PERMUTATION OF DIGITS

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Students: Boris Ciocoiu, Matei Pleșcan, Jasmin Ojog, Miruna Aungurencei, Andrei Vleoangă
School: National College Iasi, Romania
Teacher: Gabriel Popa
Researchers: Claudiu Volf, Universitatea Alexandru Ioan Cuza Iași

PROBLEM STATEMENT

In Olympus two twins were born: Bin and Ter. As children of gods, they have special powers: they already know all natural numbers (an infinity). Bin prefers base 2 and Ter base 3. They play by applying circular permutations on numbers (written in their preferred bases). For instance:

$$\mathrm{Bin}(26) = \mathrm{Bin}(10110_2) = 01011_2 = 1011_2 = 11$$
$$\mathrm{Ter}(25) = \mathrm{Ter}(221_3) = 122_3 = 17$$

a) The children enjoy more when the numbers drop. They would like to play together and transform as many numbers as possible into 1, applying each his permutation (if needed). Example:

$$7 = 21_3 \xrightarrow{T} 12_3 = 5 = 101_2 \xrightarrow{B} 110_2 \xrightarrow{B} 11_2 = 10_3 \xrightarrow{T} 1$$

Which are the numbers that the children can transform into 1 ?

b) Are there any numbers that can be made as large as desired by applying the permutations above ?
c) Generalizations.

THE MAIN IDEA

We will show that *any natural number N* can be transformed into 1 by applying circular permutations and convenient representations using bases 2 and 3.
The basic idea relies on continuously reducing the number of symbols used for representing number *N* based on a set of *key observations*.

OUR SOLUTION: KEY OBSERVATIONS

O1: Numbers 1 = 1_2, 2 = 10_2 and 3 = 10_3 can be obviously transformed into 1 by using the allowed operations.

O2: All 0 digits that may appear in the corresponding base-2 or base-3 representation of *N* can be deleted.

Proof:

After repeatedly applying circular permutations until such a 0 symbol arrives on the left-most position it can be deleted since it becomes irrelevant.

O3: If *N* > 3 admits an *all-1* base-2 representation it *cannot* admit an *all-1* base-3 representation and vice versa.

Proof:(1)

An all-1 base-2 representation of *N* > 3 has the form $2^n - 1$, while an all-1 base-3 representation has the form $\frac{3^m - 1}{2}$, with *m*, *n* > 1 natural numbers. Let us suppose that there exist *m*, *n* > 1 such that(2):

$$2^n - 1 = \frac{3^m - 1}{2} \Rightarrow 2^{n+1} = 3^m + 1$$

For even *m* we have:

$$m = 2k:\ 2^{n+1} = 3^m + 1 \qquad 2^{n+1} = 3^{2k} + 1 \qquad 2^{n+1} = 9^k + 1$$

$$2^{n+1} = (8+1)^k + 1 \qquad 2^{n+1} = M_8 + 2 \qquad 2^n = M_4 + 1 \qquad n = 0, m = 0$$

For odd *m* we have:

$$m = 2k+1:\ 2^{n+1} = 3^m + 1 \Rightarrow 2^{n+1} = 3^{2k+1} + 1 \Rightarrow 2^{n+1} = 3 \cdot 9^k + 1 \Rightarrow$$

$$\Rightarrow 2^{n+1} = 3 \cdot (8+1)^k + 1 \Rightarrow 2^{n+1} = 3 \cdot M_8 + 4 \Rightarrow 2^{n-1} = 3 \cdot M_2 + 1 \Rightarrow n = 1,\ m = 1$$

Since in both cases the condition *m*, *n* > 1 is not met we conclude that observation O3 is true.

O4: A base-3 representation of *N* that has symbol 2 on its right-most position *cannot* admit an all-1 base-2 representation.

Proof:

A base-3 representation of *N* with symbol 2 on its right-most position has the form M_3+2, while an all-1 base-2 representation of *N* has the form 2^n-1.
According to the parity of *n* we may have:

$$n=2k:\ 2^n-1=2^{2k}-1=4^k-1=(3+1)^k-1=M_3\neq M_3+2$$

$$n=2k+1:\ 2^n-1=2^{2k+1}-1=2\cdot 4^k-1=2\cdot(3+1)^k-1=M_3+1\neq M_3+2$$

OUR SOLUTION: THE ALGORITHM

Step 1: we start by generating the base-2 representation of *N* and successively eliminate all the 0 symbols that may appear, by applying circular permutation (according to O2).

Step 2: we generate the base-3 representation of the number from the previous step. According to O3, this is not an all-1 code, thus we may eliminate all 0 symbols, if any.

Step 3: if the number from *Step 2* is an all-1 code, then we generate the corresponding base-2 representation, that is not an all-1 code (according to O3). We further eliminate all 0 symbols by circular permutations, as in *Step 1*.
If the number from *Step 2* contains some symbols 2, we apply circular permutations until we get a 2 in the right-most position. Then, we generate the base-2 representation of this number, that is not an all-1 code according to O4, and eliminate the 0 symbols, as in *Step 1*.

Step 4, 5, …: we repeatedly apply *Steps 1-3* before, continuously decreasing the number of symbols used in base-2/3 representations, until we get 1.

OUR SOLUTION: EXAMPLES

We present below several examples showing the operating mode of our algorithm. The solution has also been implemented and tested in MATLAB.

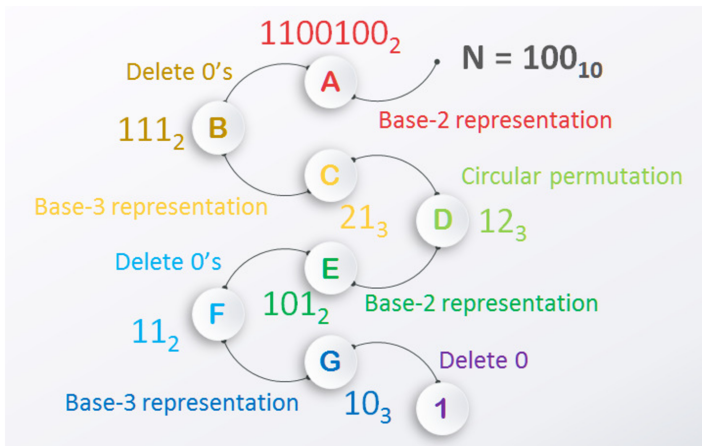
$$100_{10}=1100100_2\xrightarrow{delete\ 0}111_2\xrightarrow{base-3}21_3\xrightarrow{Circ.\,perm.}$$

$$\xrightarrow{Circ.\,perm.}12_3\xrightarrow{base-2}101_2\xrightarrow{delete\ 0}11_2\xrightarrow{base-3}10_3\rightarrow 1$$

$$1001_{10} = 1111101001_2 \xrightarrow{delete\ 0} 1111111_2 \xrightarrow{base-3} 11201_3 \xrightarrow{delete\ 0}$$

$$\xrightarrow{delete\ 0} 1112_3 \xrightarrow{base-2} 101001_2 \xrightarrow{delete\ 0} 111_2 \xrightarrow{base-3} 21_3 \xrightarrow{Circ.\ perm.}$$

$$\xrightarrow{Circ.\ perm.} 12_3 \xrightarrow{base-2} 101_2 \xrightarrow{delete\ 0} 11_2 \xrightarrow{base-3} 10_3 \to 1$$



PermutationDigits.m

```
function [out] = PermutationDigits(n);

out = [num2str(n)];
out = strcat(['N: ',out]);
N = dec2base(n,2);        % base-2 representation
strText = strcat(['Base-2 code: ',N]);
out = strvcat(out, strText);

N(N=='0') = [];           % delete any 0's
strText = strcat(['Delete 0''s: ',N]);
out = strvcat(out, strText);

while numel(N) > 1

    N = base2dec(N,2);    % base-3 representation
    N = dec2base(N,3);
    strText = strcat(['Base-3 code: ',N]);
    out = strvcat(out, strText);

    N(N=='0') = [];
    strText = strcat(['Delete 0''s: ',N]);
    out = strvc
```

Command Window

```
>> [out] = PermutationDigits(1001)

out =

N: 1001
Base-2 code: 1111101001
Delete 0's: 1111111
Base-3 code: 11201
Delete 0's: 1121
Circular permutation: 1112
Base-2 code: 101001
Delete 0's: 111
Base-3 code: 21
Delete 0's: 21
Circular permutation: 12
Base-2 code: 101
Delete 0's: 11
Base-3 code: 10
Delete 0's: 1
Base-2 code: 1
Delete 0's: 1
```

CAN WE GET ARBITRARILY LARGE NUMBERS?

We don't have a definite answer to this question, but several aspects are worth thinking over:

Under what conditions the numbers stop growing further?

Denote by $\mathrm{Max}_2(n)$ the largest number that can be obtained by circular permutation of a *n* digits code using base-2 representation and by $\mathrm{Max}_3(m)$ the largest number that can be obtained by circular permutation of a *m* digits code in a base-3 representation. Consider a natural number *N* represented in base 2 by a *n* digits code. Suppose $\mathrm{Max}_2(n)$ admits a *m* digits code in base 3, if $\mathrm{Max}_2(n) = \mathrm{Max}_3(m)$, then the growing process stops.

Should the growing process be strictly monotonic?

Should we always look for the greatest number that can be obtained by circular permutation at every iteration of the algorithm (given the base-2 or base-3 representation)?

GENERALIZATION

Generalization of O3: Consider representations of a natural number *N* in base-p and base-q, with $p,q \geq 2$ natural numbers. An all-1 base-p representation should not admit an all-1 base-q representation and vice versa, hence:

$$\frac{p^n-1}{p-1} \neq \frac{q^m-1}{q-1}$$

After processing the relation above we get:

$$\frac{p^n-1}{p-1} \neq \frac{q^m-1}{q-1} \Rightarrow \frac{q-1}{p-1}p^n - q^m \neq \frac{q-p}{p-1}$$

Choosing *p* and *q* such that $\frac{q-1}{p-1} = p^a$, $a \in ¥^*$, we get: $p^{n+a} - q^m \neq p^a - 1$. There exist results in the literature providing integer values that cannot be written in the form $p^n \pm q^m$, where *p* and *q* are prime numbers and *m*, *n* are natural numbers.

An interesting question arises: is it possible that two geometric sequences with distinct ratios (and both starting with 1) have the same sum of (a different number of) terms?

Generalization of O4: a base-q representation of *N* that has (*q*-1) on its right-most position should not admit an all-1 base-p representation.

CONCLUSIONS

"*Mathematics is the queen of the sciences and number theory is the queen of mathematics.*" is one of the most famous quotes of Gauss. The proposed problem clearly illustrated this, raises interesting challenges and generalizations worth further investigation. Software implementations of the algorithm enable simple verification and testing of our solution.

Notes d'édition

(1) Dans toute la suite, le notation M_n désigne un multiple de l'entier n

(2) Attention à l'utilisation du symbole « implication »   ; ainsi ici par exemple, page 2 on trouve : « Let us suppose that there exist m, n > 1 such that: $2^n \quad 1 = \frac{3^m \quad 1}{2} \qquad 2^{n+1} = 3^m + 1$ », il faut plutôt écrire : « Let us suppose that there exist *m, n* > 1 such that: $2^n \quad 1 = \frac{3^m \quad 1}{2}$; then there exist *m, n*> 1 such that: $2^{n+1} = 3^m + 1$