REGARDANT AUTOUR

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1. PRESENTATION OF THE RESEARCH TOPIC

We calculated the maximum distance between the base of the Eiffel Tower and the farthest visible point from the top of the tower, considering that the sky is clear, there are no clouds or other obstacles. The result is approximately 64.25 km.

We found out what percentage of France's surface can be seen from the top of the Eiffel Tower. This is approximately 2.4% of the surface.

We also found out the maximum distance we can be from the Eiffel Tower so that it can still be visible. In the end, we got a distance of 68.97 km.

We calculated the maximum distance between two people so that they can see each other. We took into account the curvature of the earth and the heights of the people (initially, we took a height of approximately 1.75 m, and later we will present a generalization). We got a maximum distance of 9.44 km.

Finally, we ask ourselves from what altitude can one observe the sea from Paris? The result is 1.44 km above the Eiffel tower.

2. BRIEF PRESENTATION OF THE CONJECTURES AND RESULTS OBTAINED

If you are on the top of the Eiffel Tower on a clear day, how far away can you see? What fraction of France is visible from the top?
On the same clear day, which is the furthest distance from the Eiffel tower that can be observed, assuming that there are no other obstructions in the way? You may take or not into account your height.
From how far away two people (say, 175 cm tall) can see each-other on a clear day?

THE SOLUTION

Part I
Maximum distance at which we can still see a point from the top of the Eiffel Tower

Let us consider the Earth a sphere with center *O* and radius $R = 6\,371$ km. We denote the Eiffel Tower by *AB* (*B* is the center of the base and *A* the top). We know that the height of the Eiffel Tower is $h = AB = 324$ m. In the conditions of the problem (clear sky, no obstacles or clouds) the farthest distance we can see is the length of the segment *AT*, where *AT* is the tangent line to Earth, and *T* is the point of tangency (see Figure 1).

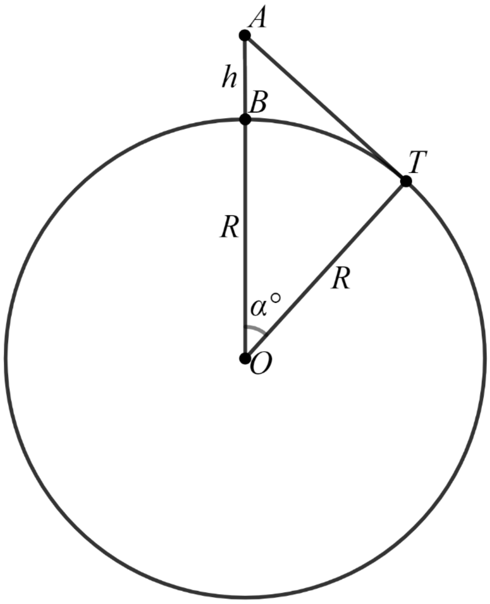

Figure 1

Using the Pythagorean theorem in triangle *AOT*, right-angled at *T*, we get:

$$d = AT = \sqrt{OA^2 - R^2} = \sqrt{(h+R)^2 - R^2} = \sqrt{h(h+2R)} = 64.253\,505 \text{ km}.$$

So, the distance from the top of the Eiffel Tower to the farthest visible point is $d = 64.253505$ km.

To determine the length of the arc *BT* we need to know a trigonometric function of the angle $\alpha° = m(\angle AOT)$ (measured in degrees).[1] In triangle *AOT* we have:

$$\cos\alpha^\circ = \frac{TO}{AO} = \frac{R}{R+h} = \frac{6371}{6371.324} = 0.999\,949\,147.$$

Thus, [2]

$$\alpha^0 = \cos^{-1}\left(\frac{R}{R+h}\right) = 0.577\,782^0$$

Now, we will calculate the length of the arc *BT*: [3]

$$L_{\text{arc } BT} = \frac{\alpha^\circ}{360^\circ} L_{\text{circle}} = \frac{0.577\,782^\circ}{360^\circ} 2\pi R = 64.251\,327 \text{ km}.$$

So, from the top of the Eiffel Tower, you can see at a distance of $64.251\,327$ km.

Part II

The percentage of France surface which is visible from the top of the Eiffel Tower

Since we are on a sphere, the visible surface is a spherical cap, not a circle. Let Q be the center of the circle determined by the base of the spherical cap and $r = QT$ its radius (see Figure 2).

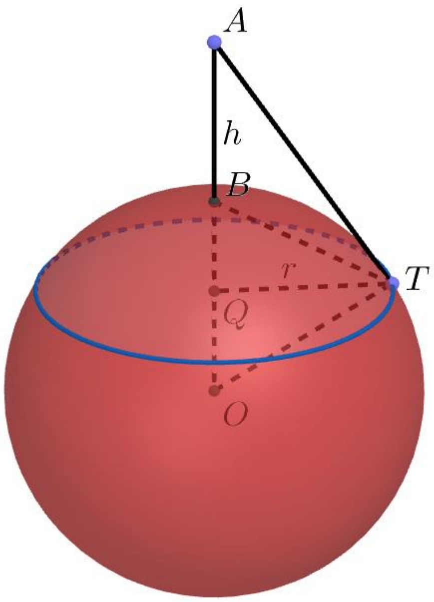

Figure 2

Applying the law of cosines in triangle *OBT*, we get: [4]

$$BT = \sqrt{OB^2 + OT^2 - 2 \cdot OB \cdot OT \cdot \cos\alpha^0} = \sqrt{2R^2\left(1-\cos\alpha\right)} = \sqrt{\frac{2R^2h}{R+h}} = 64.251\,055 \text{ km.}$$

In the triangle *OQT* right angle at Q, we have: $\sin\alpha^0 = \frac{QT}{OT} = \frac{r}{R}$,

thus

$$r = R \cdot \sin\alpha^0 = 6371 \cdot \sin\left(0.577\,826^0\right) = 64.250\,238 \text{ km.}$$

Triangle *BQT* is right-angled at Q, so

$$BQ = OB - OQ = R - R\cos\alpha^{\circ} = \frac{Rh}{R+h} = 0.323\,983 \text{ km.}$$

The area of the spherical cap is

$$A_{\text{cap}} = 2\pi R \cdot BQ = 12\,969.116\,693\,\text{km}^2.$$

The percentage of France surface (543 801 km^2 - see here https://www.cia.gov/the-world-factbook/static/fe49a53cb06ee9a6de931f4c34d23189/FR-summary.pdf) we can see from the top of the Eiffel Tower is p%, where

$$p = \frac{12\,969.116\,693 \cdot 100}{551\,500} = 2.351\,608.$$

In conclusion, from the top of the Eiffel Tower, we can see approximately 2.4% of the France surface.

Part III

Maximum distance at which we can still see the Eiffel Tower

To find the maximum distance from which a person can see the Eiffel Tower, we must calculate the distance in the field between the center B of the base of the tower and the feet of the person. Let CD be one person with height $h = CD = 1.75$ m. (see Figure 3).

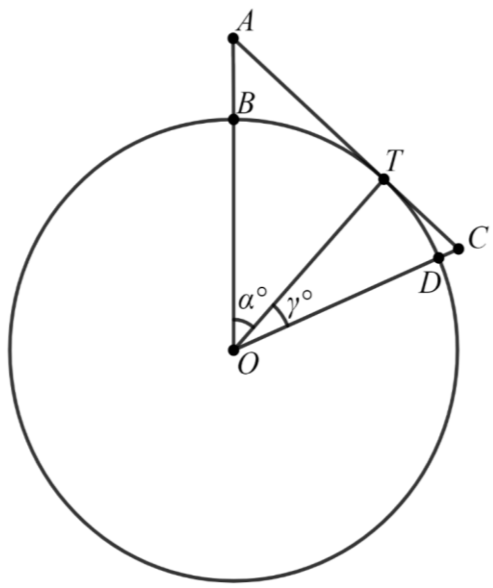

Figure 3

If $\gamma° = m(\angle TOC)$, then

$$\cos\gamma^0 = \frac{OT}{OC} = \frac{OT}{OD+CD} = \frac{6\,371}{6\,371 + 0.001\,75} = 0.999\,999\,725,$$

so $\gamma° = 0.042467119°$.

The length of the arc TD is

$$L_{\text{arc}TD} = \frac{\gamma^0}{360^0} \cdot 2\pi R = 4.722\,128 \text{ km}.$$

The maximum distance in the field between the person's feet and the base of the tower is

$$L_{\text{arc } BD} = L_{\text{arc } BT} + L_{\text{arc } TD} = 64.251\,327 + 4.722\,128 = 68.973\,260 \text{ km.}$$

Part IV-1

Maximum distance between two people that still see each other

Let O_1P_1, O_2P_2 be the two people who can see each other situated at far as possible (see Figure 4). We consider the height of the two people $O_1P_1 = O_2P_2 = 1.75$ m. The maximum distance from which the two people can still see each other is O_1O_2. This segment must be tangent to the Earth, and the tangent point will be S. The line OS and O_1O_2 are perpendicular.

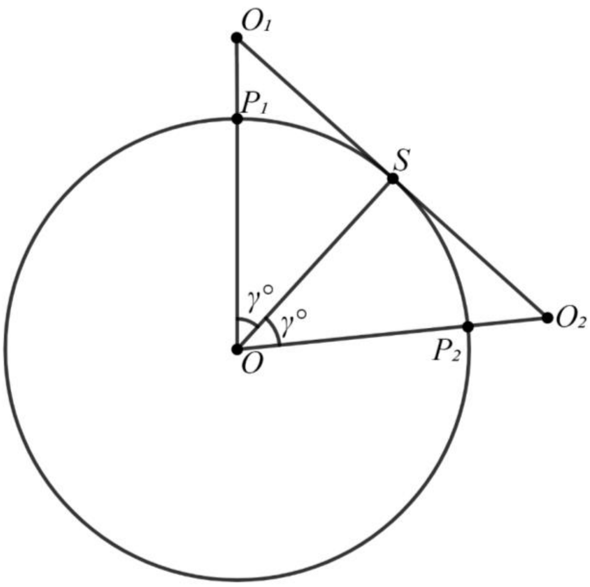

Figure 4

We have: $OO_1 = OO_2 = 6371.00175$ km. The angles $\angle O_1OS$ and $\angle O_2OS$ are both equal to $\gamma°$ because [5]

$$\cos\gamma^\circ = \frac{OS}{OO_1} = \frac{OS}{OO_2} = \frac{6\,371}{6\,371 + 0.001\,75} = 0.999\,999\,725$$

The length of the arc P_1P_2 is

$$L_{\text{arc } P_1P_2} = \frac{2\gamma^\circ}{360^\circ} \cdot L_{\text{circle}} = \frac{2\gamma^\circ}{360^\circ} \cdot 2\pi R = 9.444\,256 \text{ km.}$$

So, the maximum distance from which the two people can still see each other is $9.444\,256$ km.

Part IV-2

Generalization of the previous point

We consider now the case when the two people, O_1P_1 and O_2P_2 have different height: $O_1P_1 = u$ km and $O_2P_2 = v$ km. Suppose that the two people can see each other and they are situated as far as possible (see again Figure 4). The maximum distance from which the two people can still see each other is O_1O_2. This segment must be tangent to the Earth, and the tangent point will be S. We introduce the notations: $m(\angle O_1OS) = \theta°$ and $m(\angle O_2OS) = \varepsilon°$. It immediately follows that:

$$OO_1 = u + R,\ OO_2 = v + R,$$

$$\cos\theta = \frac{OS}{OO_1} = \frac{R}{u+R} \text{ and } \cos\varepsilon^0 = \frac{OS}{OO_2} = \frac{R}{v+R}.$$

Thus $\theta° = \cos^{-1}\left(\frac{R}{u+R}\right)$ and $\varepsilon° = \cos^{-1}\left(\frac{R}{v+R}\right)$ (we consider these values expressed in degrees, not in radians). The length of the arc P_1P_2 is

$$L_{\text{arc } P_1P_2} = \frac{\theta° + \varepsilon°}{360°} \cdot 2\pi R = \left(\cos^{-1}\left(\frac{R}{u+R}\right) + \cos^{-1}\left(\frac{R}{v+R}\right)\right) \cdot \frac{2\pi R}{360} \text{ km.}$$

So, the maximum distance from which the two people can still see each other is

$$\left(\cos^{-1}\left(\frac{R}{u+R}\right) + \cos^{-1}\left(\frac{R}{v+R}\right)\right) \cdot \frac{2\pi R}{360} \text{ km.}$$

From this final formula, for $u = v = 0.00175$ km, we obtain the final results of the previous section.[6]

Part V – Checking the altitude of the Eiffel tower [7]

We want to find out the height of the Eiffel tower while being at a far enough distance from it (we will see that we can find even the distance from us to the Eiffel Tower), having a rope d m long and a sextant (a precision instrument that measures the angle between two visible objects). We stretch the rope from point C to point D such as the point C, D and B are collinear and $CD = d$. Using the sextant, we find $m(\angle ACB) = \alpha°$ and $m(\angle ADB) = \beta°$ (see Figure 5).

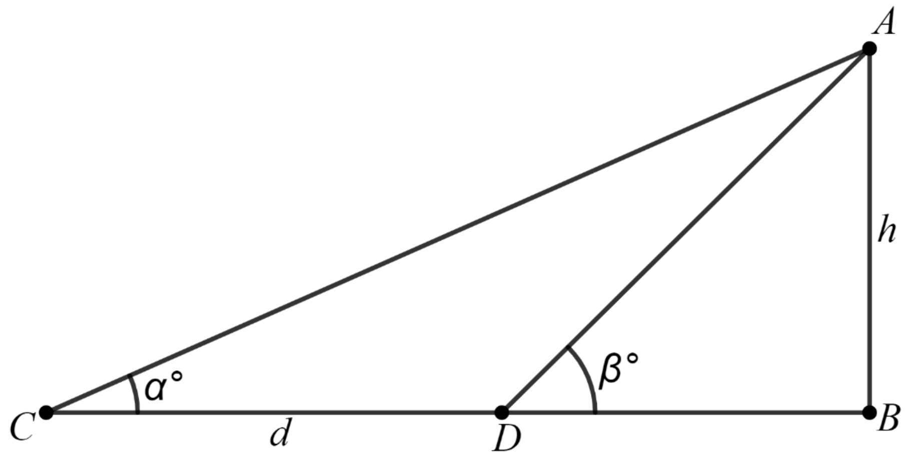

Figure 5

We have $\tan\alpha^\circ = \dfrac{AB}{CB}$ and $\tan\beta^0 = \dfrac{AB}{DB}$, so $d = CB - DB = \dfrac{AB}{\tan\alpha^\circ} - \dfrac{AB}{\tan\beta^\circ} = \dfrac{\left(\tan\beta^\circ - \tan\alpha^\circ\right)AB}{\tan\alpha^\circ \cdot \tan\beta^\circ}$.

From this formula we deduce that

$$AB = \frac{\tan\alpha^\circ \cdot \tan\beta^\circ}{\tan\beta^\circ - \tan\alpha^\circ} d ,$$

which is the height, measured in meters, of the Eiffel Tower.

Now, the distance between points C and B is $CB = \dfrac{AB}{\tan\alpha^\circ} = \dfrac{\tan\beta^\circ}{\tan\beta^\circ - \tan\alpha^\circ} d$.

So, the distance from us (point C) and the base of the Eiffel Tower, measured in meters, is

$$\frac{\tan\beta^\circ}{\tan\beta^\circ - \tan\alpha^\circ} d .$$

Part VI – Launching a weather balloon with a GO-PRO camera mounted on it – When it will spot the sea?

Let us launch in the sky a weather balloon with a GO-PRO camera mounted on in (denote the position of camera by point X) from the top A of the Eiffel Tower (with B the center of its base and the high $h = AB = 324$ m). We want to find out the minimum altitude the balloon has to gain in order to observe the sea.

Let M be the one of the nearest points of the edge of the sea which we can see from X. To minimize the distance XM we take XM to be tangent to the Earth (see Figure 6).

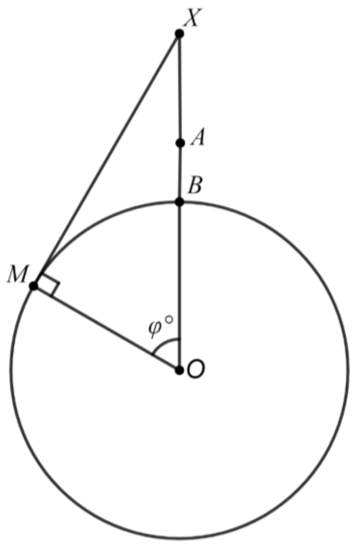

Figure 6

We will consider the minimum distance to the sea around 150 km (see Figure 7 taken from Google Earth). Let O be the center of Earth and $\varphi^\circ = m(\angle MOB)$.

Figure 7

We have

$$\frac{\varphi^\circ}{360^\circ} = \frac{L_{\text{arc } MB}}{2\pi R} = \frac{150}{40\,030.173\,592} = 0.003\,747\,173,$$

thus $\varphi^\circ = 1.348\,982\,409^\circ$.

Then,

$$AX = OX - (OB + AB) = \frac{OM}{\cos\varphi^\circ} - (OB + AB) = \frac{6\,371}{\cos(1.348\,982\,409^\circ)} - (6\,371.324) = 1.442\,221\,786 \text{ km}.$$

So, the balloon must be at least at 1.442 km above the top of the Eiffel Tower in order to observe the sea.

3. CONCLUSION

[8] We determined that from the top of the Eiffel Tower we could see a point at a maximum distance of 64 km and about 2.5% of France. We also came to the conclusion that we can see from the Eiffel Tower at a maximum distance of 68.97 km.

EDITION NOTES

[1] Expressing angles in radians, rather than degrees, is generally advisable in mathematics. In the context of this work, they would have allowed simpler expressions for the length of arcs. In a circle of radius R, the length of the arc determined by an angle of α radians is simply αR.

[2] The following formula, and others in the work, involve the function $\cos^{-1}$, whose image is the interval $[0,\pi]$ (in radians). Then, the use of that function is correct because all the angles considered in the paper belong to that interval. This fact is an obvious consequence of the context, but it might be mentioned explicitly when $\cos^{-1}$ is used for the first time.

[3] This value is approximated to millimeters, but it is only an apparent precision. It is reasonable to assume that the Earth is a perfect sphere, so that we can also assume that its radius is exactly 6371 km. But it should be taken into account that the value 324 m. for the height of the Eiffel Tower is approximated. This value can vary of some centimeters depending on the temperature, for instance. Then, the best approximation we can expect for the length of the arc BT is to centimeters.
There are mathematical techniques that, given some approximated values, allow to calculate the precision of the result of operations on those values. An interesting exercise could be to assume that the height of the Eiffel Tower is a real number in the interval $(324-\varepsilon, 324+\varepsilon)$ and to determine the approximation of the arch BT as a function of ε.

[4] The determination of BT and of r have no role in the calculation of the area and can be omitted. The calculation can start with the determination of $BQ = R - R\cos\alpha$.

[5] This explanation of the equality $m(\angle O_1OS) = (\angle O_2OS) = \gamma°$ seems to assume implicitly that two angles are equal if such are their cosines, which is not true. See also Note [2]. It should have been also observed, for instance, that the two angles are less than a right angle. In any case, the conclusion can also be reached by means of simple geometrical considerations, without involving trigonometry: OS is the height of the isosceles triangle OO_1O_2.

[6] This formula can also be used in Parts I (with $u = 0.324$ and $v = 0$) and III (with $u = 0.324$ and $v = 0.00175$). The question in Part VI can be answered using this formula too. The height of the balloon from the top of the tower is the solution of this equation

$$\left(\cos^{-1}\left(\frac{R}{0+R}\right) + \cos^{-1}\left(\frac{R}{x+0.324+R}\right)\right) \cdot R = 150\ ,$$

where $R = 6371$ and the angles are expressed in radians.

[7] In this part, the Earth surface is implicitly assumed to be flat. This should be noticed explicitly because the rest of the paper deals with issues that are closely related to the Earth curvature.

[8] This conclusion describes only the first part of the work.