Piece of cake

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1. PRESENTATION OF THE RESEARCH TOPIC

Problems of cutting a 2D shape (usually a cake) in multiple pieces of equal areas often occur in daily life. Although the approximation by eye is faster, it is not always accurate. In this paper we are presenting multiple methods of cutting various 2D shapes in 2 or even more pieces of equal area. We also present a method for cutting some shapes with holes in them, with the cake as well as the hole in it having some particular shapes.

2. BRIEF PRESENTATION OF THE CONJECTURES AND RESULTS OBTAINED[1]

A bakery produces cakes of various shapes. The height of the cakes can be ignored and thus the cakes can be considered plane geometric shapes. The shapes include circles, regular *n*-gons, various triangles and quadrilaterals, but the bakery also takes custom orders for other shapes. Study how the bakery can use as few straight-line cuts as possible to cut the cakes into two pieces of equal area. What if the bakery now needs to cut the cakes into 3 or more pieces of equal area?

3. THE SOLUTION

First, we will solve the problem of cutting the piece of cake into two equal parts by one single cut. We will separately discuss, one by one, each of the cases when the piece of cake has the shape of a triangle, a convex quadrilateral, a convex pentagon, and finally present a solution for a convex polygon with *n* sides.

Case I. The piece of cake has the shape of a triangle.

Theorem. A median of a triangle (cake, of course) splits the triangle into two other triangles of equal area.

Proof. Let *ABC* be a triangle and *AM* be the median corresponding to *A*. Let *D* be the projection of *A* on *BC* (fig. 1).

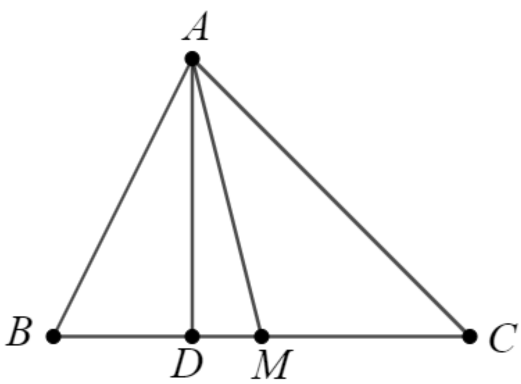

Figure 1

We have:

$$A_{ABM} = \frac{AD \cdot BM}{2} = \frac{AD \cdot MC}{2} = A_{AMC} . \blacksquare$$

So, a piece of cake with the shape of a triangle can be divided, with only one cut, into two equal parts if we cut along one of its medians. We have three possibilities of this kind.

However, the median is not the only line that satisfies our needs, as we will see in the following. Let *ABC* be a triangle and *AD* its altitude, with *D* on *BC*. We draw the right isosceles triangle *ADF* with hypotenuse *AD.* We have: $AF = DF = \frac{AD}{\sqrt{2}}$.

Let *G* be the intersection of the circle with center in *A* and radius *AF* and the altitude *AD.* Consider points *I* and *J* the intersections of the parallel to *BC* through *G* with sides *AB*, respectively *AC* (fig. 2).

Because triangles *ABC* and *AIJ* are similar, it follows that[2]

$$\frac{A_{AIJ}}{A_{ABC}} = \left(\frac{AG}{AD}\right)^2 = \left(\frac{AF}{AD}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}.$$

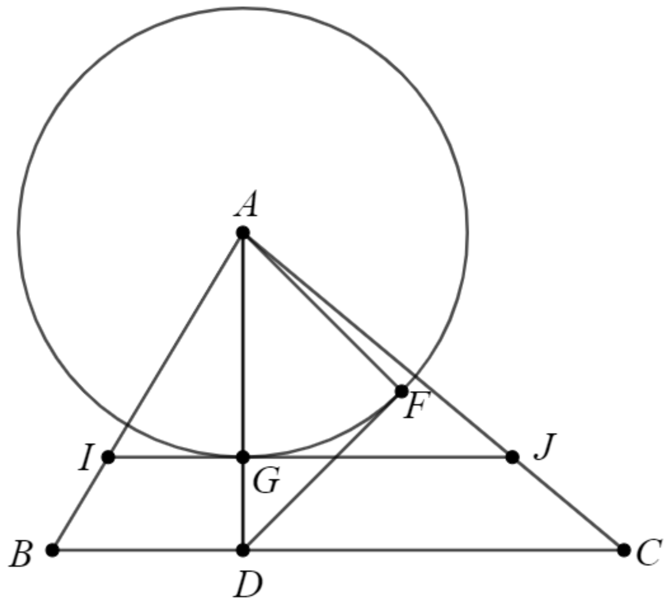

Figure 2

So, line *IJ* splits the triangle (the cake) *ABC* into two pieces of equal areas.

Case II. The piece of cake has the shape of a convex quadrilateral.

Let *ABCD* be a convex quadrilateral. We are going to convert this case to the previous one. In order to achieve this, we will draw the parallel *DE* to *AC*, with point *E* on line *AB* (fig. 3).

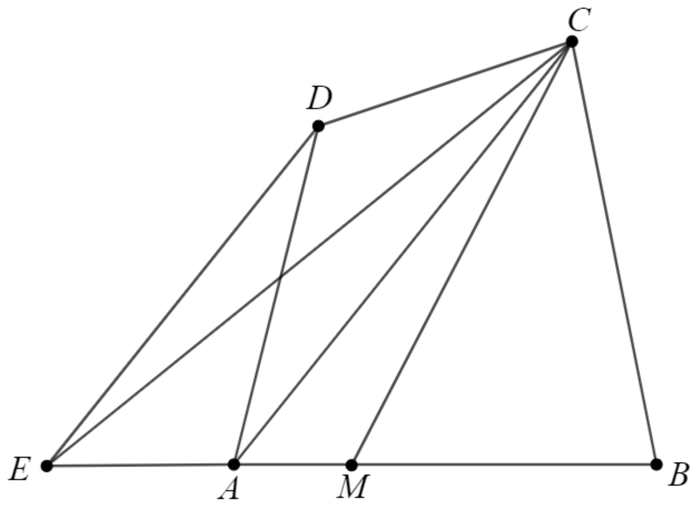

Figure 3

Because triangles *DAC* and *EAC* have the common side *AC* and the altitudes corresponding to it of equal length, we can deduce that their areas are equal. Therefore,

$$A_{ABCD} = A_{ABC} + A_{ADC} = A_{ABC} + A_{EAC} = A_{BCE}.$$

If median *CM* has its endpoint *M* lying on the side *AB*, then the problem is solved, because

$$A_{BMC} = \frac{A_{BCE}}{2} = \frac{A_{ABCD}}{2}.$$

If *M* does not belong to segment *AB* (fig. 4), then *CM* intersects side *AD* in the interior point *T* and

$$A_{ABCT} < A_{BCM} = \frac{A_{BCE}}{2} = \frac{A_{ABCD}}{2}.$$

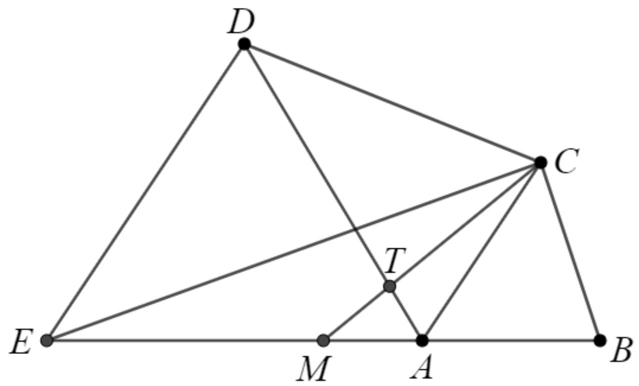

Figure 4

In this situation, line *CT* (*CM*) does not solve the problem.

Happily, we can dodge this obstacle in the following way: whatever convex quadrilateral *ABCD* is, one of the triangles separated by diagonal *AC*, *ABC* or *ADC*, has an area greater than or equal to the area of the other one.

If $A_{ABC} = A_{ADC}$, then the diagonal *AC* is the solution to our problem.

If $A_{ABC} > A_{ADC}$, then $A_{ABC} > \frac{A_{ABCD}}{2}$ and point *M* of the previous construction is surely on side *AB* (fig. 3).

If $A_{ADC} > A_{ABC}$, then $A_{ADC} > \frac{A_{ABCD}}{2}$ and we will solve the problem in the following way. We draw parallel *BF* to diagonal *AC*, point *F* being on line *AD* (fig. 5) and let *M* be the midpoint of *DF*. As $A_{DMC}=\frac{A_{CDF}}{2}=\frac{1}{2}(A_{DAC}+A_{CAF})=\frac{1}{2}(A_{DAC}+A_{CAB})=\frac{1}{2}A_{ABCD}$ and $\frac{A_{ABCD}}{2}<A_{ADC}$ we deduce that $A_{DMC}<A_{ADC}$, therefore point *M* lies on side *AD* of quadrilateral *ABCD*, which means that line *CM* is the solution to our problem.[4]

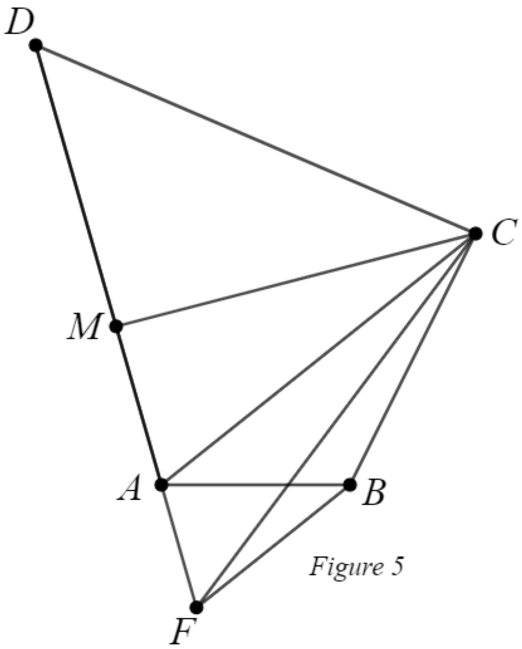

Figure 5

Let's consider now the case when the cake has the shape of a particular quadrilateral, the parallelogram.

If the cake has the shape of a parallelogram, we can cut it along any diagonal or we can choose a line that passes through the intersection of the diagonals of the parallelogram. Now, we will prove that any line which passes through the intersection of the diagonals of the parallelogram-cake splits the cake into 2 pieces of equal area.

Theorem. Let *ABCD* be a parallelogram. Any line that passes through the intersection of the diagonals splits the parallelogram *ABCD* into two regions/shapes of equal areas.

Proof. In figure 6, line *EF* with $E\in(AB)$ and $F\in(CD)$ passes through *O*, the intersection of the diagonals of the parallelogram *ABCD*.

In triangles OEB and OFD we have $\angle EOB\equiv\angle FOD$ (opposite angles), $\angle OBE=\angle ODF$ (alternate interior angles) and *OB=OD.* Therefore, triangles *OEB* and *OFD* are congruent, so *BE=DF* and then *CF=AE.*

Let *CM* be the perpendicular on *AB*, with point *M* on line *AB*, then

$$A_{BEFC}=\frac{(BE+CF)\cdot CM}{2}=\frac{(DF+AE)\cdot CM}{2}=A_{DEFA}.$$

If line *EF* is the same with one of the diagonals *AC* or *BD*, then, obviously, it splits the parallelogram into two congruent triangles, therefore, into two pieces (triangle-shaped) of equal areas. Now, the proof is complete. ■[5]

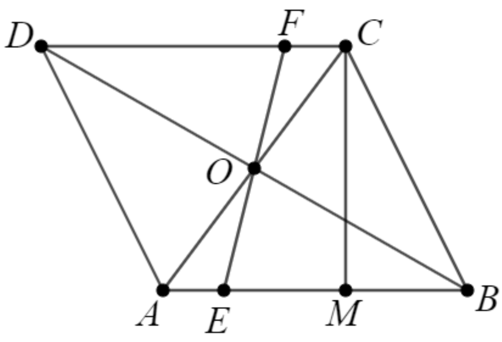

Figure 6

Case III. The cake has the shape of a convex pentagon.

Let *ABCDE* be an arbitrary convex pentagon. One of the triangles *ABC* or *ADE* must have a smaller area than half the area of the pentagon[6]. Let us presume, without loss of generality, that this is triangle *ABC*, which means $A_{ABC} < \frac{1}{2} A_{ABCDE}$. Let *F* be the point where the parallel to *AC* through *B* intersects side *CD* (fig. 7). Then

$$A_{ABC} = A_{AFC} \text{ and } A_{AEDF} = A_{AEDC} + A_{AFC} = A_{AEDC} + A_{ABC} = A_{ABCDE}.$$

Using the method from the second case, we will split quadrilateral *AEDF* into two pieces of equal areas, cutting along a line *AM*, with *M* on one of the sides *CD* or *DE*. It is certain that point *M* does not lie on segment *FC*, because $A_{AFC} = A_{ABC} < \frac{1}{2} A_{ABCDE} = \frac{1}{2} A_{AEDF}$. Therefore, line *AM* splits the pentagon *ABCDE* into two shapes of equal areas.

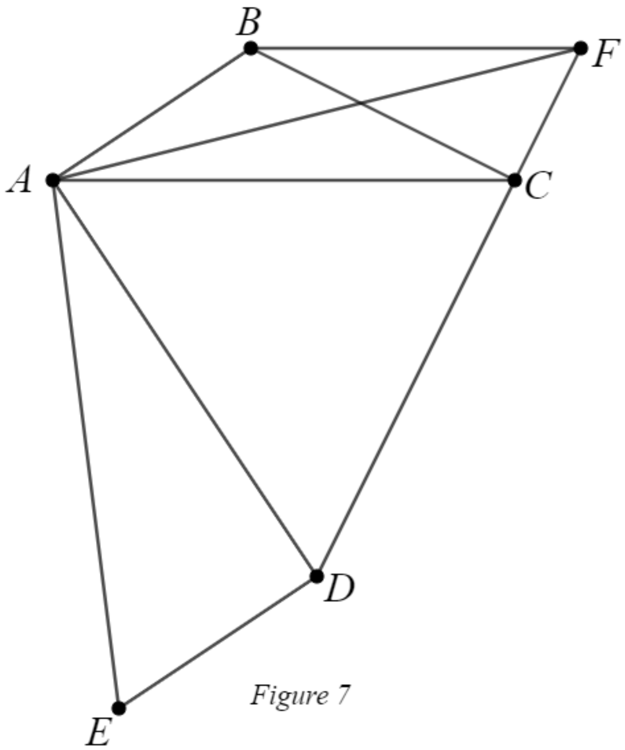

Figure 7

Case IV. The cake has the shape of a convex hexagon.

Let *ABCDEF* be an arbitrary convex hexagon. One of the triangles *ABC* or *AFE* must have a smaller area than half the area of the hexagon. Let us presume, without loss of generality, that this triangle is *ABC*, which means that $A_{ABC} < \frac{1}{2} A_{ABCDEF}$. Let *G* be the intersection of the parallel to *AC* through *B* and side *CD* (fig. 8). Therefore

$$A_{ABC} = A_{AGC} \text{ and } A_{AFEDG} = A_{AFEDC} + A_{AGC} = A_{AFEDC} + A_{ABC} = A_{ABCDEF}.$$

Using the method from the third case, we will split pentagon *AFEDG* into two pieces of equal areas cutting along a line *AM*, with *M* on one of the sides *CD*, *DE* or *FE.* It is certain that point *M* does not lie on *GC* because

$$A_{AGC} = A_{ABC} < \frac{1}{2} A_{ABCDEF} = \frac{1}{2} A_{AFEDG}.$$

Therefore, line *AM* splits the hexagon into two shapes of equal areas.

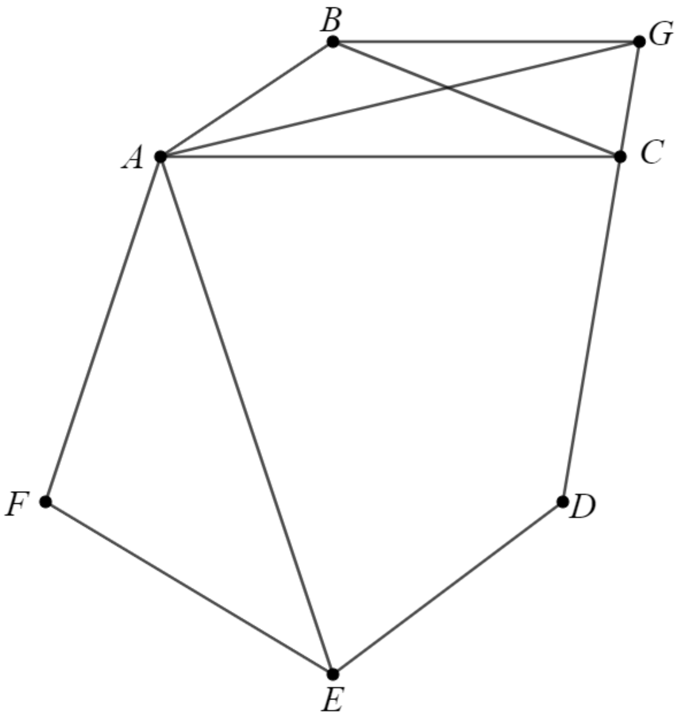

Figure 8

The method applied to cases I through IV can be used, furthermore, for convex polygons with 7, 8, 9 or more sides.[7]

We can also consider **a situation when the cake has a hole in it**. We will analyze some particular situations in which the cake and the hole have an already studied shape, using the results from the parallelogram, but also a theorem about splitting a circle into two pieces of equal area.

Theorem. In a circle, any diameter of the circle (a line that passes through the center of the circle) splits the cake into two pieces of equal area.

Proof. Let there be a circle of center *O*, radius *r* and *AB* a diameter (fig. 9). We can use the following formula:

$$A = \frac{n}{360}\pi r^2,$$

where *A* is the area of the sector, *r* is the radius of the circle and *n* is the measure of the angle between the lines that determine the sector, in degrees. We can see that *r* is the radius of the circle, and $n = 180$, because the diameter is a line, with points *A, O, B* being collinear points. From these three observations, we can deduce that the two sectors have equal areas. ■

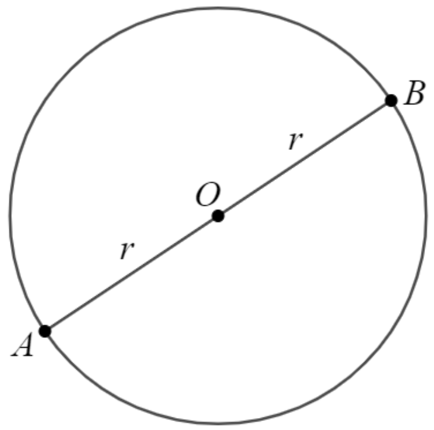

Figure 9

We will analyze the following cases:

a) The cake and the hole are circle-shaped;
b) The cake is circle-shaped and the hole is parallelogram-shaped;
c) The cake is parallelogram shaped and the hole is circle-shaped;
d) The cake and the hole are parallelogram shaped.

In cases b), c) and d), we will consider rhombuses, rectangles and squares. For these cases we will try to split the cakes with no hole in them as well as the ones with a hole into two pieces of equal area. We will use the results about splitting a circle or parallelogram shaped cake into two pieces of equal area. In all of the 4 cases, we will prove that this splitting can be done with only one cut.

a) The cake and the hole are circle-shaped.

Let *O* be the center of the larger circle and let *Q* be the center of the hole, as shown in figure 10. We will try to split both circles into two pieces of equal area. According to what we previously stated, when we discussed the case of splitting the circle, any line that passes through the center of the circle splits it into two pieces of equal areas, therefore a line that passes through *O* and *Q* will split the cake into two pieces of equal areas. If *O* and *Q* are the same (fig. 11), then any line that passes through this point satisfies the conditions of the problem.

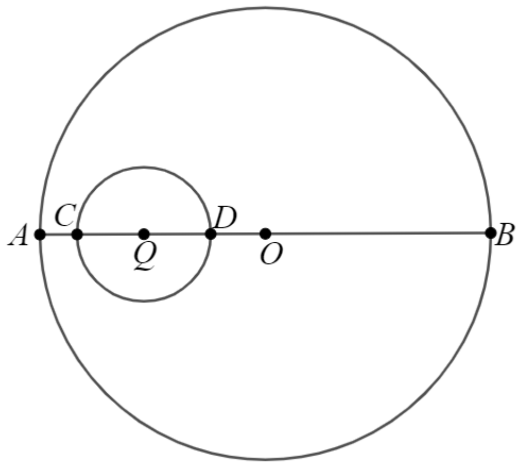

Figure 10

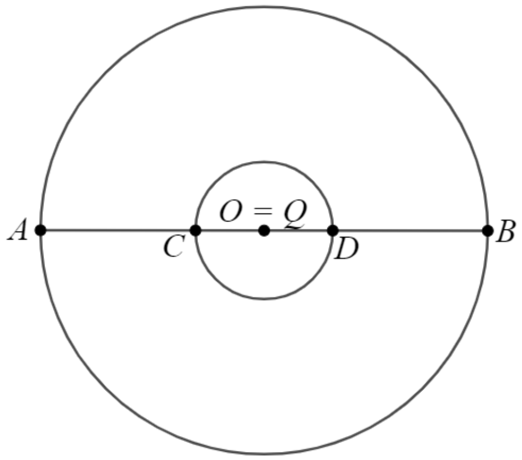

Figure 11

b) The cake is circle-shaped, and the hole in it is parallelogram-shaped.

Let *O* be the center of the circle representing the shape of the cake and let parallelogram *ABCD* be the hole in the cake (fig. 12). Consider *Q* the intersection of the diagonals of *ABCD*, then using the result from splitting a parallelogram-shaped cake, we deduce that any line passing through *Q* will split the hole into two pieces of equal areas. For splitting the circle, we will draw a line through *O*. Therefore, for cutting the cake into two pieces of equal areas, the line should pass through points *O* and *Q*. If the two points are the same (fig. 13), any line passing through *O* will solve the problem.[8]

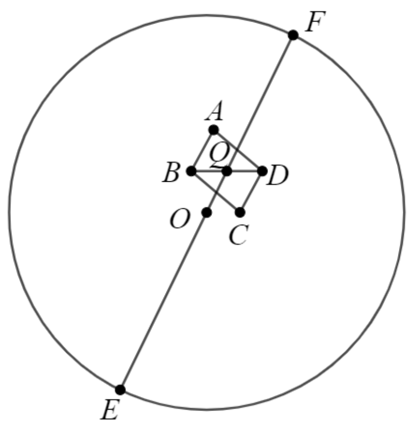

Figure 12

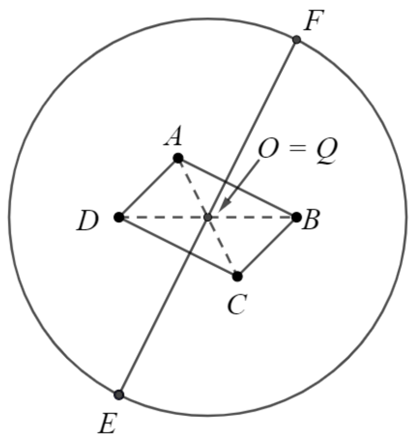

Figure 13

c) The cake is parallelogram-shaped and the hole is circle-shaped.

Let *O* be the center of the hole, *ABCD* the shape of the cake (fig. 14) and *Q* the intersection of the diagonals of *ABCD*. Using the technique from the previous case, we will deduce that the line passing through *O* and *Q* will split the cake in two pieces of equal areas. If the two points are the same (fig. 15) any line passing through *O* will solve this case.

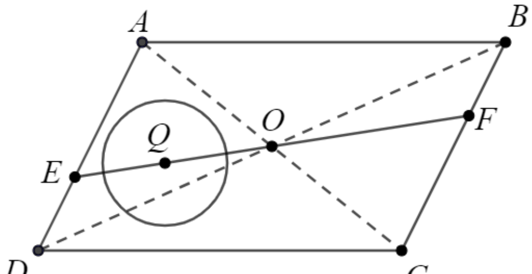

Figure 14

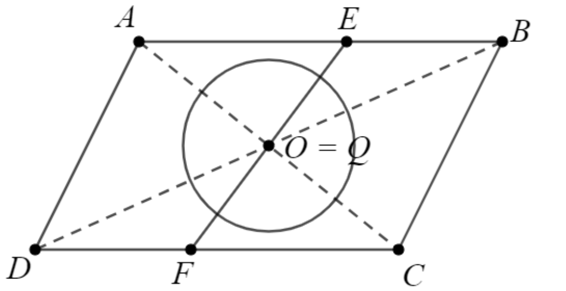

Figure 15

d) The cake and the hole both have the shape of a parallelogram.

Let *ABCD* be the shape of the cake and let *EFGH* be the shape of the hole. Considering *O* and *Q* the intersections of the diagonals of each of the two parallelograms (fig. 16), according to the results from the case of splitting a parallelogram into two pieces of equal areas, we deduce that the line passing through the two intersections (*O* and *Q*) will split the cake into two pieces of equal areas. If the points are the same, any line passing through *O* will solve this case (fig. 17).

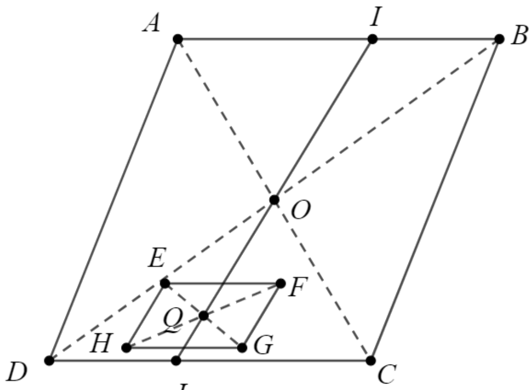

Figure 16

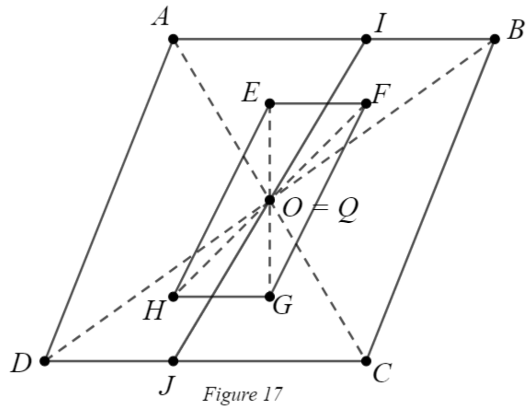

Figure 17

We will now present a method for cutting a convex heptagon in 3 pieces of equal areas, using calculus. This method can be also used for cutting in 3 pieces of equal area any convex
n-gon, with *n* greater than or equal to 3.

Let $OA_1A_2A_3A_4A_5A_6$ be an arbitrary convex heptagon. We will draw all the diagonals from point *O* (fig. 18). Using Heron's formula ($S=\sqrt{p(p-a)(p-b)(p-c)}$, where *S* is the area of the triangle, *p* is the semi-perimeter, and *a, b,* respectively *c* are the lengths of the sides of the triangle), we can compute the area of each of the 5 triangles OA_1A_2, OA_2A_3, …, OA_5A_6 and, adding them, the area of the entire heptagon. We will start adding areas of the triangles, from OA_1A_2 to OA_5A_6. When the sum first exceeds one third of the area of the heptagon, we will stop. Let's presume that

$$A_{OA_1A_2} + \ldots + A_{OA_xA_{x+1}} < \frac{1}{3}A_{OA_1A_2\ldots A_6} < A_{OA_1A_2} + \ldots + A_{OA_{x+1}A_{x+2}}.$$

Subtracting $A_{OA_1A_2} + \ldots + A_{OA_xA_{x+1}}$ from one third of the area of the heptagon, we will obtain an area *s*. Let *M* be the point on segment $A_{x+1}A_{x+2}$ such that $A_{OA_{x+1}M} = s$. Knowing the ratio of the areas of $OA_{x+1}M$ and $OA_{x+1}A_{x+2}$ and the length of $A_{x+1}A_{x+2}$, we can find the length of $A_{x+1}M$ because

$$\frac{A_{x+1}M}{A_{x+1}A_{x+2}} = \frac{A_{OA_{x+1}M}}{A_{OA_{x+1}A_{x+2}}}.$$

Now, we will have the position of *M*.[9]

Working analogously, we can determine a point *N* on segment A_yA_{y+1} such that

$$A_{OA_6A_5} + \ldots + A_{OA_yA_{y-1}} + A_{OA_yN} = \frac{1}{3}A_{OA_1\ldots A_6}.$$

Lines *OM* and *ON* are the solution to our problem and divide the heptagon in 3 shapes of equal areas.

This method can also be generalized for dividing a convex *n*-gon ($n \geq 3$) in *m* pieces of equal areas ($m \geq 2$), using $m-1$ lines.

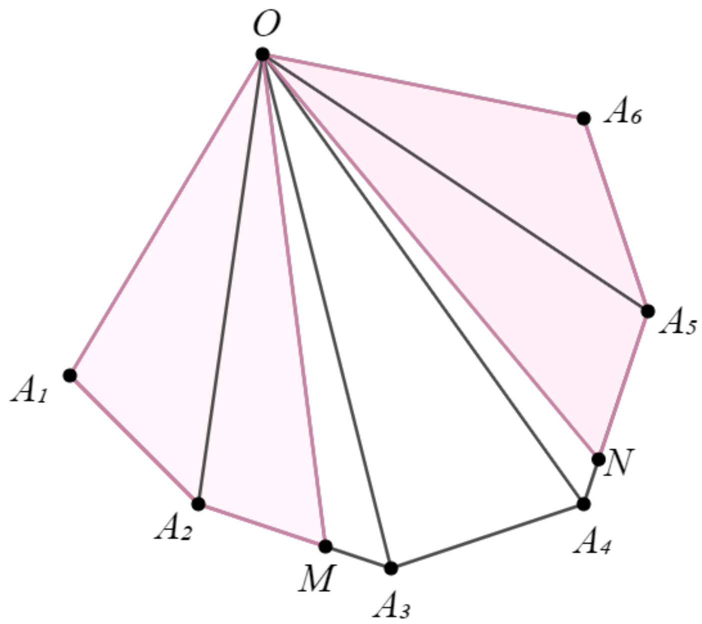

Figure 18

4. Conclusion

We have used both calculus and geometry to find ways of cutting different shapes in 2 or more pieces of equal areas and we have analyzed some particular cases in which the shapes have holes in them. An interesting study would be to see how the problem can be solved if the height of the cake is taken into consideration.

Notes d'édition

(1) Section 2 is another introduction of the research topic.

(2) The notation A_{ABC} refers to the area of the triangle ABC.

[3] A shape is convex if for any couple of points in it, say A and B, the entire segment [AB] lies inside the shape.

[4] An argument proving that M is necessarily in the quadrilateral is lacking. Actually this construction is the same construction as before where we change the name of the vertices (and the previous argument proves that M is necessarily in the quadrilateral).

[5] Once again, this boils down to change the name of vertices.

[6] **At least** one of the triangles. The reason is that the sum of the 2 areas is smaller than the area of the pentagon. Both can be strictly smaller.

[7] Yes indeed, the idea of replacing a vertex by another aligned with an adjacent side reduces the problem of cutting a N-

gon to the problem of cutting a (N-1)-gon. As the problem is solved for a triangle, by recursion it is solved for a polygon with arbitrary number of sides.

[8] To be more precise, the line should not necessarily pass through the points O and Q, but it is *enough* that the line passes through O and Q for the problem to be solved. One could imagine other solutions that avoid the hole and would not pass through the center of the circle.

[9] The position of M is generally not constructible (straighedge and compass construction) but only approximable (using the graduation of a ruler for example).