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DECOMPOSING INTEGERS

"God created the integers, all else is the work of man." Leopold Kronecker

The Problem. What are **the integers** that can be written as $\mathbf{x^2+ay^2}$, where $\mathbf{a\in Z}$ is fixed?

Remark. Finding $n\in Z$ that can be written as x^2+ay^2 is like looking for $n\in Z$ for which the equation $x^2+ay^2=n$ has solutions $(x,y)\in Z\times Z$. It will be noticed that some equations have no solutions, while others have a finite number of solutions and others have an infinite number of solutions.

Using the Scientific WorkPlace Program for graphing, we will analyze the important cases a=1, a=-1, a=2, a=-2, discovering the solvability of equations by graphical search on circles, ellipses, lines and hyperboles. For a=0, we will also mention the graphical answer.

We will present some general results demonstrated in number theory, which are found in the works in the bibliography, both for $a\in\{1,2,-1,-2,0\}$ and for other values. We will also mention open problems.

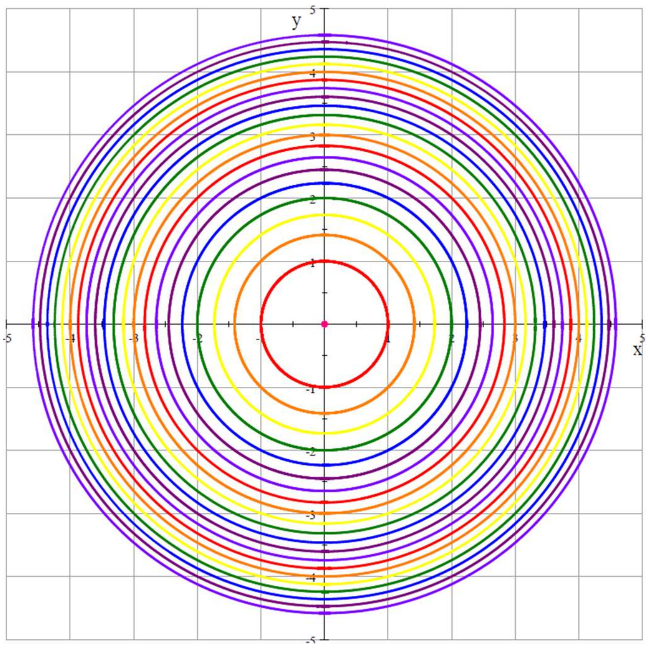
A computer program could be written to generate, for each fixed $a\in Z$, the numbers $n=x^2+ay^2\in Z$, giving values $(x,y)\in N\times N$ and then ordering the generated integers. Having an infinite number of integers, the program must be stopped running imposing an upper generation limit.

I. a>0 (a=1,2,3,5,7, other cases)

Remark. Since $x^2+ay^2\geq 0, \forall (x,y)\in Z\times Z \Rightarrow n\in Z, n<0$ cannot be written as x^2+ay^2, $(x,y)\in Z\times Z$.
We look for $n\in Z$, $n\geq 0$ for which there is $(x,y)\in Z\times Z$ such that $n=x^2+ay^2$.

QUESTION 1. a=1. What are integers $n\in Z$, $n\geq 0$ that can be written as $\mathbf{x^2+y^2}$?

Answer 1.1. We partially researched, by **graphical analysis** and algebraic verification, which of the numbers $n\in\{0,1,2,...,21\}$ can be written as x^2+y^2.



1) n=0: We are looking if the circle that becomes a double point, with the equation

$$x^2+y^2=0\Leftrightarrow(x,y)=(0,0),$$

passes through a point with integers coordinates $(x,y)\in Z\times Z$. We find: $n=0=0^2+0^2$.

2) $n\in Z, n\geq 1$: We are looking if the circle with center (0,0) and radius $\sqrt{n}$, with the equation

$$x^2+y^2=n\Leftrightarrow x^2+y^2=(\sqrt{n})^2,$$

passes through a point with integers coordinates $(x,y)\in Z\times Z$. By searching for $n\in\{1,2,...,21\}$, i.e. looking for the colored circles passing through integer coordinate points, we find *all possible writing solutions*:

red: $1=1^2+0^2=0^2+1^2=(-1)^2+0^2=0^2+(-1)^2$;
orange: $2=1^2+1^2=(-1)^2+1^2=(-1)^2+(-1)^2=1^2+(-1)^2$;
yellow: 3 cannot be written as x^2+y^2, i.e. there are no integers coordinates points (x,y) located on the circle with center (0,0) and radius $\sqrt{3}$.
green: $4=2^2+0^2=0^2+2^2=(-2)^2+0^2=0^2+(-2)^2$;
blue: $5=2^2+1^2=1^2+2^2=(-1)^2+2^2=(-2)^2+1^2=(-2)^2+(-1)^2=(-1)^2+(-2)^2=1^2+(-2)^2=2^2+(-1)^2$;
purple: 6 cannot be written as x^2+y^2, i.e. there are no integers coordinates points (x,y) located on the circle with center (0,0) and radius $\sqrt{6}$.

violet: 7 cannot be written as x^2+y^2, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{7}$.
red: $8=2^2+2^2=(-2)^2+2^2=(-2)^2+(-2)^2=2^2+(-2)^2$;
orange: $9=3^2+0^2=0^2+3^2=(-3)^2+0^2=0^2+(-3)^2$;
yellow: $10=3^2+1^2=1^2+3^2=(-1)^2+3^2=(-3)^2+1^2=$
$(-3)^2+(-1)^2=(-1)^2+(-3)^2=1^2+(-3)^2=3^2+(-1)^2$;
green: 11 cannot be written as x^2+y^2, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{(11)}$;
blue: 12 cannot be written as x^2+y^2, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{(12)}$;
purple: $13=3^2+2^2=2^2+3^2=(-2)^2+3^2=(-3)^2+2^2=(-3)^2+(-2)^2=(-2)^2+(-3)^2=2^2+(-3)^2=3^2+(-2)^2$;
violet: 14 cannot be written as x^2+y^2, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{(14)}$.
red: 15 cannot be written as x^2+y^2, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{(15)}$;
orange: $16=4^2+0^2=0^2+4^2=(-4)^2+0^2=0^2+(-4)^2$;
yellow: $17=4^2+1^2=1^2+4^2=(-1)^2+4^2=(-4)^2+1^2=(-4)^2+(-1)^2=(-1)^2+(-4)^2=1^2+(-4)^2=4^2+(-1)^2$;
green: $18=3^2+3^2=(-3)^2+3^2=(-3)^2+(-3)^2=3^2+(-3)^2$;
blue: 19 cannot be written as x^2+y^2, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{(19)}$;
purple: $20=4^2+2^2=2^2+4^2=(-2)^2+4^2=(-4)^2+2^2=(-4)^2+(-2)^2=(-2)^2+(-4)^2=2^2+(-4)^2=4^2+(-2)^2$;
violet: 21 cannot be written as x^2+y^2, i.e. there are no integers coordinates points (x,y) located on the circle with center $(0,0)$ and radius $\sqrt{(21)}$.
Graphic conclusions: For $n\in\{1,2,...,21\}$, completely traversing the circles of equations $x^2+y^2=n$ (*curves with finite length*), from the point $(\sqrt{n},0)$ counterclockwise, we found that the numbers

1,2,4,5,8,9,10,13,16,17,18,20

can be written as x^2+y^2, $(x,y)\in Z\times Z$. In addition, the writing is not unique. Due to the symmetry, it is sufficient to find the solutions $(x,y)\in N\times N$ for the equation $x^2+y^2=n$, i.e. of those points on Ox_+,Oy_+ or in I[st] quadrant that are on the circle and have coordinates in N.
Answer 1.2. Theorem in Number Theory.
a) (Fermat) The prime number $n=p>0$ is written as $p=x^2+y^2,(x,y)\in Z\times Z$ if and only if $p=2$ or $p=4k+1,k\in N$, $p\equiv1(\text{mod } 4)$. (Bănescu [3], Ionaşcu, Patterson [7])
b) The number $n\in Z$, $n\geq1$ is written as $n=x^2+y^2$, $(x,y)\in Z\times Z$ if and only if its prime factors, of the form $4k+3,k\in N$ ($\equiv3(\text{mod } 4)$), are numbers that appear at even powers. (Bănescu [3])
Remark 1.1. If $n=k^2$, $k\in Z$ then there exists the trivial writing solution $n=k^2+0^2=(-k)^2+0^2=0^2+(-k)^2=0^2+k^2$.
Remark 1.2. If there are two distinct unordered pairs of solutions $(x,y)\in N\times N$ of the equation $x^2+y^2=n$, then n is composite. (Andreescu [2], pp. 89)
Remark 1.3. Let $z=u+iv\in C$. Then

$$u^2+v^2=|z|^2=|z^2|=|u^2-v^2+i\cdot2uv|=\sqrt{((u^2-v^2)^2+(2uv)^2)}\Leftrightarrow(u^2-v^2)^2+(2uv)^2=(u^2+v^2)^2.$$

A method of generating, by a computer program, the numbers $n=x^2+y^2\in N$ with form derived from Pythagorean numbers, can be obtained by giving values for $(u,v)\in N\times N$, with $u>v$ (for such a pair (u,v) we can construct accordingly and (v,u), and $(-u,v)$ and so on). Having obtained an infinite number of integers, the program must be stopped running, imposing an upper generation limit.

QUESTION 2. a=2. What are integers $n\in Z$, $n\geq0$ that can be written as x^2+2y^2?

Answer 2.1. We partially researched, by **graphical analysis** and algebraic verification, which of the numbers $n\in\{0,1,2,...,21\}$ can be written as x^2+2y^2.
1) n=0: We are looking if the ellipse that becomes a double point, with the equation

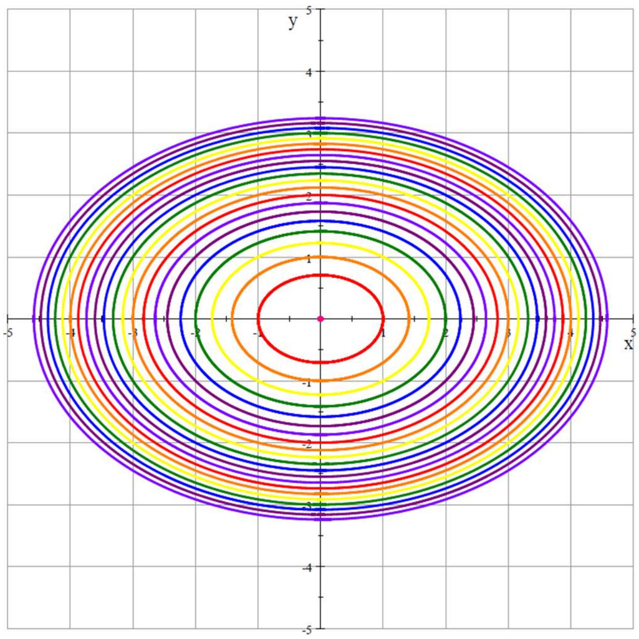
$$x^2+2y^2=0\Leftrightarrow(x,y)=(0,0),$$

passes through a point with integers coordinates $(x,y)\in Z\times Z$. We find: $n=0=0^2+2\cdot0^2$.

2) $n\in Z, n\geq 1$: We are looking if the ellipse with center (0,0) and semiaxis $\sqrt{n}$, $\sqrt{(n/2)}$, with the equation

$$x^2+2y^2=n \Leftrightarrow \frac{x^2}{(\sqrt{n})^2}+\frac{y^2}{(\sqrt{(n/2)})^2}=1,$$

passes through a point with integers coordinates $(x,y)\in Z\times Z$. By searching for $n\in\{1,2,...,21\}$, i.e. looking for the colored ellipses passing through integer coordinate points, we find *all possible writing solutions*:



red: $1=1^2+2\cdot 0^2=(-1)^2+2\cdot 0^2$;
orange: $2=0^2+2\cdot 1^2=0^2+2\cdot(-1)^2$;
yellow: $3=1^2+2\cdot 1^2=(-1)^2+2\cdot 1^2=(-1)^2+2\cdot(-1)^2= 1^2+2\cdot(-1)^2$;
green: $4=2^2+2\cdot 0^2=(-2)^2+2\cdot 0^2$;
blue: 5 cannot be written as x^2+2y^2, i.e. there are no integers coordinates points (x,y) located on the ellipse with $x^2+2y^2=5$;
purple: $6=2^2+2\cdot 1^2=(-2)^2+2\cdot 1^2=(-2)^2+2\cdot(-1)^2= 2^2+2\cdot(-1)^2$;
violet: 7 cannot be written as x^2+2y^2, i.e. there are no integers coordinates points (x,y) located on the ellipse $x^2+2y^2=7$;
red: $8=0^2+2\cdot 2^2=0^2+2\cdot(-2)^2$;
orange: $9=3^2+2\cdot 0^2=1^2+2\cdot 2^2=(-1)^2+2\cdot 2^2=(-3)^2+2\cdot 0^2=(-1)^2+2\cdot(-2)^2=1^2+2\cdot(-2)^2$;
yellow: 10 cannot be written as x^2+2y^2, i.e. there are no integers coordinates points (x,y) located on the ellipse $x^2+2y^2=10$;
green: $11=3^2+2\cdot 1^2=(-3)^2+2\cdot 1^2=(-3)^2+2\cdot(-1)^2=3^2+2\cdot(-1)^2$;
blue: $12=2^2+2\cdot 2^2=(-2)^2+2\cdot 2^2=(-2)^2+2\cdot(-2)^2=2^2+2\cdot(-2)^2$;
purple: 13 cannot be written as x^2+2y^2, i.e. there are no integers coordinates points (x,y) located on the ellipse $x^2+2y^2=13$;
violet: 14 cannot be written as x^2+2y^2, i.e. there are no integers coordinates points (x,y) located on the ellipse $x^2+2y^2=14$;
red: 15 cannot be written as x^2+2y^2, i.e. there are no integers coordinates points (x,y) located on the ellipse $x^2+2y^2=15$;
orange: $16=4^2+2\cdot 0^2=(-4)^2+2\cdot 0^2$;
yellow: $17=3^2+2\cdot 1^2=(-3)^2+2\cdot 1^2=(-3)^2+2\cdot(-1)^2=3^2+2\cdot(-1)^2$;
green: $18=4^2+2\cdot 1^2=0^2+2\cdot 3^2=(-4)^2+2\cdot 1^2=(-4)^2+2\cdot(-1)^2= 0^2+2\cdot(-3)^2=4^2+2\cdot(-1)^2$;
blue: $19=1^2+2\cdot 3^2=(-1)^2+2\cdot 3^2=(-1)^2+2\cdot(-3)^2=1^2+2\cdot(-3)^2$;
purple: 20 cannot be written as x^2+2y^2, i.e. there are no integers coordinates points (x,y) located on the ellipse $x^2+2y^2=20$;
violet: 21 cannot be written as x^2+2y^2, i.e. there are no integers coordinates points (x,y) located on the ellipse $x^2+2y^2=21$.

Graphic conclusions: For $n\in\{1,2,...,21\}$, completely traversing ellipses of equations $x^2+2y^2=n$ (*curves with finite length*), from the point $(\sqrt{n},0)$ counterclockwise, we found that the numbers

1,2,3,4,6,8,9,11,12,16,17,18,19

can be written as x^2+2y^2, $(x,y)\in Z\times Z$. In addition, the writing is not unique. Due to the symmetry, it is sufficient to find the solutions $(x,y)\in N\times N$ for the equation $x^2+2y^2=n$, i.e. of those points on Ox_+,Oy_+ or in Ist quadrant that are on the ellipse and have coordinates in N.

Answer 2.2. Theorem in Number Theory.

a) (Lagrange) The prime number $n=p>0$ is written as $p=x^2+2y^2,(x,y)\in Z\times Z$ if and only if $p=2$ or $p=8k+1,k\in N$, $p\equiv 1 \pmod 8$ or $p=8k+3,k\in N$, $p\equiv 3 \pmod 8$. (Bănescu [3], Ionaşcu, Patterson [7])

b) The number $n\in Z$, $n\geq 1$ is written as $n=x^2+2y^2,(x,y)\in Z\times Z$ if and only if its prime factors, of the form $8k+5,k\in N$ or $8k+7,k\in N$ ($\equiv 5 \pmod 8$ or $\equiv 7 \pmod 8$) are numbers that appear at even powers. (Bănescu [3])

Remark 2.1. If $n=k^2$, $k\in Z$ then there exists the trivial writing solution $n=k^2+2\cdot 0^2=(-k)^2+2\cdot 0^2$.

QUESTION 3. **$a=3,5,7$.** What are integers $n\in Z, n\geq 0$ that can be written as x^2+ay^2?

Answer 3.1. We partially researched, in the similar way, by **graphical analysis** and algebraic verification, which of the numbers $n\in\{0,1,2,...,21\}$ can be written as x^2+ay^2.

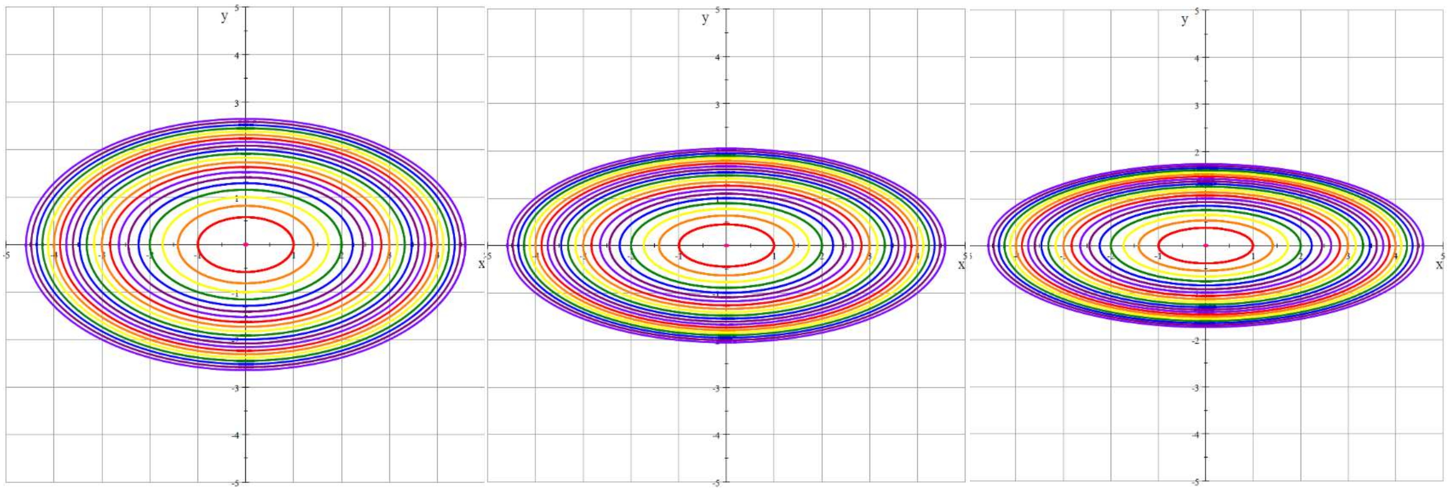
Graphic conclusions: For $n=0$, $0=0^2+a\cdot 0^2$. For $n\in\{1,2,...,21\}$, completely traversing ellipses of equations $x^2+ay^2=n$ (curves with finite length), from the point $(\sqrt{n},0)$ counterclockwise, we found that the numbers

$a=3$: 1,3,4,7,9,12,13,16,19,21

$a=5$: 1,4,5,6,9,14,16,20,21

$a=7$: 1,4,7,8,9,11,16

can be written as $x^2+ay^2,(x,y)\in Z\times Z$. In addition, the writing is not unique.



Answer 3.2.1. Theorem in Number Theory.

a) (Euler) The prime number $n=p>0$ is written as $p=x^2+3y^2,(x,y)\in Z\times Z$ if and only if $p=3$ or $p=3k+1$, $k\in N$, $p\equiv 1 (\mathrm{mod}\ 3)$ (Bănescu [3])/ $p=6k+1$, $k\in N$, $p\equiv 1(\mathrm{mod}\ 6)$ ((Ionaşcu, Patterson [7]))

b) The number $n\in Z$, $n\geq 1$ is written as $n=x^2+3y^2$, $(x,y)\in Z\times Z$ if and only if its prime factors, of the form $3k+2, k\in N$ ($\equiv 2(\mathrm{mod}\ 3)$) are numbers that appear at even powers. (Bănescu [3])

Answer 3.2.2. Theorem in Number Theory.

a) (Lagrange-Dirichlet) The prime number $n=p>0$ is written as $p=x^2+5y^2,(x,y)\in Z\times Z$ if and only if $p=5$ or $p=20k+1,k\in N$, $p\equiv 1(\mathrm{mod}\ 20)$ or $p=20k+9,k\in N$, $p\equiv 9(\mathrm{mod}\ 20)$. (Bănescu [3], Ionaşcu, Patterson [7])

b) Under what conditions $n\in N$, $n\geq 1$ is written as $n=x^2+5y^2$, $(x,y)\in Z\times Z$, the question remains open.

Answer 3.2.3. Theorem in Number Theory.

a) The prime number $n=p>0$ is written as $p=x^2+7y^2,(x,y)\in Z\times Z$ if and only if $p=7$ or $p=7k+1,k\in N$, $p\equiv 1(\mathrm{mod}\ 7)$ or $p=7k+2,k\in N$, $p\equiv 2(\mathrm{mod}\ 7)$ or $p=7k+4,k\in N$, $p\equiv 4(\mathrm{mod}\ 7)$. (Bănescu [3])

b) The number $n\in N$, $n\geq 1$ is written as $n=x^2+7y^2,(x,y)\in Z\times Z$ if and only if its prime factors, of the form $7k+3,k\in N$ or $7k+5,k\in N$ or $7k+6,k\in N$ ($\equiv 3(\mathrm{mod}\ 7)$ or $\equiv 5(\mathrm{mod}\ 7)$ or $\equiv 6(\mathrm{mod}\ 7)$) are numbers that appear at even powers. (Bănescu [3])

Remark 3.1. If $n=k^2$, $k\in Z$ then there exists the trivial writing solution $n=k^2+a\cdot 0^2=(-k)^2+a\cdot 0^2$.

QUESTION 4. Certain **$a>0$**. What are integers $n\in Z, n\geq 0$ that can be written as x^2+ay^2?

Answer 4.1. Graphical analysis and algebraic verification become difficult for large values of $a>0$ and $n>0$. For the the prime number $n=p>0$ we have the results:

Answer 4.2. Theorem in Number Theory. (Ionaşcu, Patterson [7])

a) (Fermat) The prime number $n=p>0$ is written as $p=x^2+y^2,(x,y)\in Z\times Z$ if and only if $p=2$ or $p\equiv 1(\mathrm{mod}\ 4)$.

b) (Fermat) The prime number $n=p>0$ is written as $p=x^2+2y^2,(x,y)\in Z\times Z$ if and only if $p=2$ or $p\equiv j(\mathrm{mod}\ 8)$, $j\in\{1,3\}$.

c) (Fermat-Euler) The prime number $n=p>0$ is written as $p=x^2+3y^2,(x,y)\in Z\times Z$ if and only if $p=3$ or $p\equiv 1(\mathrm{mod}\ 6)$.

d) The prime number $n=p>0$ is written as $p=x^2+4y^2,(x,y)\in Z\times Z$ if and only if $p\equiv 1(\mathrm{mod}\ 4)$.

e) (Lagrange) The prime number $n=p>0$ is written as $p=x^2+5y^2,(x,y)\in Z\times Z$ if and only if $p=5$ or $p\equiv j^2(\mathrm{mod}\ 20)$, $j\in\{1,3\}$.

f) The prime number $n=p>0$ is written as $p=x^2+6y^2,(x,y)\in Z\times Z$ if and only if $p\equiv j \pmod{24}$, $j\in\{1,7\}$.
g) The prime number $n=p>0$ is written as $p=x^2+7y^2,(x,y)\in Z\times Z$ if and only if $p=7$ or $p\equiv j^2 \pmod{14}$, $j\in\{1,3,5\}$.
h) The prime number $n=p>0$ is written as $p=x^2+8y^2,(x,y)\in Z\times Z$ if and only if $p\equiv 1 \pmod{8}$.
i) The prime number $n=p>0$ is written as $p=x^2+9y^2,(x,y)\in Z\times Z$ if and only if $p\equiv j^2 \pmod{36}$, $j\in\{1,5,7\}$.
j) The prime number $n=p>0$ is written as $p=x^2+10y^2,(x,y)\in Z\times Z$ if and only if $p\equiv j \pmod{40}$, $j\in\{1,9,11,19\}$.
k) The prime number $n=p>0$ is written as $p=x^2+12y^2,(x,y)\in Z\times Z$ if and only if $p\equiv j \pmod{48}$, $j\in\{1,13,25,37\}$.
l) The prime number $n=p>0$ is written as $p=x^2+13y^2,(x,y)\in Z\times Z$ if and only if $p\equiv j^2 \pmod{52}$, $j\in\{1,3,5,7,9,11\}$.
m) The prime number $n=p>0$ is written as $p=x^2+15y^2,(x,y)\in Z\times Z$ if and only if $p\equiv j \pmod{60}$, $j\in\{1,19,31,49\}$.
n) The prime number $n=p>0$ is written as $p=x^2+16y^2,(x,y)\in Z\times Z$ if and only if $p\equiv 1 \pmod{8}$.
Remark 4.1. If $n=k^2,k\in Z$ then there exists the trivial writing solution $n=k^2+a\cdot 0^2=(-k)^2+a\cdot 0^2$.

II. a<0 (a=-1,-2, other cases)

Remark. We look for $n\in Z$ for which there is $(x,y)\in Z\times Z$ such that $n=x^2-ay^2$.

QUESTION 1. a=-1. What are integers that can be written as x^2-y^2?

Answer 1.1. We partially researched, by **graphical analysis** in a certain region of the plane and algebraic verification, which of the numbers $n\in\{-21,...,-2,-1,0,1,2,...,21\}$ can be written as x^2-y^2.
1) $n=0$: We are looking if the hyperbola that becomes two secant lines, with the equation

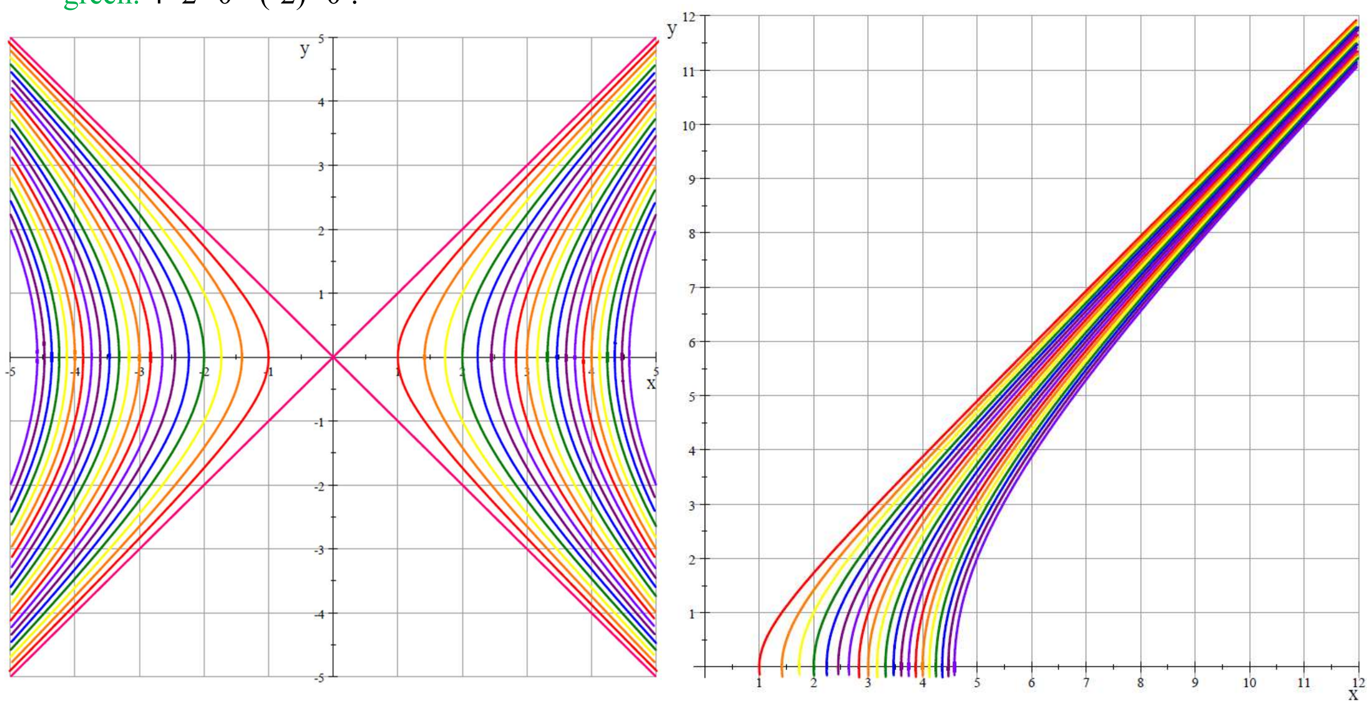
$$x^2-y^2=0 \Leftrightarrow (x=y \text{ or } x=-y)$$

passes through a point with integers coordinates $(x,y)\in Z\times Z$.
We find: $n=0=0^2-0^2=...=(\pm x)^2-(\pm x)^2=(\pm x)^2-(\mp x)^2, \forall x\in Z$.
2) $n\in Z, n\geq 1$: We are looking if the hyperbola with the equation

$$x^2-y^2=n \Leftrightarrow ((x^2)/((\sqrt{n})^2))-((y^2)/((\sqrt{n})^2))=1,$$

passes through a point with integers coordinates $(x,y)\in Z\times Z$. The graphical analysis area must be enlarged as a region in the plane to find at least one solution for writing n, if it exists, using (x,y) integer coordinates. Here, we will increase the symmetric region of the plane $[-5,5]\times[-5,5]$ only partially, relative to Ox_+,Oy_+ and I[st] quadrant, i.e. to $[0,12]\times[0,12]$, symmetrizing then the solutions found. By searching for $n\in\{1,2,...,21\}$, i.e. looking for the colored hyperbolas passing through integer coordinate points, we observe only *some of the possible writing solutions,* those from the studied plan region:

red: $1=1^2-0^2=(-1)^2-0^2$.
orange: 2=we cannot decide from the study.
yellow: $3=2^2-1^2=(-2)^2-1^2=(-2)^2-(-1)^2=2^2-(-1)^2$.
green: $4=2^2-0^2=(-2)^2-0^2$.

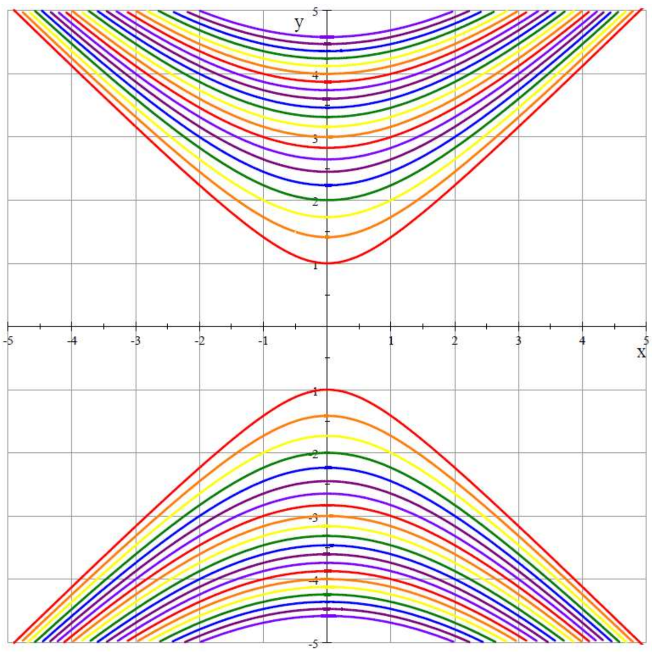


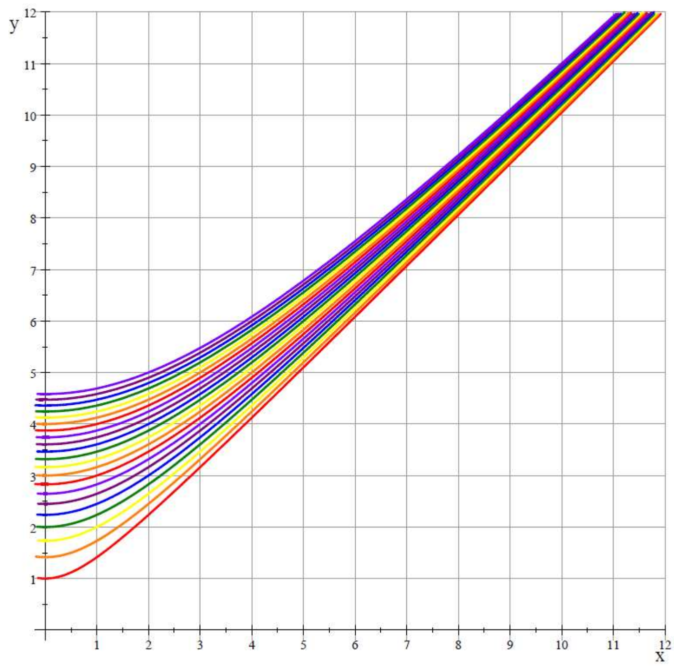
blue: $5=3^2-2^2=(-3)^2-2^2=(-3)^2-(-2)^2=3^2-(-2)^2$.
purple: 6=we cannot decide from the study.
violet: $7=4^2-3^2=(-4)^2-3^2=(-4)^2-(-3)^2=4^2-(-3)^2$.
red: $8=3^2-1^2=(-3)^2-1^2=(-3)^2-(-1)^2=3^2-(-1)^2$.
orange: $9=5^2-4^2=(-5)^2-4^2=(-5)^2-(-4)^2=5^2-(-4)^2$.
yellow: 10=we cannot decide from the study.
green: $11=6^2-5^2=(-6)^2-5^2=(-6)^2-(-5)^2=6^2-(-5)^2$.
blue: $12=4^2-2^2=(-4)^2-2^2=(-4)^2-(-2)^2=4^2-(-2)^2$.
purple: $13=7^2-6^2=(-7)^2-6^2=(-7)^2-(-6)^2=7^2-(-6)^2$.
violet: 14=we cannot decide from the study.
red: $15=8^2-7^2=(-8)^2-7^2=(-8)^2-(-7)^2=8^2-(-7)^2$.
orange: $16=4^2-0^2=(-4)^2-0^2=5^2-3^2=(-5)^2-3^2=(-5)^2-(-3)^2=5^2-(-3)^2$.
yellow: $17=9^2-8^2=(-9)^2-8^2=(-9)^2-(-8)^2=9^2-(-8)^2$.
green: 18=we cannot decide from the study.
blue: $19=10^2-9^2=(-10)^2-9^2=(-10)^2-(-9)^2=10^2-(-9)^2$.
purple: $20=6^2-4^2=(-6)^2-4^2=(-6)^2-(-4)^2=6^2-(-4)^2$.
violet: $21=11^2-10^2=(-11)^2-10^2=(-11)^2-(-10)^2=11^2-(-10)^2$.

3) $n\in Z, n\leq-1$: We are looking if the hyperbola with the equation

$$x^2-y^2=n\Leftrightarrow-((x^2)/((\sqrt{(-n)})^2))+((y^2)/((\sqrt{(-n)})^2))=1$$

passes through a point with integers coordinates $(x,y)\in Z\times Z$. The graphical analysis area must be enlarged as a region in the plane to find at least one solution for writing n, if it exists, using (x,y) integer coordinates. Here, we will increase the symmetric region of the plane $[-5,5]\times[-5,5]$ only partially, relative to Ox_+,Oy_+ and I[st] quadrant, i.e. to $[0,12]\times[0,12]$, symmetrizing then the solutions found. By searching for $n\in\{-21,...,-2,-1\}$, i.e. looking for the colored hyperbolas passing through integer coordinate points, we observe only some of the possible writing solutions, those from the studied plan region:





red: $-1=0^2-1^2=0^2-(-1)^2$.
orange: 2=we cannot decide from the study.
yellow: $-3=1^2-2^2=(-1)^2-2^2=(-1)^2-(-2)^2=1^2-(-2)^2$.
green: $-4=0^2-2^2=0^2-(-2)^2$.
blue: $-5=2^2-3^2=(-2)^2-3^2=(-2)^2-(-3)^2=2^2-(-3)^2$.
purple: -6=we cannot decide from the study.
violet: $-7=3^2-4^2=(-3)^2-4^2=(-3)^2-(-4)^2=3^2-(-4)^2$.
red: $-8=1^2-3^2=(-1)^2-3^2=(-1)^2-(-3)^2=1^2-(-3)^2$.
orange: $-9=4^2-5^2=(-4)^2-5^2=(-4)^2-(-5)^2=4^2-(-5)^2$.

yellow: -10=we cannot decide from the study.
green: $-11=5^2-6^2=(-5)^2-6^2=(-5)^2-(-6)^2=5^2-(-6)^2$.
blue: $-12=2^2-4^2=(-2)^2-4^2=(-2)^2-(-4)^2=2^2-(-4)^2$.
purple: $-13=6^2-7^2=(-6)^2-7^2=(-6)^2-(-7)^2=6^2-(-7)^2$.
violet: -14=we cannot decide from the study.
red: $-15=7^2-8^2=(-7)^2-8^2=(-7)^2-(-8)^2=7^2-(-8)^2$.
orange: $-16=0^2-4^2=0^2-(-4)^2=3^2-5^2=(-3)^2-5^2=(-3)^2-(-5)^2=3^2-(-5)^2$.
yellow: $-17=8^2-9^2=(-8)^2-9^2=(-8)^2-(-9)^2=8^2-(-9)^2$.
green: -18=we cannot decide from the study.
blue: $-19=9^2-10^2=(-9)^2-10^2=(-9)^2-(-10)^2=9^2-(-10)^2$.
purple: $-20=4^2-6^2=(-4)^2-6^2=(-4)^2-(-6)^2=4^2-(-6)^2$.
violet: $-21=10^2-11^2=(-10)^2-11^2=(-10)^2-(-11)^2=10^2-(-11)^2$.

Graphic conclusions: 1) $n=0=(\pm x)^2-(\pm x)^2=(\pm x)^2-(\mp x)^2, \forall x\in Z$.

2) For $n\in\{1,2,...,21\}$, partially traversing hyperbolas of equations $x^2-y^2=n$ (*curves with infinite length*), from the point $(\sqrt{n},0)$ in the lower left to the upper right direction, we found that the numbers

1,3,4,5,7,8,9,11,12,13,15,16,17,19,20,21

can be written as $x^2-y^2,(x,y)\in Z\times Z$. In addition, the writing is not unique. Due to the symmetry, it is sufficient to find the solutions $(x,y)\in N\times N$ for the equation $x^2-y^2=n$, i.e. of those points on Ox_+,Oy_+ or in I[st] quadrant that are on the hyperbola and have coordinates in N.

3) For $n\in\{-21,...,-2,-1\}$, partially traversing hyperbolas of equations $x^2-y^2=n$ (curves with infinite length), from the point $(0,\sqrt{(-n)})$ in the lower left to the upper right direction, we found that the numbers

-1,-3,-4,-5,-7,-8,-9,-11,-12,-13,-15,-16,-17,-19,-20,-21

can be written as $x^2-y^2,(x,y)\in Z\times Z$. In addition, the writing is not unique. Due to the symmetry, it is sufficient to find the solutions $(x,y)\in N\times N$ for the equation $x^2-y^2=n$, i.e. of those points on Ox_+,Oy_+ or in I[st] quadrant that are on the hyperbola and have coordinates in N.

Answer 1.2. Theorem in Number Theory.

a) If the integer $n\in Z$ is of the form $2k+1,k\in Z$, $n\equiv 1 \pmod 2$ or $4k,k\in Z$, $n\equiv 0 \pmod 4$ then n can be written as $n=x^2-y^2,(x,y)\in Z\times Z$.

b) If $n\in Z$ is of the form $4k+2,k\in Z$, $n\equiv 2 \pmod 4$, the question remains open.

Sketch of proof. We look for $n\in Z$ for which there exists $(x,y)\in Z\times Z$ such that $n=x^2-y^2 \Leftrightarrow n=(x-y)(x+y)$.
$\forall n=2k+1$, $k\in Z$ (so for $n=4k'+1$, $n=4k'+3$, $k'\in Z$) $\Rightarrow \exists x=k+1\in Z$, $\exists y=k\in Z$ such that $n=x^2-y^2$.
$\forall n=4k$, $k\in Z \Rightarrow \exists x=k+1\in Z$, $\exists y=k-1\in Z$ such that $n=x^2-y^2$.
$\forall n=4k+2$, $k\in Z \Rightarrow$ the question remains open. We tried

$x-y=2$, $x+y=2k+1 \Rightarrow x=((2k+3)/2)\notin Z$ and $y=((2k-1)/2)\notin Z$.

Remark 1.1. If $n=k^2$, $k\in Z$ then there exists the trivial writing solution $n=k^2-0^2=(-k)^2-0^2$.
If $n=-k^2$, $k\in Z$ then there exists the trivial writing solution $n=0^2-k^2=0^2-(-k)^2$.

QUESTION 2. a=-2. What are integers that can be written as x^2-2y^2?

Answer 2.1. We partially researched, by **graphical analysis** in a certain region of the plane and algebraic verification, which of the numbers $n\in\{-21,...,-2,-1,0,1,2,...,21\}$ can be written as x^2-2y^2.

1) n=0: We are looking if the hyperbola that becomes two secant lines, with the equation

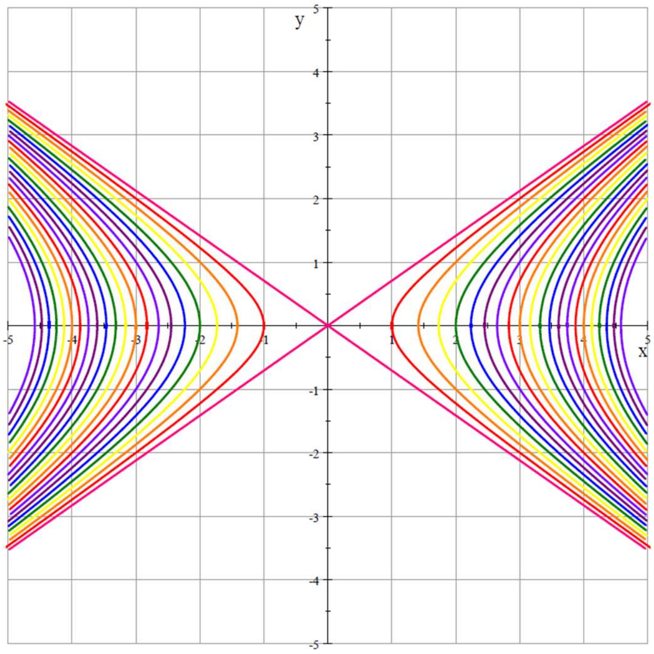
$x^2-2y^2=0 \Leftrightarrow (x=\sqrt{2}y \text{ or } x=-\sqrt{2}y)$

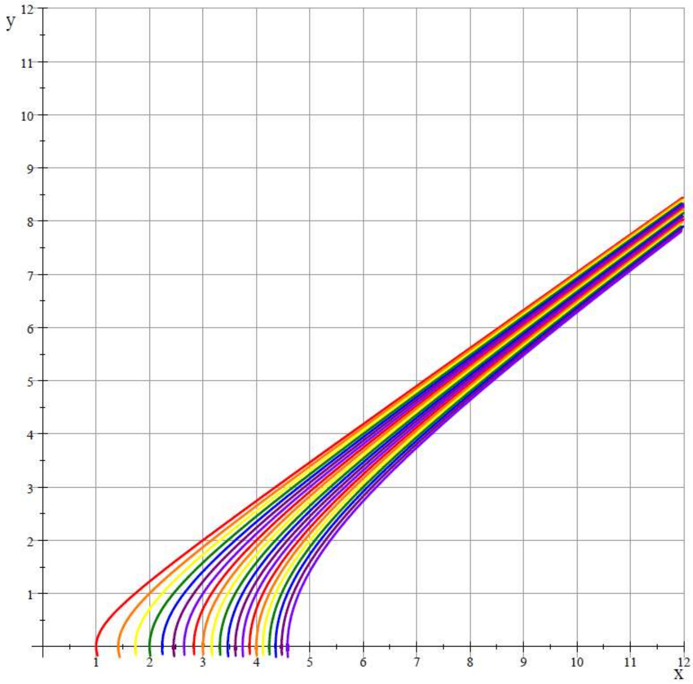
passes through a point with integers coordinates $(x,y)\in Z\times Z$. We find: $n=0=0^2-2\cdot 0^2$.

2) $n\in Z, n\geq 1$: We are looking if the hyperbola with the equation

$x^2-2y^2=n \Leftrightarrow ((x^2)/((\sqrt{n})^2))-((y^2)/((\sqrt{(n/2)})^2))=1$

passes through a point with integers coordinates $(x,y)\in Z\times Z$. The graphical analysis area must be enlarged as a region in the plane to find at least one solution for writing n, if it exists, using (x,y) integer coordinates. Here, we will increase the symmetric region of the plane $[-5,5]\times[-5,5]$ only partially, relative to Ox_+,Oy_+ and I[st] quadrant, i.e. to $[0,12]\times[0,12]$, symmetrizing then the solutions found. By searching for $n\in\{1,2,...,21\}$, i.e. looking for the colored hyperbolas passing through integer coordinate points, we observe only some of the possible writing solutions, those from the studied plan region:



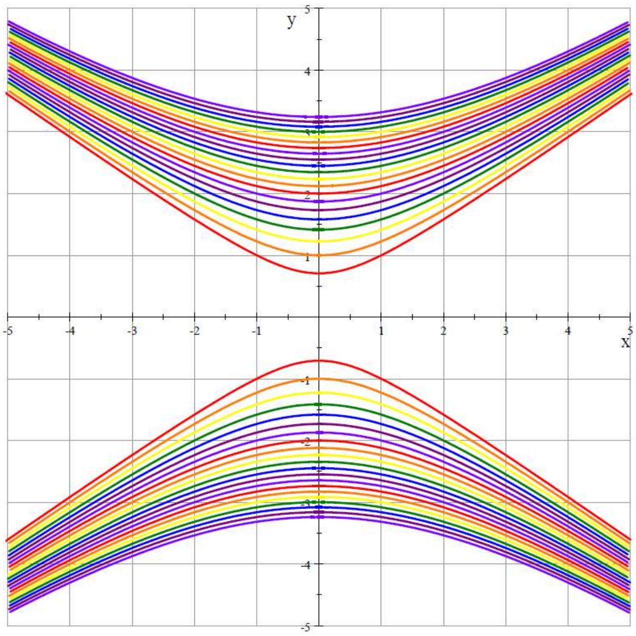


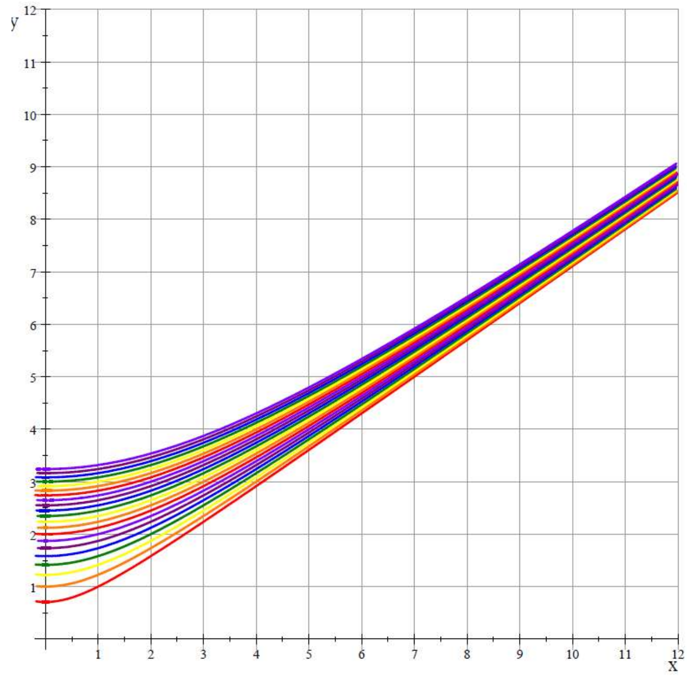
red: $1=1^2-2\cdot0^2=(-1)^2-2\cdot0^2=3^2-2\cdot2^2=(-3)^2-2\cdot2^2=(-3)^2-2\cdot(-2)^2=3^2-2\cdot(-2)^2$.
orange: $2=2^2-2\cdot1^2=(-2)^2-2\cdot1^2=(-2)^2-2\cdot(-1)^2=2^2-2\cdot(-1)^2=10^2-2\cdot7^2=(-10)^2-2\cdot7^2=(-10)^2-2\cdot(-7)^2=10^2-2\cdot(-7)^2$.
yellow: 3=we cannot decide from the study.
green: $4=2^2-2\cdot0^2=(-2)^2-2\cdot0^2=6^2-2\cdot4^2=(-6)^2-2\cdot4^2=(-6)^2-2\cdot(-4)^2=6^2-2\cdot(-4)^2$.
blue: 5=we cannot decide from the study.
purple: 6=we cannot decide from the study.
violet: $7=3^2-2\cdot1^2=(-3)^2-2\cdot1^2=(-3)^2-2\cdot(-1)^2=3^2-2\cdot(-1)^2=5^2-2\cdot3^2=(-5)^2-2\cdot3^2=(-5)^2-2\cdot(-3)^2=5^2-2\cdot(-3)^2$.
red: $8=4^2-2\cdot2^2=(-4)^2-2\cdot2^2=(-4)^2-2\cdot(-2)^2=4^2-2\cdot(-2)^2$.
orange: $9=3^2-0^2=(-3)^2-0^2=9^2-2\cdot6^2=(-9)^2-2\cdot6^2=(-9)^2-2\cdot(-6)^2=9^2-2\cdot(-6)^2$.
yellow: 10=we cannot decide from the study.
green: 11=we cannot decide from the study.
blue: 12=we cannot decide from the study.
purple: 13=we cannot decide from the study.
violet: $14=4^2-2\cdot1^2=(-4)^2-2\cdot1^2=(-4)^2-2\cdot(-1)^2=4^2-2\cdot(-1)^2=8^2-2\cdot5^2=(-8)^2-2\cdot5^2=(-8)^2-2\cdot(-5)^2=8^2-2\cdot(-5)^2$.
red: 15=we cannot decide from the study.
orange: $16=4^2-2\cdot0^2=(-4)^2-2\cdot0^2$.
yellow: $17=5^2-2\cdot2^2=(-5)^2-2\cdot2^2=(-5)^2-2\cdot(-2)^2=5^2-2\cdot(-2)^2=7^2-2\cdot4^2=(-7)^2-2\cdot4^2=(-7)^2-2\cdot(-4)^2=7^2-2\cdot(-4)^2$.
green: $18=6^2-2\cdot3^2=(-6)^2-2\cdot3^2=(-6)^2-2\cdot(-3)^2=6^2-2\cdot(-3)^2$.
blue: 19=we cannot decide from the study.
purple: 20=we cannot decide from the study.
violet: 21=we cannot decide from the study.

3) $n\in Z, n\leq-1$: We are looking if the hyperbola with center (0,0), with the equation

$$x^2-2y^2=n\Leftrightarrow-((x^2)/((\sqrt{(-n)})^2))+((y^2)/((\sqrt{(-(n/2))})^2))=1$$

passes through a point with integers coordinates $(x,y)\in Z\times Z$. The graphical analysis area must be enlarged as a region in the plane to find at least one solution for writing n, if it exists, using (x,y) integer coordinates. Here, we will increase the symmetric region of the plane $[-5,5]\times[-5,5]$ only partially, relative to Ox_+,Oy_+ and I[st] quadrant, i.e. to $[0,12]\times[0,12]$, symmetrizing then the solutions found. By searching for $n\in\{-21,...,-2,-1\}$ i.e. looking for the colored hyperbolas passing through integer coordinate points, we observe only some of the possible writing solutions, those from the studied plan region:





red: $-1=1^2-2\cdot1^2=(-1)^2-2\cdot1^2=(-1)^2-2\cdot(-1)^2=1^2-2\cdot(-1)^2=7^2-2\cdot5^2=(-7)^2-2\cdot5^2=(-7)^2-2\cdot(-5)^2=7^2-2\cdot(-5)^2$.
orange: $-2=0^2-2\cdot1^2=0^2-2\cdot(-1)^2=4^2-2\cdot3^2=(-4)^2-2\cdot3^2=(-4)^2-2\cdot(-3)^2=4^2-2\cdot(-3)^2$.
yellow: -3=we cannot decide from the study.
green: $-4=2^2-2\cdot2^2=(-2)^2-2\cdot2^2=(-2)^2-2\cdot(-2)^2=2^2-2\cdot(-2)^2$.
blue: -5=we cannot decide from the study.
purple: -6=we cannot decide from the study.
violet: $-7=5^2-2\cdot4^2=(-5)^2-2\cdot4^2=(-5)^2-2\cdot(-4)^2=5^2-2\cdot(-4)^2=11^2-2\cdot8^2=(-11)^2-2\cdot8^2=(-11)^2-2\cdot(-8)^2=11^2-2\cdot(-8)^2$.
red: $-8=0^2-2\cdot2^2=0^2-2\cdot(-2)^2=8^2-2\cdot6^2=(-8)^2-2\cdot6^2=(-8)^2-2\cdot(-6)^2=8^2-2\cdot(-6)^2$.
orange: $-9=3^2-2\cdot3^2=(-3)^2-2\cdot3^2=(-3)^2-2\cdot(-3)^2=3^2-2\cdot(-3)^2$.
yellow: -10=we cannot decide from the study.
green: -11=we cannot decide from the study.
blue: -12=we cannot decide from the study.
purple: -13=we cannot decide from the study.
violet: $-14=2^2-2\cdot3^2=(-2)^2-2\cdot3^2=(-2)^2-2\cdot(-3)^2=2^2-2\cdot(-3)^2=6^2-2\cdot5^2=(-6)^2-2\cdot5^2=(-6)^2-2\cdot(-5)^2=6^2-2\cdot(-5)^2$.
red: -15=we cannot decide from the study.
orange: $-16=0^2-2\cdot4^2=0^2-2\cdot(-4)^2=4^2-2\cdot4^2=(-4)^2-2\cdot4^2=(-4)^2-2\cdot(-4)^2=4^2-2\cdot(-4)^2$.
yellow: $-17=1^2-2\cdot3^2=(-1)^2-2\cdot3^2=(-1)^2-2\cdot(-3)^2=1^2-2\cdot(-3)^2=9^2-2\cdot7^2=(-9)^2-2\cdot7^2=(-9)^2-2\cdot(-7)^2=9^2-2\cdot(-7)^2$.
green: $-18=0^2-2\cdot3^2=0^2-2\cdot(-3)^2$.
blue: -19=we cannot decide from the study.
purple: -20=we cannot decide from the study.
violet: -21=we cannot decide from the study.

Graphic conclusions: 1) $n=0=0^2-2\cdot0^2$.

2) For $n\in\{1,2,...,21\}$, partially traversing hyperbolas of equations $x^2-2y^2=n$ (*curves with infinite length*), from the point $(\sqrt{n},0)$ in the lower left to the upper right direction, we found that the numbers

1,2,4,7,8,9,14,16,17,18

can be written as $x^2-2y^2,(x,y)\in Z\times Z$. In addition, the writing is not unique. Due to the symmetry, it is sufficient to find the solutions $(x,y)\in N\times N$ for the equation $x^2-2y^2=n$, i.e. of those points on Ox_+,Oy_+ or in I[st] quadrant that are on the hyperbola and have coordinates in N.

3) For $n\in\{-21,...,-2,-1\}$, partially traversing hyperbolas of equations $x^2-2y^2=n$ (curves with infinite length), from the point $(0,\sqrt{(-n)})$ in the lower left to the upper right direction, we found that the numbers

$-1,-2,-4,-7,-8,-9,-14,-16,-17,-18$

can be written as $x^2-2y^2,(x,y)\in Z\times Z$. In addition, the writing is not unique. Due to the symmetry, it is sufficient to find the solutions $(x,y)\in N\times N$ for the equation $x^2-2y^2=n$, i.e. of those points on Ox_+,Oy_+ or in I[st] quadrant that are on the hyperbola and have coordinates in N.

Answer 2.2. Theorem in Number Theory.

a) The odd prime number $n=p>0$ is written as $p=x^2-2y^2,(x,y)\in Z\times Z$ if and only if $p=8k+1,k\in N$, $p\equiv1 \pmod 8$ or $p=8k-1,k\in N$, $p\equiv-1 \pmod 8$. (Andreescu [1],[2], pp. 118, Ionaşcu, Patterson [7])

b) The prime number $n=-p<0$ is written as $-p=x^2-2y^2,(x,y)\in Z\times Z$ if and only if $p=2$ or $p\equiv\pm1 \pmod 8$. (Ionaşcu, Patterson [7])

Remark 2.1. If $n=k^2,k\in Z$ then there exists the trivial writing solution $n=k^2-0^2=(-k)^2-0^2$.

If $n=-k^2,k\in Z$ then there exists the trivial writing solution $n=0^2-k^2=0^2-(-k)^2$.

Remark 2.2. (Muscalu [8]) For n=1, the equation $x^2-2y^2=1$ has an infinity of integer solutions. Indeed,

-the equation has the solution (3,2), i.e. $3^2-2\cdot2^2=1$;

-if (x,y) is a solution, then also (3x+4y,2x+3y) is a solution, because

$$(3x+4y)^2-2(2x+3y)^2=9x^2+24xy+16y^2-2\cdot(4x^2+12xy+9y^2)=x^2-2y^2=1.$$

-let us define the string ((xk,yk))k by

$$(x_1,y_1)=(3,2)$$

$$(x_{k+1},y_{k+1})=(3x_k+4y_k,2x_k+3y_k),\ k\in N,k\geq1$$

According to the above statements, $((x_k,y_k))_k$ is a string of solutions.

Since $x_{k+1}>x_k$, $k\in N,k\geq1$ it follows that it is a string of distinct solutions.

The equation has an infinite number of solutions, i.e. n=1 can be written in an infinite number of ways as x^2-2y^2.

QUESTION 3. Certain **a<0**. What are integers that can be written as **x^2+ay^2**?

Answer 3.1. Graphical analysis and algebraic verification become difficult for large values of $|a|>0$ and $|n|>0$.

For the the prime number $n=p>0$ we have the results:

Answer 3.2.1. Theorem in Number Theory. (Ionaşcu, Patterson [7])

a) The prime number $n=p>0$ is written as $p=x^2-y^2,(x,y)\in Z\times Z$ if and only if $p\neq2$.

b) The prime number $n=p>0$ is written as $p=x^2-2y^2,(x,y)\in Z\times Z$ if and only if $p=2$ or $p\equiv\pm1 \pmod 8$.

c) The prime number $n=p>0$ is written as $p=x^2-3y^2,(x,y)\in Z\times Z$ if and only if $p\equiv1 \pmod{12}$. (Andreescu [1],[2])

d) The prime number $n=p>0$ is written as $p=x^2-4y^2,(x,y)\in Z\times Z$ if and only if $p\equiv1 \pmod 4$.

e) The prime number $n=p>0$ is written as $p=x^2-5y^2,(x,y)\in Z\times Z$ if and only if $p=5$ or $p\equiv\pm j^2 \pmod{20}$, $j\in\{1,3\}$.

f) The prime number $n=p>0$ is written as $p=x^2-6y^2,(x,y)\in Z\times Z$ if and only if $p=3$ or $p\equiv j \pmod{24}$, $j\in\{1,19\}$

g) The prime number $n=p>0$ is written as $p=x^2-7y^2,(x,y)\in Z\times Z$ if and only if $p=2$ or $p\equiv j \pmod{14}$, $j\in\{1,9,11\}$.

h) The prime number $n=p>0$ is written as $p=x^2-8y^2,(x,y)\in Z\times Z$ if and only if $p=7$ or $p\equiv j^2 \pmod{32}$, $j\in\{1,3,5,7\}$.

i) The prime number $n=p>0$ is written as $p=x^2-9y^2,(x,y)\in Z\times Z$ if and only if $p\equiv1 \pmod 6$.

j) The prime number $n=p>0$ is written as $p=x^2-10y^2,(x,y)\in Z\times Z$ if and only if $p\equiv j \pmod{40}$, $j\in\{1,9,31,39\}$.

k) The prime number $n=p>0$ is written as $p=x^2-11y^2,(x,y)\in Z\times Z$ if and only if $p\equiv j^2 \pmod{44}$, $j\in\{1,3,5,7,9\}$.

For the the prime number $n=-p<0$ we have the results:

Answer 3.2.2. Theorem in Number Theory.(Ionaşcu, Patterson [7])

a) The prime number $n=-p<0$ is written as $-p=x^2-y^2,(x,y)\in Z\times Z$ if and only if $p\neq2$.

b) The prime number $n=-p<0$ is written as $-p=x^2-2y^2,(x,y)\in Z\times Z$ if and only if $p=2$ or $p\equiv\pm1 \pmod 8$.

c) The prime number $n=-p<0$ is written as $-p=x^2-3y^2,(x,y)\in Z\times Z$ if and only if $p=2$ or $p=3$ or $p\equiv11 \pmod{12}$.

d) The prime number $n=-p<0$ is written as $-p=x^2-4y^2,(x,y)\in Z\times Z$ if and only if $p\equiv3 \pmod 4$.

e) The prime number $n=-p<0$ is written as $-p=x^2-5y^2,(x,y)\in Z\times Z$ if and only if $p=5$ or $p\equiv\pm j^2 \pmod{20}$, $j\in\{1,3\}$.

f) The prime number $n=-p<0$ is written as $-p=x^2-6y^2,(x,y)\in Z\times Z$ if and only if $p=2$ or $p\equiv j \pmod{24}$, $j\in\{5,23\}$

g) The prime number $n=-p<0$ is written as $-p=x^2-7y^2,(x,y)\in Z\times Z$ if and only if $p=7$ or $p\equiv j \pmod{14}$, $j\in\{3,5,13\}$.

h) The prime number $n=-p<0$ is written as $-p=x^2-8y^2,(x,y)\in Z\times Z$ if and only if $p=7$ or $p\equiv-j^2 \pmod{32}$, $j\in\{1,3,5,7\}$.

i) The prime number $n=-p<0$ is written as $-p=x^2-9y^2,(x,y)\in Z\times Z$ if and only if $p\equiv-1 \pmod 6$.

j) The prime number $n=-p<0$ is written as $-p=x^2-10y^2,(x,y)\in Z\times Z$ if and only if $p\equiv j \pmod{40}$, $j\in\{1,9,31,39\}$.
k) The prime number $n=-p<0$ is written as $-p=x^2-11y^2,(x,y)\in Z\times Z$ if and only if $p=2$ or $p=11$ or $p\equiv -j^2 \pmod{44}$, $j\in\{1,3,5,7,9\}$.
Remark. Recall that the search for $n\in Z$ that can be written as x^2+ay^2 is the search for $n\in Z$ for which the equation $x^2+ay^2=n$ has solutions $(x,y)\in Z\times Z$.
Answer 3.3. Theorem in Number Theory. (Caşu, Bényi [5], Rosen [12]) For $n \in Z$ the equation

$$x^2+ ay^2 = n, \text{ with } a <0, -a=A^2, A \in N, A \geq 1,$$

has at most a finite number of solutions $(x,y)\in Z\times Z$.
Sketch of proof. $x^2-A^2y^2 = n \Leftrightarrow (x - Ay)(x + Ay) = n \Leftrightarrow$

$$\begin{cases} x - Ay = u \\ x + Ay = v \end{cases}, \text{ where u and v are integers such that } n= u\cdot v.$$

Remark. (Caşu, Bényi [5]) For $n = 1$, the equation

$$x^2 + ay^2 = 1, \text{ with } a < 0, -a = A^2, A \in N, A\geq 1,$$

has only one natural solution, the trivial one $(1,0)$, so two integers.
That is, $n=1$ can be written in two ways as $x^2 - ay^2 = 1^2 - a\cdot 0^2 = (-1)^2 - a\cdot 0^2$.
Remark. For $n \in Z$ the equation

$$x^2+ ay^2 = n, \text{ with } a <0, -a=A^2, A \in N, A \geq 1, \boxed{-a = D}$$

is called *Diophantine equations*, even *general Pell equations*.
Partial Answer 3.4. Theorem in Number Theory. (Andreescu [1], [2], Caşu. Bényi [5])
For $n= 1$, the positive Pell equation $x^2 - Dy^2 = 1$ has an infinity of solutions $(x_k, y_k)\in Z\times Z$ given by

$$\begin{cases} x_{k+1} = x_1x_k + Dy_1y_k \\ y_{k+1} = y_1x_k + x_1y_k \end{cases}$$

where (x_1, y_1) is a fundamental solution, i.e. nontrivial solution with the smallest $x_1> 0$.
The general solution can be written, using matrix calculation, in the form

$$\begin{cases} x_k = \frac{1}{2}\left(\left(x_1 + y_1\sqrt{D}\right)^k + \left(x_1 - y_1\sqrt{D}\right)^k\right) \\ y_k = \frac{1}{2\sqrt{D}}\left(\left(x_1 + y_1\sqrt{D}\right)^k - \left(x_1 - y_1\sqrt{D}\right)^k\right) \end{cases}, k \in N^*$$

(check the relationship $x_k + y_k\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)^k, k \in N^*$).
That is, $n=1$ can be written in an infinite number of ways as $x^2 + ay^2$, with $a < 0$, $-a \neq A^2$, $A \in N^*$. $\boxed{-a = D}$
Example. (Andreescu [1], [2])
a) For $n=1$, the positive Pell equation $x^2 - 2y^2 = 1$ has the fundamental solution $(3, 2)$ and the general solution

$$\begin{cases} x_k = \frac{1}{2}\left(\left(3 + 2\sqrt{2}\right)^k + \left(3 - 2\sqrt{2}\right)^k\right) \\ y_k = \frac{1}{2\sqrt{2}}\left(\left(3 + 2\sqrt{2}\right)^k - \left(3 - 2\sqrt{2}\right)^k\right) \end{cases}, k \in N^*.$$

b) For $n = 1$, the positive Pell equation $x^2 - 3y^2 = 1$ has the fundamental solution $(2, 1)$ and the general solution

$$\begin{cases} x_k = \frac{1}{2}\left(\left(2 + \sqrt{3}\right)^k + \left(2 - \sqrt{3}\right)^k\right) \\ y_k = \frac{1}{2\sqrt{3}}\left(\left(2 + \sqrt{3}\right)^k - \left(2 - \sqrt{3}\right)^k\right) \end{cases}, k \in N^*.$$

c) For $n = 1$, the positive Pell equation $x^2 - 5y^2 = 1$ has the fundamental solution $(9, 4)$ and the general solution

$$\begin{cases} x_k = \frac{1}{2}\left(\left(9 + 4\sqrt{5}\right)^k + \left(9 - 4\sqrt{5}\right)^k\right) \\ y_k = \frac{1}{2\sqrt{5}}\left(\left(9 + 4\sqrt{5}\right)^k - \left(9 - 4\sqrt{5}\right)^k\right) \end{cases}, k \in N^*.$$

Remark. (Andreescu [2], Caşu, Bényi [5]) For n=1, the fundamental solutions for the positive Pell equations

$x^2+ay^2=1$, $-a=D$

can be found in the following table:

-a	x_1	y_1	-a	x_1	y_1	-a	x_1	y_1
2	3	2	38	37	6	71	3480	413
3	2	1	39	25	4	72	17	2
5	9	4	40	19	3	73	2281249	267000
6	5	2	41	2049	320	74	3699	430
7	8	3	42	13	2	75	26	3
8	3	1	43	3482	531	76	57799	6630
10	19	6	44	199	30	77	351	40
11	10	3	45	161	24	78	53	6
12	7	2	46	24335	3588	79	80	9
13	649	180	47	48	7	80	9	1
14	15	4	48	7	1	82	163	18
15	4	1	50	99	14	83	82	9
17	33	8	51	50	7	84	55	6
18	17	4	52	649	90	85	285769	30996
19	170	39	53	66249	9100	86	10405	1122
20	9	2	54	485	66	87	28	3
21	55	12	55	89	12	88	197	21
22	197	42	56	15	2	89	500001	53000
23	24	5	57	151	20	90	19	2
24	5	1	58	19603	2574	91	1574	165
26	51	10	59	530	69	92	1151	120
27	26	5	60	31	4	93	12151	1260
28	127	24	61	1766319049	226153980	94	2143295	221064
29	9801	1820	62	63	8	95	39	4
30	11	2	63	8	1	96	49	5
31	1520	273	65	129	16	97	62809633	6377352
32	17	3	66	65	8	98	99	10
33	23	4	67	48842	5967	99	10	1
34	35	6	68	33	4	101	201	20
35	6	1	69	7775	936	102	101	10
37	73	12	70	251	30	103	227528	22419

Remark. Unlike the positive Pell equation, which admits solutions for any a, the negative Pell equation admits solutions for certain numbers a.

Partial Answer 3.5. Theorem in Number Theory. (Andreescu [1], [2], Caşu. Bényi [5]) For n = –1, the negative Pell equation $x^2 - Dy^2 = -1$, if it is solvable, then the equation has an infinity of solutions

$$\begin{cases} x'_k = x'_0 x_k + D y'_0 y_k \\ y'_k = y'_0 x_k + x'_0 y_k \end{cases}$$

where (x'_0, y'_0) is the nontrivial minimal solution, i.e. with the smallest $x'_1 > 0$, and (x_k, y_k) is the solution of the positive Pell equation.

The general solution can be written, using matrix calculation, in the form

$$\begin{cases} x'_k = \frac{1}{2}\left(\left(x'_0 + y'_0\sqrt{D}\right)\left(x_1 + y_1\sqrt{D}\right)^k + \left(x'_0 - y'_0\sqrt{D}\right)\left(x_1 - y_1\sqrt{D}\right)^k\right) \\ y'_k = \frac{1}{2\sqrt{D}}\left(\left(x'_0 + y'_0\sqrt{D}\right)\left(x_1 + y_1\sqrt{D}\right)^k - \left(x'_0 - y'_0\sqrt{D}\right)\left(x_1 - y_1\sqrt{D}\right)^k\right) \end{cases}, k \in N^*.$$

(check the relationship $x_k + y_k\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)^k, k \in N^*$).

That is, if n= –1 can be written in a way like $x^2 + ay^2$, with a <0, $-a \neq A^2$, A ∈N*, $\boxed{-a = D}$, then it can be written in an infinity of ways.

Remark. (Andreescu [2], Caşu, Bényi [5]) For n=-1, the minimal solutions for the negative Pell equations

$x^2+ay^2=-1$, $-a=D$

can be found in the following table:

-a	x'_0	y'_0	-a	x'_0	y'_0	-a	x'_0	y'_0
2	1	1	37	6	1	73	1068	125
5	2	1	41	32	5	74	43	5
10	3	1	50	7	1	82	9	1
13	18	5	53	182	25	85	378	41
17	4	1	58	99	13	89	500	53
26	5	1	61	29718	3805	97	5604	569
29	70	13	65	8	1	101	10	1

Example. a) (Andreescu [1], [2]) For n = –1, the negative Pell equation $x^2 - 34y^2 = -1$ it is not solvable.
b) (Caşu, Bényi [5]) For n = –1, the negative Pell equation $x^2 - 3y^2 = -1$ it is not solvable.
c) (Andreescu [1], [2]) For n = –1, the negative Pell equation $x^2 - 5y^2 = -1$ has the minimum solution (2, 1) and the general solution described by the formula.

Partial Answer 3.6. Theorem in Number Theory. (Andreescu [1], [2]) For n= –1, let the negative Pell equation $x^2 + ay^2 = -1$, with $a < 0$, $-a \neq A^2$, $A \in N^*$, $\boxed{-a = D.}$ If D is a prime number, then the equation is solvable if and only if $D = 2$ or $D = 4k + 1$, $k \in N$ $D \equiv 1 \pmod 4$.
Partial Answer 3.7. Theorem in Number Theory. (Andreescu [1], [2]) Let $k \in N$. $k \geq 2$. For $n = -1$, the negative Pell equation $x^2 - (k^2 - 4)\, y^2 = -1$ is solvable if and only if $k = 3$.
Remark. (Andreescu [1], [2]) For $n = \pm 1$, the general solution of the positive / negative Pell equation

$$x^2 + ay^2 = \pm 1, \text{ with } a < 0, -a \neq A^2, A \in N^*, \boxed{-a = D},$$

can be determined using methods related, for the most part, to higher mathematics: method of continuous fractions; method of the square rings; other methods.
Remark. (Andreescu [1], [2]) For $n \in Z$, the general solution of the general positive / negative Pell equation

$$x^2 + ay^2 = n, \text{ with } a < 0, -a \neq A^2, A \in N^*, \boxed{-a = D},$$

can be determined using methods related, for the most part, to higher mathematics: direct search method, which is effective when $|n|$ is small (we studied graphically for $|n| \geq 21$) and when the fundamental solution of the attached Pell equation $x^2 + ay^2 = 1$ is small (for graphical analysis in a suitable region); Lagrange method; method of the square rings; cyclic method; other methods.
Answer 3.8. **Theorem in Number Theory** (Andreescu [1], [2], pp. 323, 324). For $n \in N$, the general positive Pell equation $x^2 + ay^2 = n$, with $a < 0$, $-a \neq A^2$, $A \in N^*$, $\boxed{-a = D}$, if it has a nontrivial solution, it has an infinity of solutions $(x,y) \in N \times N$.

Answer 3.9. Theorem in Number Theory. (Andreescu [1], [2]) For $n = p \in N$ prime number, the general positive Pell equation $x^2 + ay^2 = \pm p$, with $a < 0$, $-a \neq A^2$, $A \in N^*$, $\boxed{-a = D}$ has at most a minimal solution $(x,y) \in N \times N$ for which $x \geq 0$. If the equation is solvable, then it has one or two classes of solutions, as p divides or not the number $-2a$.

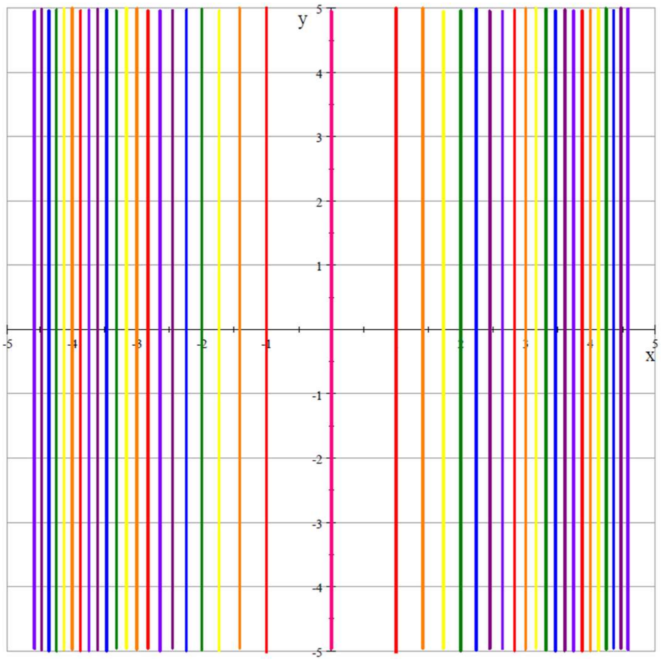
Remark. (Andreescu [1], [2], pp. 117) The general positive / negative Pell equation either has no solutions or has an infinity of solutions. Determining the solvability of the general positive / negative Pell equation requires notions of higher mathematics that we do not introduce in this presentation.

III. a=0

QUESTION 1. a=0. What are integers that can be written as $x^2+0 \cdot y^2$?

Remark. Solution. Since $x^2+0 \cdot y^2 \geq 0, \forall (x,y) \in Z \times Z \Rightarrow n \in Z, n<0$ cannot be written as $x^2+0 \cdot y^2$, $(x,y) \in Z \times Z$.
We look for $n \in Z, n \geq 0$ for which there is $(x,y) \in Z \times Z$ such that $n=x^2+0 \cdot y^2$.
Answer 1.1. We partially researched, by **graphical analysis** in a certain region of the plane and algebraic verification, which of the numbers $n \in \{0,1,2,...,21\}$ can be written as $x^2+0 \cdot y^2$.
1) n=0: We are looking if the two coincident lines, with the equation $x^2=0 \Leftrightarrow x=0$ passes through a point with integers coordinates $(x,y) \in Z \times Z$. We find: $n=0=0^2+0 \cdot 0^2=0^2+0 \cdot y^2, \forall y \in Z$.
2) $n \in Z, n \geq 1$: We are looking if the lines with the equation $x^2+0 \cdot y^2=n \Leftrightarrow$ ($x=\sqrt{n}$ or $x=-\sqrt{n}$) passes through a point with integers coordinates $(x,y) \in Z \times Z$. By searching for $n \in \{1,2,...,21\}$, i.e. looking for the colored lines passing through integer coordinate points, we observe only *some of the possible writing solutions*, those from the studied plan region:
red: $1=1^2+0 \cdot y^2=(-1)^2+0 \cdot y^2, y \in Z$
orange: 2 it cannot be written as $x^2+0 \cdot y^2$, i.e. there is no integer coordinate point (x,y) located on the lines of equations $x=\sqrt{2}$ or $x=-\sqrt{2}$.
yellow: 3 it cannot be written as $x^2+0 \cdot y^2$, i.e. there is no integer coordinate point (x,y) located on the lines of equations $x=\sqrt{3}$ or $x=-\sqrt{3}$.
green: $4=2^2+0 \cdot y^2=(-2)^2+0 \cdot y^2, y \in Z$

blue: 5 it cannot be written as $x^2+0\cdot y^2$, i.e. there is no integer coordinate point (x,y) located on the lines of equations $x=\sqrt{5}$ or $x=-\sqrt{5}$.



purple: 6 it cannot be written as $x^2+0\cdot y^2$...
violet: 7 it cannot be written as $x^2+0\cdot y^2$, i.e. there is no integer coordinate point (x,y) located on the lines of equations $x=\sqrt{7}$ or $x=-\sqrt{7}$.
red: 8 it cannot be written as $x^2+0\cdot y^2$...
orange: $9=3^2+0\cdot y^2=(-3)^2+0\cdot y^2, y\in Z$
yellow: 10 it cannot be written as $x^2+0\cdot y^2$...
green: 11 it cannot be written as $x^2+0\cdot y^2$...
blue: 12 it cannot be written as $x^2+0\cdot y^2$...
purple: 13 it cannot be written as $x^2+0\cdot y^2$...
violet: 14 it cannot be written as $x^2+0\cdot y^2$...
red: 15 it cannot be written as $x^2+0\cdot y^2$...
orange: $16=4^2+0\cdot y^2=(-4)^2+0\cdot y^2, y\in Z$
yellow: 17 it cannot be written as $x^2+0\cdot y^2$...
green: 18 it cannot be written as $x^2+0\cdot y^2$...
blue: 19 it cannot be written as $x^2+0\cdot y^2$...
purple: 20 it cannot be written as $x^2+0\cdot y^2$...
violet: 21 it cannot be written as $x^2+0\cdot y^2$...

Graphic conclusions: For $n\in\{1,2,...,21\}$, partially traversing hlines of equations $x^2+0\cdot y^2=n$ (*curves with infinite length*), from bottom to top direction, we found that the numbers

1,4,9,16

can be written as $x^2+0\cdot y^2, (x,y)\in Z\times Z$. In addition, the writing is not unique.

Answer 1.2. Theorem in Number Theory. The number $n\in Z$, $n\geq 1$ is written as $n=x^2+0\cdot y^2, (x,y)\in Z\times Z$ if and only if its prime factors are numbers that appear at even powers.

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