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LEAKY CHOICE

2018-2019

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1. Introduction

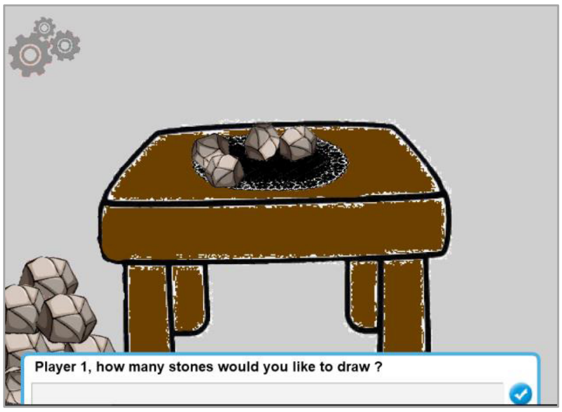
During the school year 2018/2019 we participated in the "Math&Languages" project. Our school was paired with high school in France. We solved mathematics problems together and presented the results on the MATh.en.JEANS Congress in Toulouse.

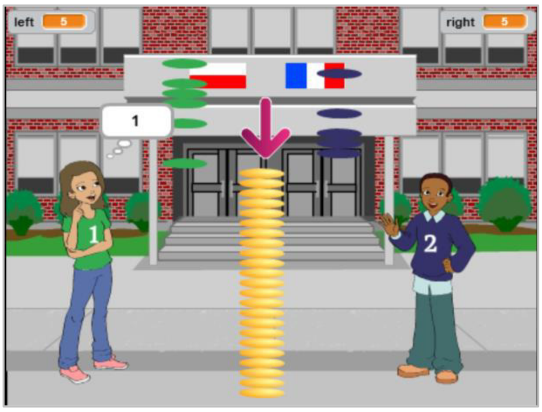
2. The rules

Our problem was to find the winning strategies for a game, called „The leaky choice". Two players play and each of them draws the defined number of stones (1). They cannot draw two stones, that is why the choice is called leaky.

3. Scratch

To help us resolve our task, we created two programs in Scratch. They were supposed to automatically draw a given number of stones. The program on the left was made by a member of the French team - Victor Lebriquer, and the other one by Julia Brodowska and Aleksandra Wrzosek.





4.

5. Notation

M - allowed set of moves

n – total number of stones

6. Strategy for $n = 7$, $M = \{1; 3\}$

We started working on our project by inventing a strategy when the number of stones was 7. Each player could draw either one or three stones. It turned out that if player 1 begins, no matter what he chooses, he always wins.

Pair of turn 1		Pair of turn 2		Pair of turn 3		Pair of turn 4		
Player 1	Player 2	Player 1	Player 2	Player 1	Player 2	Player 1	Player 2	
3	3	1	-	-	-	-	-	P1 wins
3	1	3	-	-	-	-	-	P1 wins
1	3	3	-	-	-	-	-	P1 wins
3	1	1	1	1	-	-	-	P1 wins
1	3	1	1	1	-	-	-	P1 wins
1	1	3	1	1	-	-	-	P1 wins
1	1	1	3	1	-	-	-	P1 wins
1	1	1	1	3	-	-	-	P1 wins
1	1	1	1	1	1	1	-	P1 wins

The proof for that is simple:

The number of all stones taken by both of players in one round is an even number (because sum of two odd numbers is always an even number). Hence, the sum of all stones taken in all not completed rounds is an odd number. Total number of stones (7) is also odd, so the game always ends after a not completed round. That means that the first player wins.

7. Strategy for M={1;3} and $n = 8$

Pair of turn 1		Pair of turn 2		Pair of turn 3		Pair of turn 4		
Player1	Player2	Player1	Player2	Player1	Player2	Player1	Player2	
3	3	1	1	-	-	-	-	P2 wins
1	3	1	3	-	-	-	-	P2 wins
1	3	3	1	-	-	-	-	P2 wins

Conclusion for $M = \{1; k\}$, k odd number

Our discoveries led us to create **2 universal theorems**:

THEOREM 1:

If the number of stones is even and allowed moves are 1 and k (k is odd), player 2 always wins.

Proof:

The sum of two odd numbers is always even, so the number of stones taken in one round is even. The game cannot be ended after an incomplete round, because the sum of some even numbers and one odd number is always odd. So player 2 is the one to finish the game.

THEOREM 2:

If the number of stones is odd and allowed moves are 1 and k (k is odd), player 1 always wins.

Proof:

It's similar to the proof presented in the first theorem - the game cannot be ended after a full round because the sum of even numbers is always even.

8. Strategies for $M = \{1; 4\}$

We found the winning strategies for $n = 5;\ 6;\ 7;\ 8;\ 9;\ 10;\ 11;\ 12$

$n = 5$ strategy for player 2

Player	P1	P2
Stones taken	1	4
	4	1

$n = 6$ strategy for player 1

Player	P1	P2	P1
Stones taken	1	4	1
	4	1	1

$n = 7$ strategy for player 2

Player	P1	P2	P1	P2
Stones taken	1	4	1	1
	4	1	1	1

$n = 8$ strategy for player 1

Player	P1	P2	P1	P2	P1
Stones taken	1	4	1	1	1
		1	4	1	1

$n = 9$ strategy for player 1

Player	P1	P2	P1
Stones taken	4	1	4
		4	1

$n = 10$ strategy for player 2

Player	P1	P2	P1	P2
Stones taken	1	4	4	1
			1	4
	4	4	1	1

$n = 11$ strategy for player 1

Player	P1	P2	P1	P2	P1
Stones taken	4	1	4	1	1
		4	1	1	1

$n = 12$ strategy for player 2

Player	P1	P2	P1	P2	P1	P2
Stones taken	1	4	1	4	1	1
			4	1	1	1
	4	1	1	4	1	1
			4	1	1	1

9. Conclusion for $M = \{1, 4\}$

After having created strategies from previous point, we found out that players do not have to memorise the sequence of moves they have to make. Strategies when $M = \{1; 4\}$ depend on divisibility of n by 5.

- **When n is divisible by 5**

When the number of stones is divisible by 5 then the second player has a winning strategy. It works because the number of stones taken in one round has to be 5; so if the first player takes one stone, then the second player takes 4. However, when the first player takes four stones, the second player takes 1. We called this strategy the "complement to 5" rule.

- **Rest of dividing n by 5 is 1**

First player has a winning strategy. They have to start by taking one stone (2). Then they use the "complement to 5" rule after each turn of the second player.

- **Rest of dividing n by 5 is 2**

Regarding the case when the rest of dividing the number of stones by 5 is 2, the second player has a winning strategy. The person repeats the "complement to 5" rule after each move of the first player until only two stones are left on the table and then the second player wins.

- **Rest of dividing n by 5 is 3**

When the rest of dividing is 3 the first player has a winning strategy. In order to win, the first player has to start by taking one stone. Then the person repeats the "complement to 5" rule after each move of the second player until only two stones are left and then he/she wins.

- **Rest of dividing n by 5 is 4**

The first player has also a winning strategy when the rest of dividing the number of stones by 5 is 4. The person starts by taking four stones and then he/she repeats the "complement to 5" rule after each move of the second player up to the end of the game when the person wins.

10. Non-leaky games

When we managed to solve this problem, we could think about similar games for slightly different n and *non-leaky* choice

Winning strategy for $M = \{1, 2, 3\}$

We use the « complement to 4 » rule, because

for 1: $1 + 3 = 4$

for 2: $2 + 2 = 4$

for 3: $3 + 1 = 4$

If n is divisible by 4 then player 2 has a winning strategy: he uses the « complement to 4 » rule.

If the rest of dividing n by 4 is $r \neq 0$ then player 1 has a winning strategy: he starts by drawing r stones and then he uses the « complement to 4 » rule.

Winning strategy for $M = \{1, 2, \dots, s\}$

Same solution, but instead of using 3 stones, let's use s.

Therefore we use the "complement to $s + 1$" rule.

11. Conclusion

We have found winning strategies for $M = \{1; k\}$, where k is odd. We also created, based on divisibility by 5, the winning strategies for $M = \{1; 4\}$ and we think that it is possible to find analogous strategies for $M = \{1, k\}$, where k is even number ($k \neq 2$).

However, we do not know if the same method works for 3 or more allowed moves, for example $M = \{1,4,7\}$ (3).

Open problems:

- Is it true that for any n there exists a winning strategy for one of the players? (4)
- If the answer is NO: find all the numbers n for which a strategy exists.

Edition Notes

(1) More precisely, at the beginning of this two-players game, we have a certain number of stones, and each player alternately removes some stones, but the number of removed stones must be chosen in some given "leaky" set M. It will be assumed that M contains 1, so the game can continue until there are no more stones. Then the winner is the player who takes the last stone.

(2) The first player *can* take 1 stone but he does not *have to*. As shown before in the cases n=6 and n=11, he could also take 4 stones. Then the number of stones left to the second player divided by 5 yields a rest 2, and the first player can apply the "complement to 5" strategy until only two stones are left.

(3) The case $M = \{1; k\}$ with k even is not very different from the case $M = \{1; 4\}$, using a "complement to $k + 1$ rule" for $n \geq k + 1$ instead of a "complement to 5 rule". The case of a leaky set M of 3 or more integers looks more difficult.

(4) It is not difficult to show that one of the players has a winning strategy if we assume, as in note 1, that the rule is such that game always ends and one of the players wins. Indeed, suppose that player 1 has no winning strategy; then, whatever he will play either player 2 will be able to win directly, or he will have at least one choice after which again player 1 cannot be sure of winning; continuing like this, at the end of the game player 2 wins, and so he has a winning strategy.