Match the Maths

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Abstract

Our research deals with the mathematical construction of a soccer ball, that is made of regular pentagons and hexagons. Given the diameter of the ball, we shall approximate the length of the stitch and the area of the ball. Conversely, if the length of the pentagons is given, we shall approximate the size of the ball.

Figure 1. The official FIFA World Cup ball used in Mexico, 1970

The problem

a) Determine the number of pentagons and the number of hexagons on the soccer ball.
b) Find the approximate total length of the stitch?
c) Calculate the sum of the areas of all patches that build it. Is it equal to the area of a sphere of diameter 25 cm? Why?
d) Suppose that a soccer ball and a sphere have the same surface area. Decide which of these two will enclose the highest volume. Why?
e) If the pentagon side on a soccer ball is 4.5 cm, determine the diameter of the ball (by approximation).

For the first point, we saw that we have 12 Pentagons and 20 Hexagons, working with the principle of the symmetry. Secondly, we approximate the soccer ball by a regular truncated icosahedron and after some mathematical calculations, we obtain that the length of the stitch is about 476.1 cm.
In the solution of problem, we have approximated the soccer ball by a truncated regular icosahedron. So, we had to calculate and compare the two areas of the solids and we have founded that $A_{sphere} < A_{ball}$, because the surface of the ball is covered with patches, each patch having edges and so the surface area is not optimized as in the case of a sphere. Doing the same thing with the volume, we have founded that $V_{ball} < V_{sphere}$, where we have approximate the volume of the ball by the volume of a truncated polyhedron.
Finally, we just had to do opposite way because it was given the length of the stitch and we needed to get the diameter.

Solution of the problem

a) After several unsuccessful attempts, we decided to count the edges by the number of vertices as shown down:

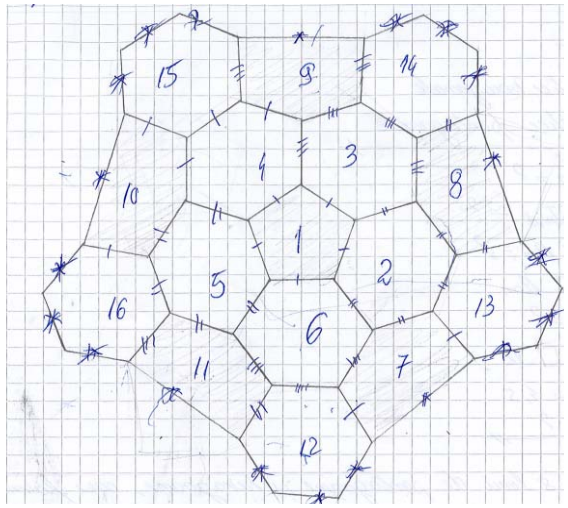

Figure 2. The ball's development in plan

We denote by
P = number of pentagons, H = number of hexagons and E = number of edges.

We observe that (1)

$$2E = 3V \Rightarrow V = \frac{2}{3} E \ (1)$$

When counting the edges by the number of faces, we have

$$2E = 5P + 6H \ \ (2)$$

By employing the relation (1), we obtain: (2)

$$\frac{2}{3} \times E - E + H = 2 \Leftrightarrow 3H - E = 6 \Leftrightarrow 6H - 2E = 12$$

And thus,

$$6(P+H) - (5P + 6H) = 12 \Leftrightarrow 6P + 6H - 5P - 6H = 12,$$

from where

$$\boxed{P = 12 \ \textit{(total number of pentagons)}}$$

A soccer ball is a truncated icosahedron, having 60 vertices. As V = 60, we find from (2) that E = 90 and, from (3),

$$\boxed{H = 20.}$$

Otherwise, as a soccer ball has 32 patches, we have:

$$\boxed{H = 32 - P = 32 - 12 = 20}$$

(b) We shall approximate the soccer ball by a regular truncated icosahedron, as in the figure. Each of the 12 vertices of the regular icosahedron is removed by cutting it with a plane, such that the section (the intersection of the plane with the icosahedron) is a pentagon. We repeat this procedure for all 12 vertices of the icosahedron, such that all the pentagons that we obtain are congruent. The red solids that are cut off from the icosahedron are all pentagonal pyramids.

Each face of the icosahedron is an equilateral triangle. After the cut, the equilateral triangle turns into a regular hexagon. If we denote by l the sidelength of the triangle (which is the side of the icosahedron), then the side of the hexagon is $\frac{l}{3}$ (see Figure (5a) below).

As a consequence, the pentagonal pyramid is regular (see figure (5b) below). Therefore, the volume of the soccer ball can be approximated by the volume of the truncated icosahedron, which is the volume of the icosahedron minus 12 times the volume of a regular pentagonal pyramid (the red cut from the icosahedron).

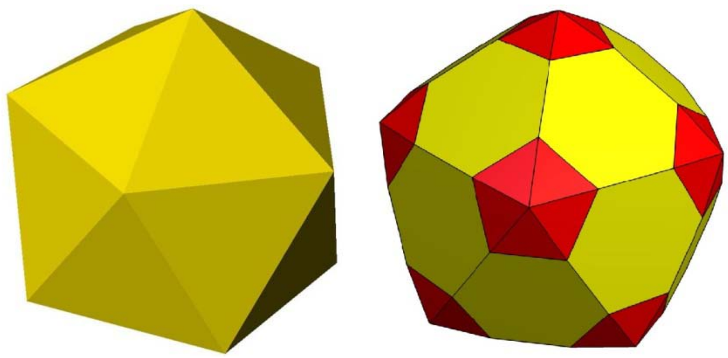

Figure 3. *Cutting* of the vertices of a regular icosahedron

In order to approximate the total length of the stitch, we shall express the volume of the soccer ball in two ways (see Figure 4), as follows:

- the volume of a sphere with diameter $d(V_{sphere})$ and
- the volume of a truncated icosahedron ($V_{tr} = V_{ico} - 12V_{pyramid}$).

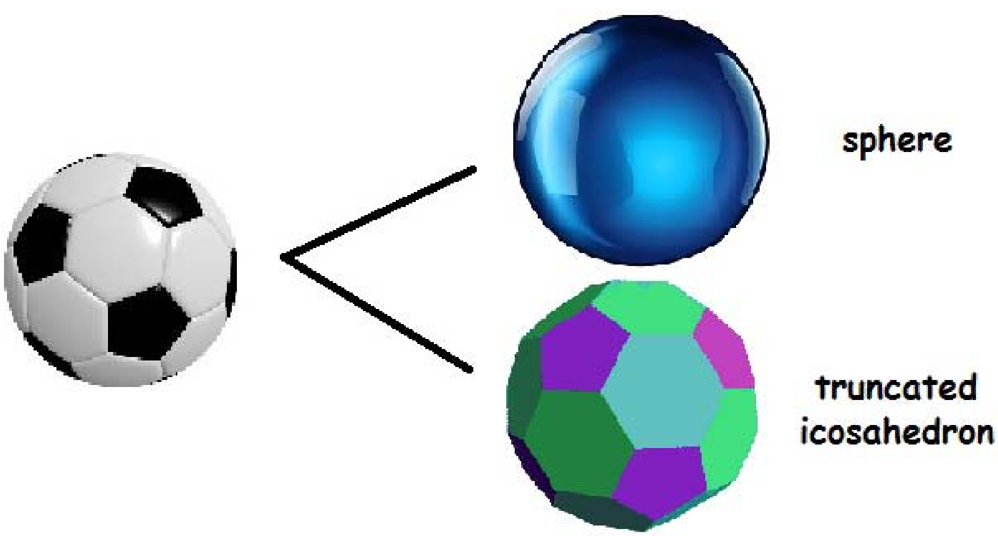

Figure 4. Approximations of a real soccer ball

Firstly, the volume of the sphere of diameter d is:

$$V_{sphere} = \frac{4\pi R^3}{3} = \frac{\pi d^3}{6}.$$

The volume of a regular icosahedron of edge length l is:

$$V_{ico} = \frac{5}{12}(3+\sqrt{5})l^3.$$

The volume of the regular pentagonal pyramid of edge length a is

$$V_{pyramid} = \frac{5+\sqrt{5}}{24}a^3.$$

As here $a = \frac{l}{3}$, we get that

$$V_{pyramid} = \frac{5+\sqrt{5}}{648}l^3.$$

The volume of the truncated icosahedron is

$$\begin{aligned} V_{tr} &= V_{ico} - 12\,V_{pyramid} \\ &= \frac{5}{12}(3+\sqrt{5})l^3 - \frac{5+\sqrt{5}}{54}l^3 \\ &= \frac{125+43\sqrt{5}}{108}l^3. \end{aligned}$$

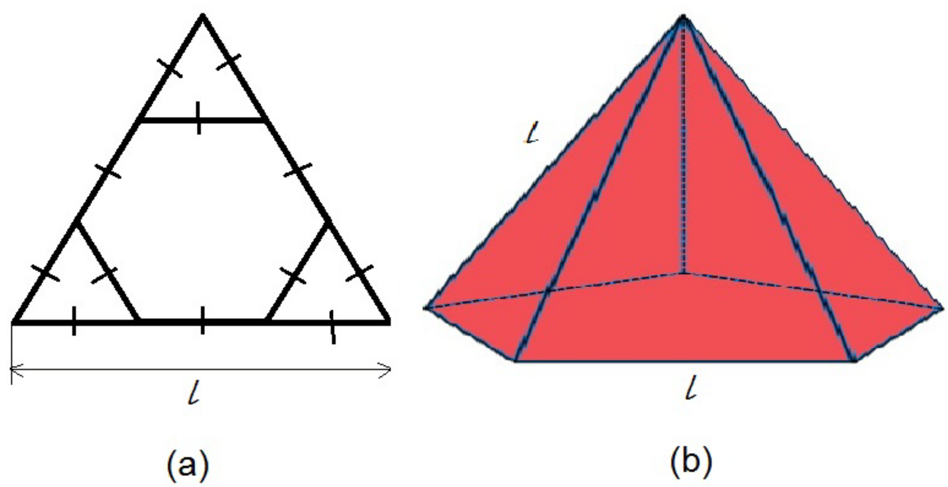

Figure 5. The echilateral triangle and the pentagon pyramid

By equating V_{sphere} and V_{tr} , we obtain:

$$\frac{125+43\sqrt{5}}{108}l^3 = \frac{\pi d^3}{6},$$

hence the side of a triangle is (3)

$$l = d\left(\frac{108\pi}{125+43\sqrt{5}}\right)^{\frac{1}{3}} \cong 15.87cm.$$

The sidelength a of the truncated icosahedron is the length of a side of a pentagon, which is

$$a = \frac{l}{3} \cong 5.29\ cm.$$

As the soccer ball has 90 edges, the total length of the stitch is

$$L = 90\ a\ \cong 476.1\ cm.$$

(c) As calculated previously, the approximate sidelength a of a pentagon (or hexagon) on the soccer ball for which the volume of the soccer ball (which we approximated with a truncated icosahedron) is equal to the volume of a sphere of diameter $25\ cm$ is $a \cong 5.29\ cm$. We are now comparing the areas of these two solids, the sphere and the soccer ball.

Area of a sphere of diameter d is

$$A_{sphere} = \pi d^2 \cong 1963.5\ cm^2$$

On the other side, the area of the soccer ball is

$$A_{ball} = 12 \times \text{Area of a pentagon} + 20 \times \text{Area of a hexagon.}$$

The area of a pentagon of side a is

$$A_p = \frac{a^2}{4}\sqrt{25 + 10\sqrt{5}}.$$

The area of a hexagon of side a is

$$A_h = \frac{3a^2}{2}\sqrt{3}.$$

Figure 6.A flat soccer ball

Therefore, the area of a soccer ball is

$$A_{ball} = 12A_p + 20A_h = 3a^2(\sqrt{25 + 10\sqrt{5}} + 10\sqrt{3}) \cong 2031.3\ cm^2.$$

As we can see, $A_{ball} > A_{sphere}$, because the surface of the soccer ball is covered with patches, each patch having edges, and so the surface area is not optimized as in the case of a sphere. Also, a ball which has a smaller area is significatively faster than a bigger one because of the contact area.

With other words, our calculation is consistent with the following result:

Out of all surfaces that enclose a given volume, the sphere has the smallest surface area.

Figure 7. Kicking a soccer ball

(d) We shall compare now the volumes.We know that fact that the areas of the two solids are equal, so

$$A_{ball} = 3a^2(\sqrt{25+10\sqrt{5}}+ 10\sqrt{3}) = \pi d^2 = A_{sphere}.$$

From this, we find that

$$\frac{a}{d} = 1\Big/\sqrt{\frac{3}{\pi}\left(\sqrt{25+10\sqrt{5}} + 10\sqrt{3}\right)} \cong 0.2046.$$

On the other hand,

$$V_{sphere} = \frac{4\pi d^2}{3} = \frac{\pi d^3}{6}$$

$$V_{ball} = \frac{125+43\sqrt{5}}{4}a^3$$

From the last two relations and using the approximate value of $a/_d$, we obtain

$$\frac{V_{ball}}{V_{sphere}} = \frac{3}{2\pi}\left(125+43\sqrt{5}\right)\left(\frac{a}{d}\right)^3 \cong 0.8297.$$

We observe that this fraction is smaller than 1, and so $V_{ball} < V_{sphere}$.

As we can see, $V_{ball} < V_{sphere}$, due to the fact that the surface of the soccer ball is covered with patches, each patch having edges, and the space in which the air is placed is smaller than in a perfect sphere.

With other words, our calculation is consistent with the following result:

Out of all solids that have the same surface area, the sphere has the largest volume.

(e) We are now given the sidelength $a = {}^{l}/_{3} = 4.5\ cm$ of a pentagon and we need to find the diameter of the ball. We equate the volumes of the sphere and of the truncated icosahedron:

$$\frac{(125+43\sqrt{5})}{108}l^3 = \frac{\pi d^3}{6} \Rightarrow d^3 = \frac{l^3}{108\pi} \times \left(125 + 43\sqrt{5}\right)$$

$\Rightarrow d^3 \cong 9816.6$

and so, $d \cong 21.41cm$.

Conclusion

In the beginning, we thought that it was easy to build a soccer ballusing some mathematical and logical forms. Once we have started to solve the problem and after we have searched for some information about it, we realised that it was much harder. It required a lot of work and exact calculations, a lot of time spend on research and, obviously, quite a few failures. After finishing the solution, we have seen that the problem was just a normal geometry problem, working with a solid having 20 faces (icosahedron), cutting all its vertices and, finally, obtaining the desired soccer ball.

References

- **Bogdan Enescu –** *How to make a soccer ball*, online paper
- **Wikipedia –** *Icosahedron, Platonic solids, isoperimetric property*

Notes d'édition

(1) V = number of vertices

(2) From Euler's polyhedron formula we know that $V = E - F + 2$ for any polyhedron, whith F the number of faces, which is equal to $(P + H)$ in this case.

(3) We recall that $d = 25\ cm$.