Le sujet dont vous êtes l'auteur

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Introduction:

You probably know what a prime number is. These numbers fascinated the mathematicians, especially Euclid. There are a lot of questions about them, which seem simple to solve at the beginning, but some of them haven't been solved by mathematicians for centuries. I suggest you ask yourself a subject, which you think is interesting, which you do not already know the answer to, that you had never encountered before and on which you can start by doing simulations with numbers. It's your turn.

The first problem

Description:

Let it be p, a natural prime number. There is a table of values (an array) with n lines and n columns with integers. We have to find out in how many ways we can fill this array, so that the product of the elements from every line and column is equal to $+p$ or $-p$.

The Beginning:

We started by doing some simulations. Let's give more examples, when n is equal to 2 or 3 and write the possibilities:

First example:

For n = 2:

±p	±1
±1	±p

±1	±p
±p	±1

There are only 2 possibilities to build the $2 \, x \, 2$ array. As you can see, the product of the elements from every line or column is $\pm p$. The total number of possibilities is: $2^4 \cdot 2 = 32$

Second example:

For n = 3:

$\pm p$	± 1	± 1
± 1	$\pm p$	± 1
± 1	± 1	$\pm p$

$\pm p$	± 1	± 1
± 1	± 1	$\pm p$
± 1	$\pm p$	± 1

± 1	$\pm p$	± 1
$\pm p$	± 1	± 1
± 1	± 1	$\pm p$

± 1	$\pm p$	± 1
± 1	± 1	$\pm p$
$\pm p$	± 1	± 1

± 1	± 1	$\pm p$
$\pm p$	± 1	± 1
± 1	$\pm p$	± 1

± 1	± 1	$\pm p$
± 1	$\pm p$	± 1
$\pm p$	± 1	± 1

We can build 6 tables of values where the 3 positions of ±p differ from each other. For each array we have 2^9 possibilities to fill it. In conclusion, for n = 3, we get a total of $2^9 \cdot 6$ ways to write the table of values.

Demonstration:

From the examples, we can realise that there are 2 very important things to think about:

- For a line to have the product of the elements equal to $+p$ (or $-p$), then one element from the line must be equal to $+p$ (or $-p$) and all of the other ones from the same line must be equal to $+1$ (or -1)
- For a column to have the product of the elements equal to $+p$ (or $-p$), then we have to be very careful how to arrange the elements equal to $+p$ (or $-p$) from the lines above.

Let's calculate in how many ways we can arrange the elements of $+p$ (or $-p$), which we will multiply with the number of possibilities where we arrange the elements of $+1$ (or -1).

For the first line, we have n places where we can put $+p$ (or $-p$), so $2 \cdot n$ possibilities.

For the second line, we have only $(n-1)$ places left to put $+p$ (or $-p$), so $2 \cdot (n-1)$ possibilities, because of the fact that we can't place $+p$ (or $-p$) on the same column as we placed it on the first line.

For the third line, we have only $(n-2)$ places left to put $+p$ (or $-p$), so $2 \cdot (n-2)$ possibilities, because of the fact that we can't place $+p$ (or $-p$) on the same column as we placed it on the first two lines.

We will continue, based on the same algorithm, until we will be on the n^{th} line. We have only 1 place left to put $+p$ (or $-p$), so $2 \cdot 1$ possibilities, because of the fact that we can't place $+p$ (or $-p$) on the same column as we placed it on the first $(n-1)$ lines.

Now, we have to make the product of every number of possibilities from all of the lines. Therefore, the number of ways we can place $+p$ (or $-p$) in the table of values is:
$(2 \cdot n) \cdot [2 \cdot (n-1)] \cdot [2 \cdot (n-2)] \cdot \ldots \cdot (2 \cdot 1) = 2^n \cdot n!$, $n \geq 1$, this is **relation number I**.

Now, we have to find a way to count the number of possibilities to place $+1$ (or -1) in the table of values. Let's see in how many ways we can place +1 (or -1) in a line with n integers.

± 1	± 1	± 1	………	±p	………	± 1	± 1
↓ 2 ways	↓ 2 ways	↓ 2 ways				↓ 2 ways	↓ 2 ways

We have $2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^{n-1}$ possibilities to fill a line and there are n lines in the table of values. In total, we have $2^{n-1} \cdot 2^{n-1} \cdot \ldots \cdot 2^{n-1} = 2^{n(n-1)}$ possibilities to place ± 1 in the array, $n \geq 1$, this is **relation number II.**

Final Answer:

From multiplying equations I and II, we will obtain the total number of ways we can fill this array, so that the product of the elements from every line and column is equal to $+p$ or $-p$:

$$\boldsymbol{2^n \cdot n! \cdot 2^{n(n-1)} = 2^{n^2} \cdot n!, n \geq 1} \quad (1)$$

(2)

The second problem

Description:

A natural number *N* is the product of n distinct prime numbers. In how many distinct ways can this number be represented as the difference of two nonzero perfect squares ?

The Beginning:

Let $N = p_1 \cdot p_2 \cdot \ldots \cdot p_n$, where $p_1, p_2, \ldots, p_n$ are distinct prime numbers

We find $x, y \epsilon \mathbb{N}^*$, so that $N = x^2 - y^2 \Leftrightarrow N = (x - y)(x + y)$

We started by doing some simulations. Let's give more examples, when n is equal to 2 (or 3) and we chose 2 (or 3) random prime numbers, then we write the possibilities:

First example:

For $n = 2$

- If we choose 3 and 5 as our prime numbers $\Rightarrow N = 15$
  - $x + y = 5$ and $x - y = 3 \Rightarrow 2x = 8 \Longrightarrow x = 4$ and $y = 1$
  - $x + y = 15$ and $x - y = 1 \Rightarrow 2x = 16 \Longrightarrow x = 8$ and $y = 7$
  - $x + y = 3$ and $x - y = 5 \Rightarrow 2x = 8 \Longrightarrow x = 4$ and $y = -1 \Rightarrow y \notin \mathbb{N}$
  - $x + y = 1$ and $x - y = 15 \Rightarrow 2x = 16 \Longrightarrow x = 8$ and $y = -7 \Rightarrow y \notin \mathbb{N}$

  In this case we have only 2 possibilities.
- If we chose 2 and 3 as our prime numbers $\Rightarrow N = 6$
  - $x + y = 3$ and $x - y = 2 \Rightarrow 2x = 5 \Rightarrow x \notin \mathbb{N}$
  - $x + y = 6$ and $x - y = 1 \Rightarrow 2x = 7 \Rightarrow x \notin \mathbb{N}$
  - $x + y = 2$ and $x - y = 3 \Rightarrow 2x = 5 \Rightarrow x \notin \mathbb{N}$
  - $x + y = 1$ and $x - y = 6 \Rightarrow 2x = 7 \Rightarrow x \notin \mathbb{N}$

  In this case we have zero possibilities

Second example:

For $n = 3$

- If we choose 3, 5 and 7 as our prime numbers $\Rightarrow N = 105$
  - $x + y = 15$ and $x - y = 7 \Rightarrow 2x = 22 \Longrightarrow x = 11$ and $y = 4$
  - $x + y = 21$ and $x - y = 5 \Rightarrow 2x = 26 \Longrightarrow x = 13$ and $y = 8$
  - $x + y = 35$ and $x - y = 3 \Rightarrow 2x = 38 \Longrightarrow x = 19$ and $y = 16$
  - $x + y = 105$ and $x - y = 1 \Rightarrow 2x = 8 \Longrightarrow x = 53$ and $y = 52$
  - $x + y = 7$ and $x - y = 15 \Rightarrow 2x = 22 \Longrightarrow x = 11$ and $y = -4 \Rightarrow y \notin \mathbb{N}$
  - $x + y = 5$ and $x - y = 21 \Rightarrow 2x = 26 \Longrightarrow x = 13$ and $y = -8 \Rightarrow y \notin \mathbb{N}$

- $x + y = 3$ and $x - y = 35 \Rightarrow 2x = 38 \Longrightarrow x = 19$ and $y = -16 \Rightarrow y \notin \mathbb{N}$
- $x + y = 1$ and $x - y = 105 \Rightarrow 2x = 106 \Longrightarrow x = 53$ and $y = -52 \Rightarrow y \notin \mathbb{N}$

As shown we have 4 possibilities.

Demonstration:

$N = x^2 - y^2 = p_1 \cdot p_2 \cdot p_3 \cdot .... \cdot p_n \Leftrightarrow (x - y) \cdot (x + y) = p_1 \cdot p_2 \cdot p_3 \cdot .... \cdot p_n$

If one of the prime numbers, $p_1, p_2, \dots, p_n$ is equal to 2, then $(x + y)$ is even and $(x - y)$ is odd or $(x + y)$ is odd and $(x - y)$ is even. From these cases, we obtain that $(x + y) + (x - y)$ is odd $\Rightarrow 2x$ is odd $\Rightarrow x \notin \mathbb{N}$.

Therefore, there are no solutions when one of the numbers $p_1, p_2, \dots, p_n$ is 2. This means that $p_1, p_2, \dots, p_n$ are distinct and odd prime numbers.

Therefore:

$p_1 \mid (x + y)$ or $p_1 \mid (x - y) \rightarrow 2$ possibilities
$p_2 \mid (x + y)$ or $p_2 \mid (x - y) \rightarrow 2$ possibilities
$p_3 \mid (x + y)$ or $p_3 \mid (x - y) \rightarrow 2$ possibilities
....
$p_n \mid (x + y)$ or $p_n \mid (x - y) \rightarrow 2$ possibilities

We have 2 possibilities for each prime number. Therefore, we have $2 \cdot 2 \cdot 2 \cdot ... \cdot 2 = 2^n$ possibilities in total to write N as a product of two positive numbers, $x + y$ and $x - y$.

If $x + y \leq x - y$, then $2 \cdot y \leq 0 \Rightarrow y \leq 0$, which is false $\Rightarrow x + y > x - y$

Let it be $N = a \cdot b,\ a > b$ and $a, b\ odd$.

If $x + y = a$ and $x - y = b \Rightarrow 2 \cdot x = a + b \Rightarrow x = \frac{a+b}{2} \in \mathbb{N}^* \Rightarrow y = \frac{a-b}{2} \in \mathbb{N}^*$

If $x + y = b$ and $x - y = a \Rightarrow 2 \cdot x = a + b \Rightarrow x = \frac{a+b}{2} \in \mathbb{N}^* \Rightarrow y = \frac{b-a}{2} < 0$, which is false

As we can see, the number of possibilities is cut in half $\Rightarrow$ we will obtain $2^n : 2 = 2^{n-1}$ possibilities. (3)

Final answer:

We have 2^{n-1} possibilities to write a natural number formed as a product of distinct prime numbers, as a difference of 2 squares.

C++ Program

The problem is very difficult to solve, when n is big. Of course, technology can help us a lot and that's why we made a C++ program.

You can choose how many different prime numbers you want to multiply and after that which are these.

Our program will show all the solutions $x, y \in \mathbb{N}^*$, where $x^2 - y^2 = p_1 \cdot p_2 \cdot p_3 \cdot .... \cdot p_n$ and the number of possibilities.

```cpp
#include <iostream>
#include <fstream>
#include <array>
#include <vector>
using namespace std;

int x, cnt;

void f(int target) {

    for (int diff = 1; diff * diff <= target; ++diff) {
        if (target % diff != 0)
            continue;

        int sum = target / diff;

        if (sum % 2 != diff % 2)
            continue;

        int b = (sum - diff) / 2;
        int a = sum - b;

        cnt++;
        std::printf("%d %d\n", a, b);

    }
}

void case_1(){
    int randVal, p = 1;
    cout << "Chose how many numbers you want to multiply: ";
    cin >> randVal;
    cout << '\n';
    cout << "The numbers are: ";
    while(randVal != 0){
        cin >> x;
        p *= x;
        randVal--;
    }
    cout << '\n';
    cout << "These are the results: " << "\n";
    f(p);
    cout << '\n';
    cout << "The number of posibilities: ";
    cout << cnt << '\n';
}

int main() {
    case_1();
    return 0;
}
```

If $n = 5$ and $p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$, the program shows the 16 solutions in less than a second. It would be too hard to find all the solutions without the help of the program.

```
Chose how many numbers you want to multiply:5

The numbers are:3 5 7 11 13

These are the results:
7508 7507
2504 2501
1504 1499
1076 1069
688 677
584 571
508 493
368 347
244 211
232 197
212 173
164 109
148 83
136 59
128 37
124 19

The number of posibilities: 16
```

EDITING NOTES

(1) This result could have been reached in a simpler way. Assume first that we insert only p or 1 in the cells of the table. Then we have $n!$ possibilities. At this point we can observe that, for each of the n^2 cells, we have two possibilities: $+$ or $-$. Then, the total number of possibilities is $n! \cdot 2^{n^2}$.

(2) There are variants of the problem that might be interesting for future works. For instance, one may look for the number of possibilities that always give $+p$ as the product of the elements of each row and each column. Or, more difficult, the number of possibilities that always give p^2 as product. Of course, many other variants can be considered.

(3) There are some points in the proofs above that need to be explained or fixed.

(a) Regarding the 2^n possibilities for $x + y$ and $x - y$, it should be said that, when $x + y$ is $p_1 \cdot p_2 \cdot \ldots \cdot p_n$, $x - y$ is 1 .

(b) Perhaps, the reader should be reminded that a and b are odd because such are $p_1, p_2, \ldots, p_n$.

(c) The line
If $x + y = b$ and $x - y = a \Rightarrow 2 \cdot x = a + b \Rightarrow x = \frac{a+b}{2} \in \mathbb{N}^* \Rightarrow y = \frac{b-a}{2} < 0, \ldots$
can be dropped. This possibility is excluded by the result $x + y > x - y$ and the assumption $a > b$.

(d) In the final sentence, it should be said explicitly that the number of possibilities is “cut in half” because the number of cases in which $x - y > x + y$ is the same as that in which $x + y > x - y$. It should also be said that the equality $x - y = x + y$ never holds because $p_1, p_2, \ldots, p_n$ are different prime numbers.