

Check-mate

Année 2018 – 2019

Students: MACOVEI Nicolae-Cristian, NICOLAE Mario-Alexandru, VOICILĂ Tristan-Ștefan (8th Grade), MASSIF Damien, ANGLADE Cassandre, GOURBIERE Yohan (1^{ère} S), GIRARDIN Lila, PELTIER Maelys (1^{ère} L).

Schools: “Bogdan Petriceicu Hasdeu” National College, Buzau, Romania
Lycée François Arago, Perpignan, France

Teachers: Melania NICOLAE, Marie DIUMENGE, Christelle QUERU

Researchers: Robert BROUZET (Université de Perpignan), Bogdan ENESCU

The subject:

When you play the chess, a king can attack any piece placed in an adjacent square, either horizontally, vertically, or in diagonal. Find the number of ways in which n non-attacking kings can be placed on a $2 \times 2n$ chess board.

The solutions found:

We have to prove that we can place $2^n(n+1)$, for $n \geq 1$ non-attacking kings on a $2 \times 2n$ chess board, where „non-attacking kings” means that the kings are not attacking each other.

I. First method [\(1\)](#).

We divide the chess board in n squares of the type 2×2 .

First, we place the kings on the first column for each square. For every square we have two possibilities to put the kings in this way. The number of the cases is $2 \cdot 2 \cdot 2 \cdot \dots \cdot 2 = 2^n$



Then, we move the king which is placed on the last square on the second column.

We have $2 \cdot 2 \cdot 2 \cdot \dots \cdot 2 = 2^n$ possibilities, too.

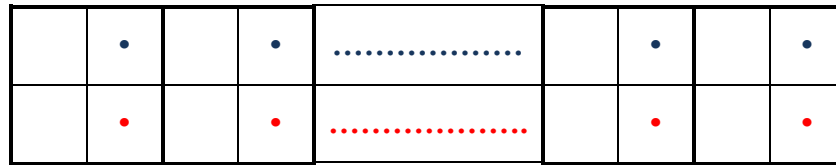


After we have done that we also move the king from the $(n-1)^{\text{th}}$ square on the second column.

Again, we will have $2 \cdot 2 \cdot 2 \cdot \dots \cdot 2 = 2^n$ possibilities



We will repeat this method until every king from each square will be placed on the second column. We will also have 2^n solutions.



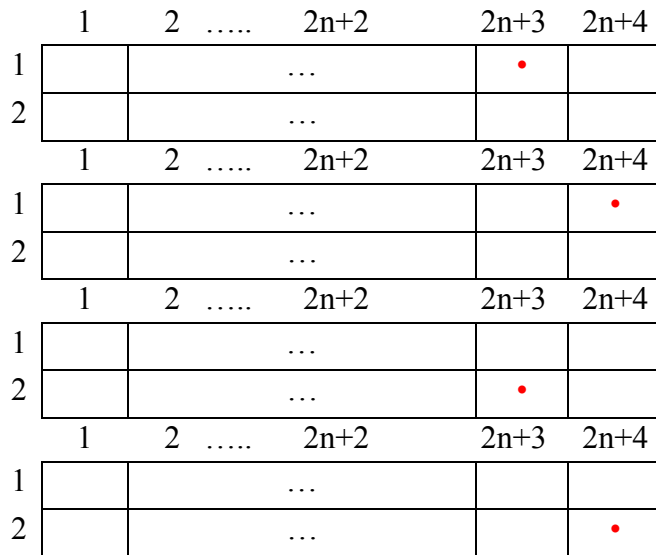
So, for every case we have 2^n possibilities. There are $n+1$ cases, so we have $2^n \cdot (n + 1)$ ways to place n non-attacking kings on a $2 \times 2n$ chess board.

II. Equation of induction (2)

We name a_n the number of possibilities for placing n kings on a $2 \times 2n$ board.

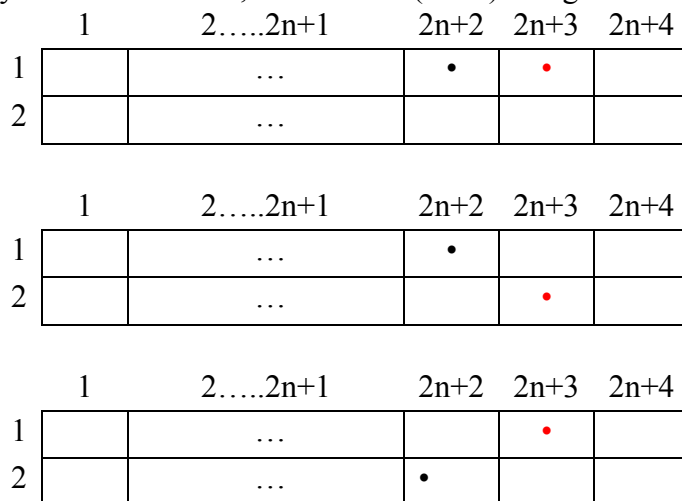
We want to find an equation between a_n , a_{n+1} and a_{n+2} .

The total number of cases for $n + 2$ kings is $a_{n+1} \cdot 4$, including the cases in which the kings attack each other, because for each way we place the first $n + 1$ kings we can place the $(n + 2)^{\text{th}}$ in 4 different ways:



We have to consider the cases in which the $(n+1)^{\text{th}}$ king and $(n+2)^{\text{th}}$ king attack each other.

For each way we place the first n kings in a_n cases, we have 4 ways to place the last two ones so that they attack each other, because the $(2n+2)^{\text{th}}$ king will attack the $(2n+3)^{\text{th}}$ king (3).



So, we have to subtract $4 \cdot a_n$ (the number of cases in which the kings attack each other).

Therefore, the number of ways we can place $n+2$ kings is

$$a_{n+2} = 4 \cdot a_{n+1} - 4 \cdot a_n \Rightarrow a_{n+2} = 4 \cdot (a_{n+1} - a_n)$$

III. Second method **(4)**

	1	2
1		
2		

We can place a king in any row: (1,1), (1,2), (2,1), (2,2) \Rightarrow 4 possibilities

	1	2	3	4
1				
2				

We can place two kings in 12 ways, and no attack: (1,1) and (1,3), (1,1) and (1,4), (1,1) and (2,3), (1,1) and (2,4), (1,2) and (1,4), (1,2) and (2,4), (2,1) and (1,3), (2,1) and (1,4), (2,1) and (2,3), (2,1) and (2,4), (2,2) and (1,4), (2,2) and (2,4). **(5)**

We want to prove that $p(n): a_n = 2^n \cdot (n + 1)$ is true, where a_n is the number of possibilities which n kings are not attacking each other, can be placed on a chess board $2 \cdot 2n$:

$$p(1): a_1 = 2^1 \cdot (1 + 1) = 4 \text{ possibilities}$$

$$p(2): a_2 = 2^2 \cdot (2 + 1) = 12 \text{ possibilities}$$

$p(1)$ and $p(2)$ are true

We suppose that $p(k)$ and $p(k + 1)$ are true, $k \in \mathbb{N} \setminus \{0\}$ **(6)**

$$p(k): a_k = 2^k \cdot (k + 1) \text{ possibilities (1)}$$

$$p(k + 1): a_{k+1} = 2^{k+1} \cdot (k + 2) \text{ possibilities (2)}$$

We want to demonstrate that $p(k + 2)$ is true, $k \in \mathbb{N} \setminus \{0\}$ **(7)**

$$p(k + 2): a_{k+2} = 2^{k+2} \cdot (k + 3)$$

We know that $a_{k+2} = (a_{k+1} - a_k) \cdot 4$ (**)

We are replacing (1) and (2) in (**) \Rightarrow

$$a_{k+2} = [2^{k+1} \cdot (k + 2) - 2^k \cdot (k + 1)] \cdot 4 \Rightarrow$$

$$a_{k+2} = 2^k \cdot (2k + 4 - k - 1) \cdot 4 \Rightarrow$$

$$a_{k+2} = 2^k \cdot (k + 3) \cdot 4 \Rightarrow$$

$$a_{k+2} = 2^{k+2} \cdot (k + 3), k \in \mathbb{N} \setminus \{0\} \Rightarrow \text{**(8)**}$$

$\Rightarrow p(k + 2)$ is true for $k \in \mathbb{N} \setminus \{0\}$ **(9)** $\Rightarrow p(n)$ is true for $n \in \mathbb{N} \setminus \{0\}$

(10)

IV. Third method **(11)**

We know that: $a_1=4$ and $a_2=12$ and $a_{n+2} = 4 \cdot a_{n+1} - 4 \cdot a_n, n \geq 1$

$$\Leftrightarrow a_{n+2} - 2 \cdot a_{n+1} = 2 \cdot (a_{n+1} - 2 \cdot a_n)$$

We note : $a_{n+1} - 2 \cdot a_n = y_n, n \geq 1$

$$\Rightarrow a_{n+2} - 2 \cdot a_{n+1} = y_{n+1}$$

$$\Rightarrow y_{n+1} = 2 \cdot y_n, n \geq 1$$

$$\text{If } n = 1 \Rightarrow y_1 = a_2 - 2 \cdot a_1 = 12 - 8 = 4 \Rightarrow y_1 = 4$$

$$y_n = 2 \cdot y_{n-1} \Rightarrow y_n = 2 \cdot y_{n-1}$$

$$y_{n-1} = 2 \cdot y_{n-2} \cdot 2^1 \Rightarrow 2^1 \cdot y_{n-1} = 2^2 \cdot y_{n-2}$$

$$y_{n-2} = 2 \cdot y_{n-3} \cdot 2^2 \Rightarrow 2^2 \cdot y_{n-2} = 2^3 \cdot y_{n-3}$$

$$y_2 = 2 \cdot y_1 \cdot 2^{n-2} \Rightarrow 2^{n-2} \cdot y_2 = 2^{n-1} \cdot y_1$$

Summing these equations, we will have:

$$y_n = 2^{n-1} \cdot y_1, \text{ where } y_1 = 4 \Rightarrow$$

$$y_n = 2^{n-1} \cdot 4 \Rightarrow y_n = 2^{n+1}, n \geq 1$$

$$\text{So, } a_{n+1} - 2 \cdot a_n = 2^{n+1} \Rightarrow a_{n+1} = 2^{n+1} + 2 \cdot a_n$$

$$\Rightarrow a_n = 2 \cdot a_{n-1} + 2^n \Rightarrow a_n = 2 \cdot a_{n-1} + 2^n$$

$$a_{n-1} = 2 \cdot a_{n-2} + 2^{n-1} \cdot 2^1 \Rightarrow 2^1 \cdot a_{n-1} = 2^2 \cdot a_{n-2} + 2^n$$

$$a_{n-2} = 2 \cdot a_{n-3} + 2^{n-2} \cdot 2^2 \Rightarrow 2^2 \cdot a_{n-2} = 2^3 \cdot a_{n-3} + 2^n$$

.

$$a_2 = 2 \cdot a_1 + 2^2 \cdot 2^{n-2} \Rightarrow 2^{n-2} \cdot a_2 = 2^{n-1} \cdot a_1 + 2^n$$

Summing these equations, we will have:

$$a_n = 2^{n-1} \cdot a_1 + 2^n \cdot (n-1) \Rightarrow a_n = 2^{n-1} \cdot (4 + 2n - 2) \Rightarrow$$

$$a_n = 2^{n-1} \cdot (2n + 2) \Rightarrow a_n = 2^n \cdot (n + 1), n \geq 1$$

Notes d'édition

(1) Rather: "First computation"

(2) Rather: "Induction relationship"

(3) Rather: $2n+1$ and $2n+2$

(4) This is not a direct proof of the computation of part I as one are using the recurrence relation of Part II.

One could notice that the computation made in part I do not show that all the cases are described by this way, so we cannot be sure that the part I compute all cases.

The part II will be useful to get a complete answer of this problem.

As a conclusion, one expect that the value of a_n is the same as the number which is computed in part I. This is the goal of part III.

(5) This case is not very useful, and does not explain why all the cases are described. The nice explanation in part 1 could be used instead.

(6) Some redactional remark : we have to be a little bit more careful : "first choose k in \mathbb{N}^* , then suppose p_k and p_{k+1} ."

(7) This is useless

(8) These 5 preceding signs \Rightarrow are useless

(9) We should write: then p_{k+2} is true. Therefore, we have shown that $p(n)$ is true for all n in \mathbb{N}^*

(10) A conclusion would have been nice. The problem introduced at the beginning. It should be interesting to think about connected questions.

For example, what happens if we change the form of the chess board, and take a square? What happens with knights instead of kings? ...

(11) This part is dealing with technical computations. There is no generalization neither some other comment about extension of this problem. This could be now done by the reader.