constant and the coordinates of the first built circle as variables. This way by using the formulas for distance between two points we formed a system of equations that helped us determine the coordinates of the center of the new circle, tangent to the initial left circle, based on the center's coordinates of the last constructed circle.

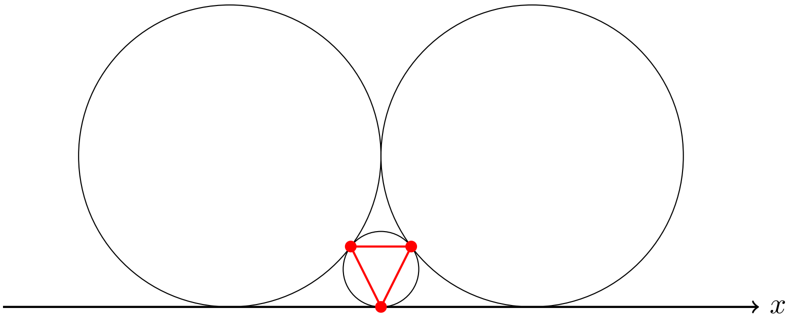


Figure 1: The circumcircle of the triangle formed by the points of tangency

Before starting the aforementioned process, we shall find the center's coordinates of the first built circle, which is tangent to both initial circles. Also, by the definition of a line tangent to a circle we know that the y-coordinate of any circles' center in our problem is equal to the radius of that circle.

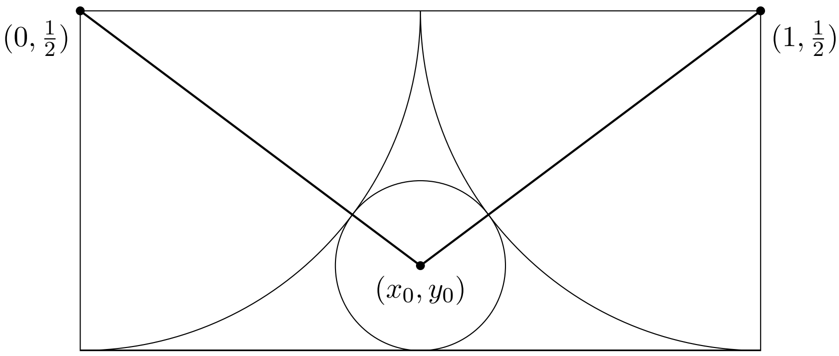


Figure 2: The first built circle tangent to both initial circles

$$\begin{cases} x_0^2 + (\frac{1}{2} - y_0)^2 = (r_0 + \frac{1}{2})^2 \\ (1 - x_0)^2 + (\frac{1}{2} - y_0)^2 = (r_0 + \frac{1}{2})^2 \end{cases} \implies \begin{cases} y_0 = \frac{x_0^2}{2} \\ y_0 = \frac{(x_0 - 1)^2}{2} \end{cases} \implies \begin{cases} x_0 = \frac{1}{2} \\ y_0 = \frac{1}{8} \end{cases}$$

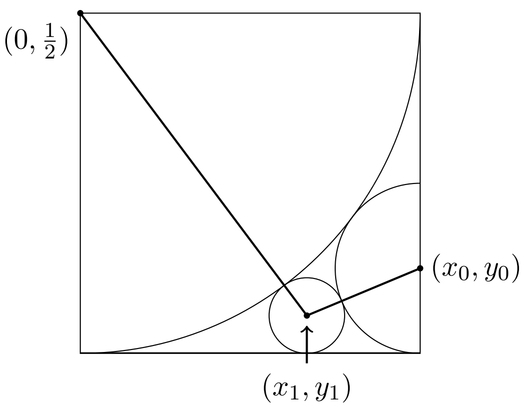


Figure 3: The first circle tangent to the left initial circle and the centered circle

$$\begin{cases} x_1^2 + (\frac{1}{2} - y_1)^2 = (r_1 + \frac{1}{2})^2 \\ (x_0 - x_1)^2 + (y_0 - y_1)^2 = (r_0 + r_1)^2 \end{cases} \implies \begin{cases} \frac{x_1^2}{2} = y_1 \\ (1 - x_0^2)x_1^2 - 2x_0x_1 + x_0^2 = 0 \end{cases} \implies \begin{cases} y_1 = \frac{x_1^2}{2} \\ x_1 = \frac{x_0}{1+x_0} \end{cases}$$

Because (x_0, y_0) are the center's coordinates of the last constructed circle for the circle of center (x_1, y_1) we can replace them with (x_{n-1}, y_{n-1}) and get the following recursive formula fie $x_n, y_n)$:

$$x_n = \frac{x_{n-1}}{1 + x_{n-1}} \qquad y_n = \frac{x_n^2}{2}$$

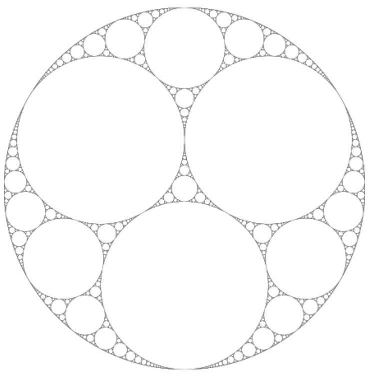
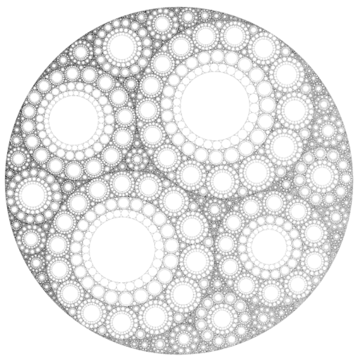
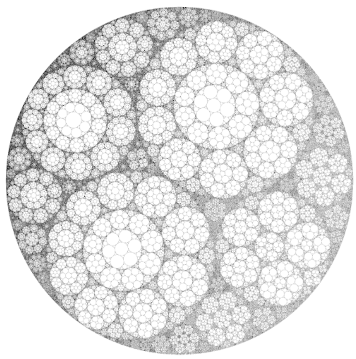
The same process can be applied to find out the center's coordinates of the circles tangent to the right initial circle. By solving the system of equations we get the following recursive formulas for the right side:

$$x_n = \frac{1}{2 - x_{n-1}} \qquad y_n = \frac{(x_n - 1)^2}{2}$$

This approach can be used to find recursive formulas for the center's coordinates of any circle tangent to one specific one, though it is not very efficient.

2.3 Apollonian Gasket

The Apollonian gasket or Apollonian Circle Packing is a type of fractal named after the Greek mathematician Apollonius of Perga. It starts off with three circles tangent to each other, and the two possible circles tangent to all the three circles simultaneously.
In an attempt to fill out as much of the surface as possible, we generate an infinity of circles that get progressively smaller, thus creating a drawing that is quite beautiful. There are plenty of variations on the Apollonian gasket, depending on the initial shapes. In the images below you can see a few examples.

If you are curious about creating your own Apollonian gasket through Python code, we recommend watching this video, which also includes relevant information on the gasket, Ford circles, and plenty more: Apollonian Gasket Fractal.

2.4 Descartes Theorem

In order to create the fourth circle tangent to three others in the previously mentioned problem, we need to know its radius. And for that there is a theorem.

Descartes' theorem states that there is a quadratic equation that connects the radii of four mutually tangent circles, also called "kissing" circles. The theorem was stated in 1643, and is named after René Descartes.

$$(k_1 + k_2 + k_3 + k_4)^2 = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2)$$

$$k_4 = k_1 + k_2 + k_3 \pm 2\sqrt{k_1k_2 + k_2k_3 + k_3k_1}$$

There is a poem called The Kiss Precise, written by Frederick Soddy in 1936, that condenses the formula into something easier to remember, using bends (k), the multiplicative inverse of the radii:

"The sum of the squares of all four bends
Is half the square of their sum"

Descartes tackles Apollonius' problem using analytic geometry, a field pioneered by him and Pierre de Fermat. The former discussed the dilemma with Princess Elisabeth of the Palatinate through letters. After circling back and forth a few times and trying different approaches, Descartes finally reached a result that now bears his name. Except he did not provide any reasoning for his new-found relation [**brilliant_descartes_theorem**]. The lack of reasoning prompted several mathematicians to prove the theorem, and they did so through very different methods .
There are a lot of special cases, and the theorem can also be used in three dimensions, using hyperbolic geometry. One of those special cases can actually be found in our problem, in which one of the three circles is a line, therefore its radius tends to infinity. Descartes uses curvature, which is the same as the bend from the poem (the multiplicative inverse of the radius), which means that the curvature is nearly 0. So one less radius to take into account. After not that many calculations, we arrive at a formula that connects the radius of our mystery circle to those of the previous two.

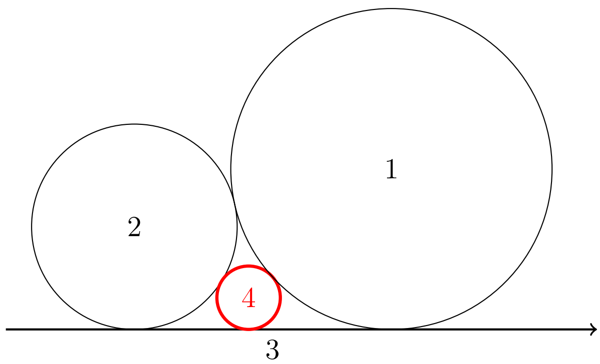


Figure 4: The circle tangent to two other circles and a line

$$k_4 = \lim_{r_3 \to \infty} \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm 2\sqrt{\frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1}}\right) = \frac{r_1 + r_2}{r_1r_2} \pm \frac{2\sqrt{r_1r_2}}{r_1r_2} \Rightarrow r_4 = \frac{r_1r_2}{(\sqrt{r_1} \pm \sqrt{r_2})^2}$$

2.5 Ford circles

A Ford circle is a geometrical representation of a rational number in a Cartesian coordinate system defined by Lester R. Ford. For each rational number expressed in lowest terms, we have a Ford circle with the radius equal to $\frac{1}{2q^2}$ and with the center at the point $\left(\frac{p}{q}, \frac{1}{2q^2}\right)$ **[ford1938fractions]**.
By constructing two tangent Ford circles of centers $(\frac{P}{Q}, \frac{1}{2Q^2})$ and $(\frac{p}{q}, \frac{1}{2q^2})$, we can easily demonstrate by using a geometric property of two tangent Ford circles: $|pQ - Pq| = 1$.
To demonstrate this propriety, we started by noting the distance between the x coordinates of the centers of the two circles with the letter 'a', the distance between the y coordinates of the centers with the letter 'b', and the distance between the two centers with the letter 'c'.

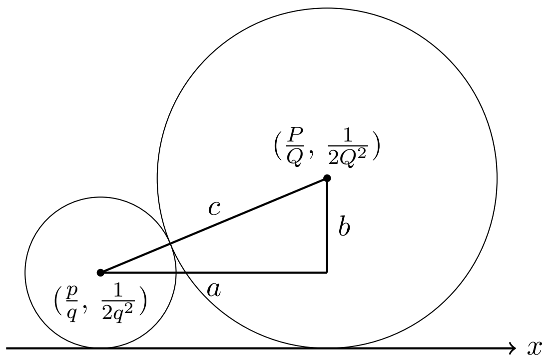


Figure 5: Two tangent Ford circles

$$a = \frac{P}{Q} - \frac{p}{q} \qquad b = \frac{1}{2Q^2} - \frac{1}{2q^2} \qquad c = \frac{1}{2Q^2} + \frac{1}{2q^2}$$

Also by using the Pythagorean theorem, we can find 'a' as equal to:

$$a = \sqrt{c^2 - b^2} = \sqrt{(\frac{1}{2Q^2} + \frac{1}{2q^2})^2 - (\frac{1}{2Q^2} - \frac{1}{2q^2})^2} = |\frac{1}{qQ}|$$

By forming an equation with the two values of 'a', we can find the relation between the two circles:

$$|pQ - Pq| = 1$$

2.6 Farey sequence

Farey sequence F_n is a sequence of sets of rational numbers $\frac{p}{q}$ with p and q being coprime integers and $0 \leqslant p < q \leqslant n$, ordered by value.

$$F_1 = \{\frac{0}{1}, \frac{1}{1}\}$$
$$F_2 = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$$
$$F_3 = \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}$$
$$F_4 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\}$$

The history of the irreductible fractions sequence begins with a simple question that was addresed to the public in an editon of the Lady's Diary publication in 1747: "It is required to find (by a general theorem) the number of fractions of different values, each less than unity, so that the gratest denominator does be less than 100". The first to publish a correct solution was a writer going by the name of Flitcoin, responding that there are 3003 fractions. His solution was using something similar to Euler's totient function, which counts the number of integers less than or equal to n that are coprime to n. The solution to this problem raised another question in mathematics: "How can we generate these irreductible fractions by a simple method/formula?". The answer was given by the geologist John Farey who published in The Philosophical Magazine and Journal a propriety of what was later called the Farey sequence, but without any mathematical proof. It was Cauchy that later provided a proof that the mediant property holds, crediting John Farey for the discovery [**ainsworth2012farey**]. The mediant, also known as the Freshman's sum is the operation defined as it follows:

$$\frac{a}{b} \bigoplus \frac{c}{d} = \frac{a+c}{b+d}$$

For any two fractions $\frac{a}{b}$ and $\frac{c}{d}$, with $\frac{a}{b} < \frac{c}{d}$ the mediant of the two $\frac{a+c}{b+d}$ lies between them in value, $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. By knowing this propriety, we remark that the mediant of two irreductible fractions will always generate a new fraction in the Farey sequence. By starting with the two fractions $\frac{0}{1}$ and $\frac{1}{1}$, we can generate the Farey sequence by applying the mediant operation. Two neighbouring fractions of the Farey sequence of a certain order $\frac{a}{b}$ and $\frac{c}{d}$, have the following propriety: $|bc - ad| = 1$, which is easily demonstrable by mathematical induction. This property also tells us that any two neighbouring fractions of the Farey sequence can be represented as two tangent Ford circles.

2.7 Functions

The first cases were created tangent at all times to the initial two circles. We will refer to them later on as the 'first generation' and the 'second generation'. Essentially, 'the first generation' circle's centers are getting closer to $x = 0$ and 'the second generation circle's centers are getting closer to $x = 1$.

These centers form two functions:
$f : [0, 1] \to \mathbb{R}$ the function which contains all the points from the first generation

$$f(x) = \frac{1}{2}x^2$$

$g : [0, 1] \to \mathbb{R}$ the function which contains all the points from the second generation

$$g(x) = \frac{1}{2}x^2 - x + \frac{1}{2}$$

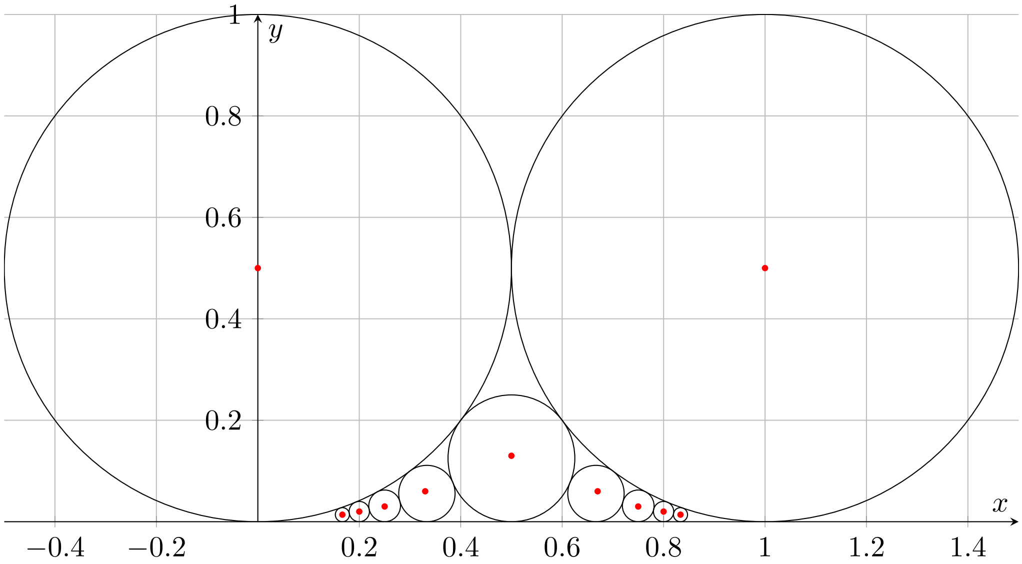


Figure 6: The first generation and the second generation circles and points

Following the idea of creating polynomial functions, a new type is showing up. It•s general formula is:
$f_n(x) : [0, 1] \to \mathbb{R}$

$$f_n(x) = \frac{n+2}{2}x^2 - \frac{n+2}{2}x + \frac{1}{2}$$

Demonstration:

Starting from the general formula of a quadratic function
$f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ and $f(x) : \mathbb{R} \to \mathbb{R}$
Given that the points $A(0, \frac{1}{2})$ and $B(1, \frac{1}{2})$ lie on its graph, we have:

$$f(0) = a \cdot 0^2 + b \cdot 0 + c = \tfrac{1}{2} \Rightarrow c = \tfrac{1}{2}$$

$$f(1) = a + b + \tfrac{1}{2} = \tfrac{1}{2} \Rightarrow a + b = 0 \Rightarrow a = -b$$

Therefore, because A and B are always on all functions' graphs, all formulas will have the form: $f(x) = ax^2 - ax + \frac{1}{2}$

Obviously, the general formula depends on how we begin to number the functions, but because at least two of the points are part of f(x) (defined above) a will forever have 2 as numerator. Also what is to be noticed is that for each function a increases with $\frac{1}{2}$.

$f_1 : [0, 1] \to \mathbb{R}$ goes through A, B and C

$$f_1(x) = \frac{3}{2}x^2 - \frac{3}{2}x + \frac{1}{2}$$

$f_2 : [0, 1] \to \mathbb{R}$ goes through A, D, E and B

$$f_2(x) = 2x^2 - 2x + \frac{1}{2}$$

$f_3 : [0, 1] \to \mathbb{R}$ goes through A, F, G and B

$$f_3(x) = \frac{5}{2}x^2 - \frac{5}{2}x + \frac{1}{2}$$

$f_4 : [0, 1] \to \mathbb{R}$ goes through A, H, I and B

$$f_4(x) = 3x^2 - 3x + \frac{1}{2}$$

$f_5 : [0, 1] \to \mathbb{R}$ goes through A, J, K and B

$$f_5(x) = \frac{7}{2}x^2 - \frac{7}{2}x + \frac{1}{2}$$

$f_6 : [0, 1] \to \mathbb{R}$ goes through A, L, M and B

$$f_6(x) = 4x^2 - 4x + \frac{1}{2}$$

$f_7 : [0, 1] \to \mathbb{R}$ goes through A, N, O and B

$$f_7(x) = \frac{9}{2}x^2 - \frac{9}{2}x + \frac{1}{2}$$

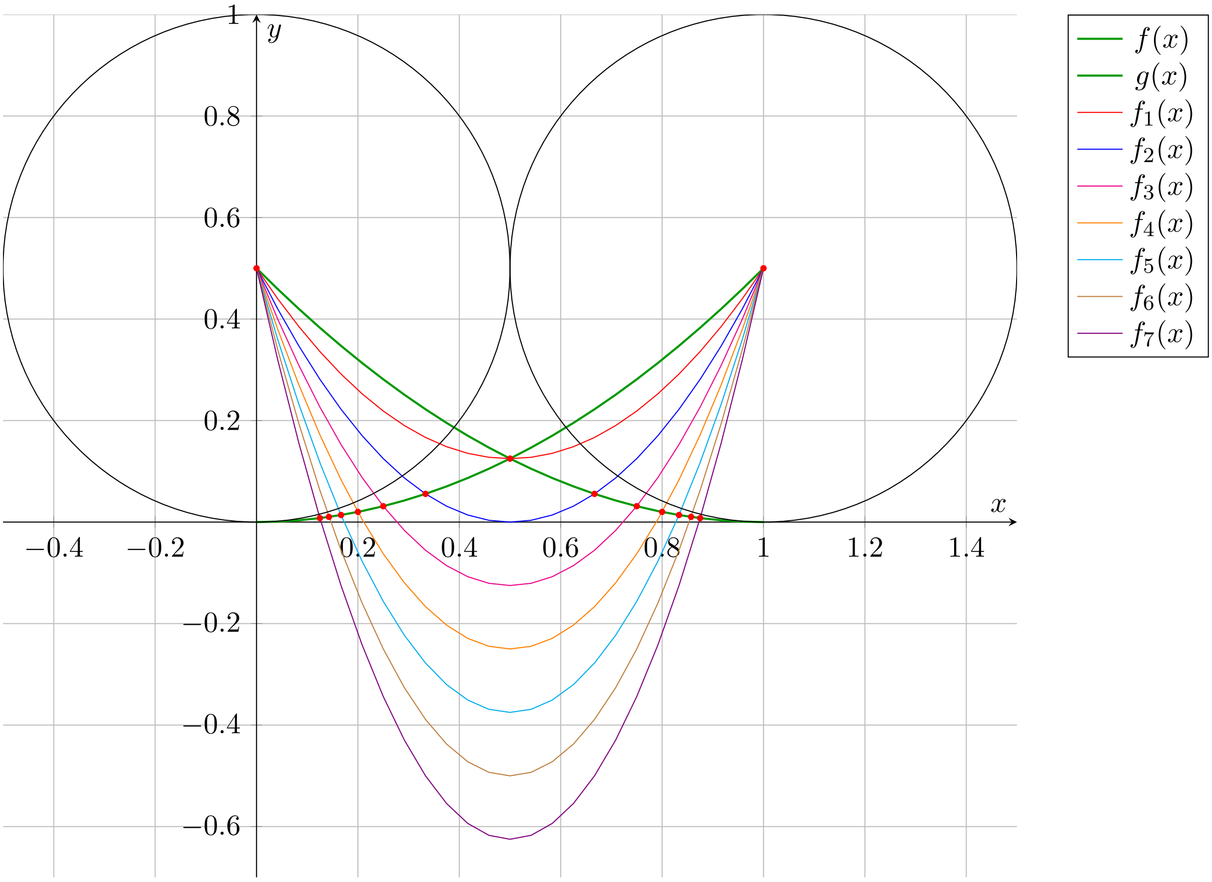


Figure 7: The polynomial functions that contain the points of the first and second generations

2.8 Total area

There is a method for finding the area covered by the Ford Circles, which doesn't take into consideration the surface of the two big initial circles. The process of finding the area is, however, quite complicated, and it includes some interesting functions (such as Euler's totient function, the Riemann zeta function and Apéry's constant), so we're not going to include the demonstration here. To get an idea though, the end result is:

$$A = \frac{45}{2} \cdot \frac{\zeta(3)}{\pi^3} \simeq 0.872284041$$

3 Visualisation

There are plenty of contexts, other than the ones previously mentioned, in which you can find these interesting circles. Some are in real life, others a bit more abstract. Below are a few examples.

3.1 Python code

For some visual representation of the problem, we wrote a Python code that starts with the two initial fractions of the Farey sequence and by increasing what we defined as a generation number, it calculates the mediants of the already existing fractions and it adds them to the sequence. After computing a certain generation, the code will print the fractions of that generation and it represents them with Ford circles in a cartesian coordinate system. It also prints the y coordinate of the center of the Ford circles. The code is written in Python 3 and it uses the libraries matplotlib, mpl-interactions and numpy.

```
import matplotlib.pyplot as plot
import numpy as np
from matplotlib.patches import Circle
from mpl_interactions import zoom_factory, panhandler
from matplotlib.scale import LinearScale, register_scale


class RestrictedLinearScale(LinearScale):
    name = 'restricted_linear'

    def __init__(self, axis, min=-np.inf, max=np.inf):
        super().__init__(axis)
        self.min = min
        self.max = max

    def limit_range_for_scale(self, vmin, vmax, minpos):
        return max(vmin, self.min), min(vmax, self.max)


register_scale(RestrictedLinearScale)

figure, axis = plot.subplots()

generation_index = int(input("Input the generation: "))

x_numerator_set = [0, 1]

x_denominator_set = [1, 1]

y_denominator_set = []
```

```
for i in range(generation_index - 1):
    j = 1
    while j < len(x_numerator_set):
        x_numerator_set.insert(j, x_numerator_set[j - 1] + x_numerator_set[j])
        x_denominator_set.insert(j, x_denominator_set[j - 1] +
    x_denominator_set[j])
        j += 2

for denominator in x_denominator_set:
    y_denominator_set.append(2 * (denominator ** 2))

for (numerator, denominator) in zip(x_numerator_set, x_denominator_set):
    print(f"{numerator}/{denominator}", end=" ")

print()

for denominator in y_denominator_set:
    print(f"1/{denominator}", end=" ")

for (x_numerator, x_denominator, y_denominator) in zip(x_numerator_set,
    x_denominator_set, y_denominator_set):
    circle = Circle((x_numerator / x_denominator, 1 / y_denominator), 1 /
    y_denominator, fill=False, edgecolor="black",
                    linewidth=1)
    axis.add_patch(circle)

axis.set_xlim(-0.5, 1.5)
axis.set_ylim(0, 1)

axis.set_aspect('equal', adjustable='box')

zoom_factory(axis)
panhandler = panhandler(figure, button=1)

axis.set_xscale('restricted_linear', min=-0.5, max=1.5)
axis.set_yscale('restricted_linear', min=0, max=1)
plot.tight_layout()

plot.show()
```

3.2 Real life applications

Here are a few real life situations in which Ford Circles can be seen:

3.2.1 Polonceau bridge

There are several bridges that include Ford Circles in their design. The first one was the Carrousel bridge in Paris, near the Louvre, designed by Antoine Rémy Polonceau. Unfortunately, due to it being too narrow for Paris' emerging traffic, the structure was completely changed in 1930, no longer featuring the Ford Circles.
There are, however, some bridges still standing, such as the Pont à la Polonceau (over the River Lanterne, near Bourguignon-lès-Conflans) and the Pont Saint-Thomas (in Strasbourg).

3.2.2 Bath bubbles

Another unexpected situation in which Ford geometry can be found is in bath bubbles. Looking from up close, you can see tiny Ford Spheres, tangent to one another and occupying almost all the space. It is a really good example of sphere packing, a type of problem that requires filling up as much of a volume as possible by using spheres of different sizes.

3.2.3 Pavements

You've probably wondered about the reason why this article is called "Circle pavements". And I hope that by now, after reading this article, you can understand the title's meaning. Either way, the point of paving something is to cover its entire surface. And we have to do that by using circles, which is a bit more difficult. That's how we end up doing circle packing, adding circles as small as physically possible, until there is no more grass visible. We're not sure if anyone's actually used Descartes' theorem to pave their garden, but perhaps now that you've read this article you might actually try it.

4 The process

All in all, we researched this topic for about half a year. We started with only the statement at the beginning of the article, struggling to find a good path to follow. We tried several ones, such as looking for a recursive formula between the circles' centers' coordinates, or for functions connecting the centers. One day, we stumbled upon the Apollonian gasket, from which it was easy to find the Ford Circles and Descartes' theorem. We have also had the occasion to meet some people at the Math en Jeans congress, who have given us a few good suggestions on some interesting aspects of our problem.

5 Conclusion

We have had a limited amount of time, but even so, we have managed to tackle recursivity and functions for our problem, the Apollonian gasket, Descartes' theorem, Ford circles, and the Farey sequence. We have created a Python code to ease visualisation, and a wooden puzzle to be able to grasp the perhaps more difficult mathematical concepts. There are plenty of things left to discover, and many more ideas to explore. Some aspects that we would like to cover in the future, such as Ford spheres, the Rauzy gasket, some interesting functions and others.