The Big Gap

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1 Topic of Research

There are 4 natural numbers a, b, c, d written on a row. The differences $|a-b|$, $|b-c|$, $|c-d|$, $|d-a|$ are written on the next row. The process continues. For example:

224	23	6	1
201	17	5	223
184	12	218	22
172	206	196	162
34	10	34	10
24	24	24	24
0	0	0	0

In this example, $a = 224$, $b = 23$, $c = 6$ and $d = 1$

It can be observed that the null row 0, 0, 0, 0 has been obtained.

- Is this a coincidence or can the null row be obtained for any 4 natural numbers?
- What happens if the 4 numbers are real?
- What happens if there are not 4 numbers, but 3, 5, 6 or more?

Note: This research topic was also discussed and the results were published in the article *"Game of Differences"* in 2017. In our work, we will use some of those results and ideas, marking them accordingly, the rest of the paper being original.

2 Solutions

2.1 Solution for 4 natural numbers case

Let's replace every number with its remainder of the division by 2 (named *modulo 2*).
Let's use a simpler example than the one in the Topic of Research (section 1):

7	3	9	2
4	6	7	5
2	1	2	1
1	1	1	1
0	0	0	0

$\longrightarrow$

1	1	1	0
0	0	1	1
0	1	0	1
1	1	1	1
0	0	0	0

There are $2^4 = 16$ ways to distribute the remainders 1 and 0 on the first row.
After maximum 4 steps the null row is obtained, meaning that after maximum 4 steps, all numbers written will be divisible by 2. This is proved in the following table:

Note: The 16 cases are the cases in **bold**

1	**0**	**0**	**0**
OR			
0	**1**	**1**	**1**
1	**0**	**0**	**1**
1	**0**	**1**	**0**
1	**1**	**1**	**1**
0	**0**	**0**	**0**

0	**1**	**0**	**0**
OR			
1	**0**	**1**	**1**
1	**1**	**0**	**0**
0	**1**	**0**	**1**
1	1	1	1
0	0	0	0

0	**0**	**1**	**0**
OR			
1	**1**	**0**	**1**
0	**1**	**1**	**0**
1	0	1	0
1	1	1	1
0	0	0	0

0	**0**	**0**	**1**
OR			
1	**1**	**1**	**0**
0	**0**	**1**	**1**
0	1	0	1
1	1	1	1
0	0	0	0

Having now only numbers that are divisible by 2, we repeat the process replacing each number with its modulo 4. They can be 0 or 2. Similarly, after maximum 4 steps, every number will be divisible by 4.

After maximum $4k$ steps, where k is a natural number, all numbers will be divisible by 2^k. As we can repeat this process indefinitely, after a certain number of steps, the null row will be obtained. Indeed, the maximum of the numbers does not increase from one step to the next, so that we are sure to get the null row, at the latest when 2^k exceeds this maximum.
*The method used above was proposed in the *"Game of Differences"* article.

Looking at the table we can also make a series of observations:

- As the subtractions are circular, having n numbers, the cases in which the numbers are circularly permuted are the same.
- The cases in which we replace the 1 with 0 and the 0 with 1 are the same. We will name these two rows "complementary rows".

2.2 Solution for 4 integer numbers case

This case is trivial because after one step, the numbers will become natural.

2.3 Solution for 4 rational numbers case

We can reduce again our demonstration to the natural numbers case. We consider a, b, c and d four rational numbers:

$a = p_1/q_1, \quad (p_1, q_1) = 1$ (1)

$b = p_2/q_2, \quad (p_2, q_2) = 1$

$c = p_3/q_3, \quad (p_3, q_3) = 1$

$d = p_4/q_4, \quad (p_4, q_4) = 1$

$\frac{p_1}{q_1}$	$\frac{p_2}{q_2}$	$\frac{p_3}{q_3}$	$\frac{p_4}{q_4}$
$p_1 \times q_2 \times q_3 \times q_4$	$q_1 \times p_2 \times q_3 \times q_4$	$q_1 \times q_2 \times p_3 \times q_4$	$q_1 \times q_2 \times q_3 \times p_4$
...	...	...	...
...	...	...	...
...	...	...	...

We multiply each number by the smallest common multiple of the denominators or simply by the product of denominators, thus reducing the problem to the natural numbers case. Multiplying does not influence the behavior of differences since we will obtain a row of zeros at the end.
*This is also how this case is handled in the 2017 article.

2.4 Solution for 4 real numbers case

In this case we can choose a combination of 4 irrational numbers such that we will never achieve the 0-row. Let us consider $a > 1$ and the numbers 1, a, a^2 and a^3 for the first row.
After the first step we obtain the row: $a-1$, $a(a-1)$, $a^2(a-1)$ and a^3-1 which can be written as $(a-1)(a^2+a+1)$.

Let us consider the cubic equation $a^3 = a^2 + a + 1$. If we find the real root of this cubic equation and we replace $a^2 + a + 1$ with a^3 we obtain the initial numbers multiplied by $a - 1$ which does not affect the behavior of the path of steps. Hence, after every step, we will obtain a new row with the same "properties" as the previous one.

Row
1
2
3
...
...
$n+1$

Numbers			
1	a	a^2	a^3
$a-1$	$a(a-1)$	$a^2(a-1)$	$(a-1)(a^2+a+1)$
$(a-1)^2$	$a(a-1)^2$	$a^2(a-1)^2$	$a^3(a-1)^2$
...	...	...	...
...	...	...	...
$(a-1)^n$	$a(a-1)^n$	$a^2(a-1)^n$	$a^3(a-1)^n$

Now, we find the solution of the equation: $a^3 = a^2 + a + 1 \Leftrightarrow a^3 - a^2 - a - 1 = 0$.
The discriminant of the cubic equation $ax^3 + bx^2 + cx + d = 0$ has the formula $\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$. The discriminant of our equation is equal to -44, which is negative, so, our equation has one real root and two non-real complex conjugated roots.
Solving it we obtain the following real root: $a \approx 1.8392867552141612 > 1$.
So, we do not obtain the 0-row for every combination of 4 real numbers.
*Choosing this set of numbers and using the third degree equation was the method presented in the article *"Game of Differences"*. In the following lines we try to provide a generalization of this result using continuous functions' properties.
Using a similar combination of numbers, we will now propose a demonstration for a more general case. Let n be the number of numbers on the first row ($n > 2$). As in the previous case, let us consider $a > 1$ and the numbers 1, a, a^2, a^3 ...a^{n-1} for the first row.
After the first step we obtain the row: $a-1$, $a(a-1)$, $a^2(a-1)$, $a^3(a-1)$,...,$a^{n-2}(a-1)$ and $a^{n-1}-1$, which can be written as $(a-1)(a^{n-2} + a^{n-3} + ... + a^2 + a + 1)$.
As in the case for 4 numbers, we search for the existence of an irrational number $a > 1$ such that $a^{n-1} = a^{n-2} + a^{n-3} + ... + a^2 + a + 1$. However, we can not compute the value of a, so in order to prove that such a number exists, we use the *Intermediate value theorem* and one of its corollaries (Bolzano's Theorem).
Intermediate value theorem states that if f is a continuous function whose domain contains the interval $[b, c]$, then it takes on any value between $f(b)$ and $f(c)$ at some point within the interval.
If a continuous function has values of opposite sign inside an interval, then it has a zero in that interval (Bolzano's theorem).
Let take the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^{n-1} - x^{n-2} - x^{n-3} - ... - x^2 - x - 1$. It is a polynomial function, so it is continuous. We observe that on the interval $[1, 2]$ the function changes its sign (2). Therefore, it has a root within this interval and we choose a to be this root, obtaining that $a^{n-1} = a^{n-2} + a^{n-3} + ... + a^2 + a + 1$.
Using the same reasoning as in the case of 4 numbers, we can conclude that for any number $n > 2$, there is a set of irrational numbers (3) so that the 0-row will never be obtained.

2.5 Solutions for any number of columns

We return to the case with natural numbers.
Let n be the number of numbers written on the first row. We suppose that at least one written number is different from the others. We work with the remainders modulo 2 of the numbers.
We distinguish 3 different cases.

2.5.1 n is odd

We prove that it is impossible to get the null row for any n numbers randomly chosen, at least one being different from the others.

It is impossible that after a step the number of 1 on a row is odd

If we suppose that the last affirmation is true, then we consider a row having an odd number of ones and the previous row from which it descends

b_{n-1}	b_n	$b_1 = 0$	b_2	b_3	b_4	b_5	b_6	$\ldots$	$b_n - 2$
a_{n-1}	a_n	$a_1 = 1$	a_2	a_3	a_4	a_5	a_6	$\ldots$	$a_n - 2$

Without losing generality, we suppose $a_1 = 1$. A number depends only on 2 numbers of the previous row: the one above and the one above and right.

For example, a_1 depends on b_1 and b_2.

$a_1 = 1$, then $b_1 \neq b_2$. The two cases conduct to complementary rows, so we consider that $b_1 = 0$. Then $b_2 = 1$. If $a_2 = 1$, then $b_3 = 0$. Otherwise, $b_3 = 1$.

Continuing, we observe that for every 1 on the last row, the number above it is different from the above-right number. Here is an example

1	0	1	1	1	0	0	0	0	1	1	1	1	1	1	1
1	1	0	0	1	0	0	0	1	0	0	0	0	0	0	0

Definition: For a row (a_1, a_2, ..., a_n) of 1's and 0's, the index $i \in \{1, 2, 3, \ldots, n\}$ is called a change if $a_i \neq a_{i+1}$ (we consider $a_{n+1} = a_n$).

So, for an odd number of ones, there will be an odd number of changes on the previous line.

Observation: The number of changes in a row must be even.

Proof: Without losing generality, we assume $b_1 = 1$. Moving to the right, circularly n places, we come back to b_1. If the number of changes is odd, then b_1 should be 0, which is a contradiction.

So, starting with $b_1 = 1$, an odd number of 1, which is equivalent to an odd number of changes on the previous line, transforms b_1 into 0, which is a contradiction.

Except if the initial row is already the null row, before the null row there will always be a row full of 1. If n is odd, as we proved, it is impossible to get this row with an odd value of 1s after any step, so the only possibility of this row to exist is to be the starting one. Thus, after at most one step, we obtain that all numbers are divisible by 2. Repeating the argument, we need that all numbers have the same remainder modulo 4 (4). In conclusion, we obtain that the numbers must be all equal.

2.5.2 n is even, but not a power of 2

Using backtracking, the row full of 1 must be obtained in order to obtain the null row. This row can be also only obtained from the complementary rows in which 1 and 0 alternate. Then, the number of ones on this row is $n/2$.

If this number is odd, we apply the last result and eventually obtain that the only numbers that conduct to the null row are an alternating sequence of 2 numbers.

If it is not, by continuing using backtracking we eventually reach a row with an odd number of ones: contrarily to the case of an odd number of 1, any row with an even number of 1 can be obtained from another row (5), so that the backtracking can continue until we get an odd number of 1. For example, when $n = 12$:

0	0	1	0	0	0	1	0	0	0	1	0	3 of 1
0	1	1	0	0	1	1	0	0	1	1	0	6 of 1
1	0	1	0	1	0	1	0	1	0	1	0	6 of 1
1	1	1	1	1	1	1	1	1	1	1	1	12 of 1
0	0	0	0	0	0	0	0	0	0	0	0	null row

In this case, we can limit the possible sets of numbers to the ones in which their remainder modulo 2 complies with the configuration shown in the table above (6). However, there are still cases in which, even by respecting this rule, the 0-row is never reached and the rows become periodic. Below is an example:

2	4	5	6	2	8	7	2	6	4	9	6
2	1	1	4	6	1	5	4	2	5	3	4
1	0	3	2	5	4	1	2	3	2	1	2
1	3	1	3	1	3	1	1	1	1	1	1
2	2	2	2	2	2	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0	2

Here is an example of a set that reaches the 0-row:

18	16	5	28	20	0	11	6	8	16	33	4
2	11	23	8	20	11	5	2	8	17	29	14
9	12	15	12	9	6	3	6	9	12	15	12
3	3	3	3	3	3	3	3	3	3	3	3
0	0	0	0	0	0	0	0	0	0	0	0

As a conclusion, in most of the cases it is impossible to reach the 0-row. However there are sets that after a finite number of steps reach the 0-row, sets whose numbers must respect the modulo 2 configuration shown in the "backtracking table".

2.5.3 n is a power of 2

This is the only case in which for every n natural numbers the null row is obtained.

To prove this, we created a C++ program that test all the possible combinations of 0 and 1, no matter how they are arranged. By doing this, we proved that after maximum n steps the null row is reached, meaning that all numbers will be divisible by 2.

Then, as we described in the case of 4 numbers, the numbers will eventually become 0.

In order to assure that we have all combinations of 0 and 1 in our program, we generate the numbers between 1 and $n-1$ and we convert them to binary.

However, as n gets bigger, it becomes more and more difficult to compute and test all the necessary combinations (7), so we found a theoretical and more general approach.

Another approach for the same case: We observe that instead of subtractions we can use additions, the outcomes being the same considering that we work with modulo 2 remainders.

Let $(a_1, a_2, \dots)$ a binary row having the length 2^k. We note $a(r, i)$ the element on the "i" column and at the "r" step. For example: $a(2,1) = a_1 + a_2$; $a(2,2) = a_2 + a_3$; $a(1,i) = a_i$; $a(3,1) = a(2,1) + a(2,2) = a_1 + 2 \cdot a_2 + a_3 = a_1 + a_3$.

We prove that we reach the null row for $k = 3$ (a row of length 8), then using the same method and mathematical induction, we prove this for any $k > 3$. In the table below, we note $ai =$"i" and $ai+aj =$"ij".

Step	Numbers							
1	1	2	3	4	5	6	7	8
2	12	23	34	45	56	67	78	81
3	13	24	35	46	57	68	71	82
4	1234	2345	3456	4567	5678	6781	7812	8123
5	15	26	37	48	51	62	73	84
6	1256	2367	3478	1458	1256	2367	3478	1458
7	1357	2468	1357	2468	1357	2468	1357	2468
8	12345678	12345678	12345678	12345678	12345678	12345678	12345678	12345678

We observe that:

- $a(r, i+1)$ can be obtained from $a(r, i)$ by circularly permuting the indexes one position. For example:

  $a(4,1) = a1 + a2 + a3 + a4$

  $a(4,2) = a2 + a3 + a4 + a5$

- If we note $a(4,i) = b(1,i)$, by repeating the same process we obtain $a(7,1) = b(4,1) = b_1 + b_2 + b_3 + b_4 = a_1 + a_3 + a_5 + a_7$ etc. (see step 7)

  After the eighth step we will have all numbers equal to the sum of the all 8 numbers on the first row, which means that we will obtain on the next row the null row.

There are configurations where the null row appears earlier, however, after maximum 8 steps we are sure that we will have the null row (the row where all the numbers are divisible by 2).

Theorem: $a(2^s, 1) = a_1 + a_2 + \cdots + a_{2^s}$ for every $s \in \{1, 2, ...k\}$
Proof: For s=1, $a(2^1, 1) = a_1 + a_2$ by definition. We suppose that the theorem is true for a fixed number p and we prove that it is also true for $p+1$.
By definition, $a(2^p + 1, 1) = a(2^p, 1) + a(2^p, 2)$. Then, $a(2^p + 2, 1) = a(2^p + 1, 1) + a(2^p + 1, 2) = a(2^p, 1) + 2a(2^p, 2) + a(2^p, 3) = a(2^p, 1) + a(2^p, 3)$
By continuing this reasoning, we obtain that $a(2^p + 2^p, 1) = a(2^p, 1) + a(2^p, 1 + 2^p) \Leftrightarrow a(2^{p+1}, 1) = (a_1 + a_2 + ... + a_{2^p}) + (a_{2^p+1} + a_{2^p+2} + \cdots + a_{2^{p+1}})$ which is the conclusion.

Particularly, $a(2^k, 1) = a(2^k, i) = a1 + a2 + \cdots + a_{2^k}$for every $i \in \{1, 2, \ldots, 2^k\}$, so $a(2^k + 1, i) = 0$ for every $i \in \{1, 2, \ldots, 2^k\}$.

3 Development of the Research Topic

3.1 The Fractal

Let's imagine an empty row of an infinite number of columns with a 1 at the top right and apply the instructions:

...	0	0	0	0	0	0	0	0	0	0	1
...	0	0	0	0	0	0	0	0	0	1	1
...	0	0	0	0	0	0	0	0	1	0	1
...	0	0	0	0	0	0	0	1	1	1	1
...	0	0	0	0	0	0	1	0	0	0	1
...	0	0	0	0	0	1	1	0	0	1	1
...	0	0	0	0	0	0	0	0	0	0	1

To make it more visual, let's focus on the upper right part of the table and color the 0 in white and the 1 in black (Figure 1).

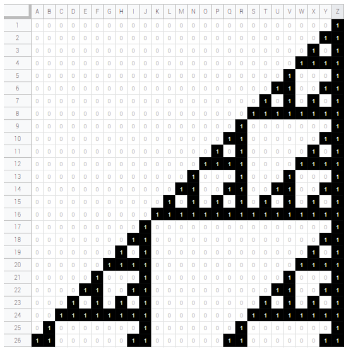

Figure 1: After 26 steps

After an infinite number of steps, we would have a self-similar fractal. This means that the structure and shape would not change according to the scale. By applying a $\times 2$ or $\times 0.5$ zoom, we would see exactly the same triangle..., so this tells us that when the triangle repeats itself in the table, it will double its size.

Step	Width
0	1
1	2
2	4
3	8
4	16
$\dots$	$\dots$
n	2^n

Indeed, this is a case of elementary cellular automaton, called *rule 102*. (For more information consult the reference) Additionally, our fractal represents Pascal's Triangle modulo 2, also known as Sierpiński's Triangle or Sierpiński's Sieve. The only difference is the way in which the cells are disposed: in ours the cells form a right triangle, whereas the original fractal takes the form of an equilateral triangle.

3.2 The Third Line

The goal here was to start from the line of zeros and try to get to the lines before, but not with the modulos of the numbers, with any natural numbers.

3.2.1 Hypothesis

Consider the following:

a, b, c, $d \in \mathbb{N}$ *such that the following table is respected*

$y \in \mathbb{N} \setminus \{0\}$

$\dots$	$\dots$	$\dots$	$\dots$
a	b	c	d
$\|a-b\| = y$	$\|b-c\| = y$	$\|c-d\| = y$	$\|d-a\| = y$
$y - y = 0$	$y - y = 0$	$y - y = 0$	$y - y = 0$

3.2.2 Thesis

We can then prove that the naturals a, b, c, d can only fall in three separate cases:

(Case 1) x-y-x-y

$$a = c$$
$$b = d$$

Or

(Case 2.1) x-y-x-z

$$a = c$$
$$b,\ d \in \{a \pm y\}$$
$$b \neq d$$

Or

(Case 2.2) x-y-z-y

$$b = d$$
$$a,\ c \in \{b \pm y\}$$
$$a \neq c$$

3.2.3 Demonstration

$$\begin{cases} |a-b| = |b-c| \\ |b-c| = |c-d| \\ |c-d| = |d-a| \\ |d-a| = |a-b| \end{cases}$$

$$\Leftrightarrow \begin{cases} (a-b)^2 = (b-c)^2 \\ (b-c)^2 = (c-d)^2 \\ (c-d)^2 = (d-a)^2 \\ (d-a)^2 = (a-b)^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} (a-b)^2 - (b-c)^2 = 0 \\ (b-c)^2 - (c-d)^2 = 0 \\ (c-d)^2 - (d-a)^2 = 0 \\ (d-a)^2 - (a-b)^2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} [(a-b) - (b-c)] \times [(a-b) + (b-c)] = 0 \quad (A) \\ [(b-c) - (c-d)] \times [(b-c) + (c-d)] = 0 \quad (B) \\ [(c-d) - (d-a)] \times [(c-d) + (d-a)] = 0 \quad (C) \\ [(d-a) - (a-b)] \times [(d-a) + (a-b)] = 0 \quad (D) \end{cases}$$

Let's focus on the first two cases.
If the result of a product is zero, one of the two factors is equal to zero:

(A_1) IF $(a-b)-(b-c)=0$	**(A_2) IF** $(a-b)+(b-c)=0$
$\Leftrightarrow a-b-b+c=0$ $\Leftrightarrow a-2b+c=0$ $\Leftrightarrow a+c=2b$ $\Rightarrow b=\dfrac{a+c}{2}$	$\Leftrightarrow a-b+b-c=0$ $\Leftrightarrow a-c=0$ $\Rightarrow a=c$

(B_1) **IF** $(b-c)-(c-d)=0$	(B_2) **IF** $(b-c)+(c-d)=0$
$\Leftrightarrow b-c-c+d=0$ $\Leftrightarrow b-2c+d=0$ $\Leftrightarrow b+d=2c$ $\Rightarrow c=\dfrac{b+d}{2}$	$\Leftrightarrow b-c+c-d=0$ $\Leftrightarrow b-c=0$ $\Rightarrow b=d$

A_1 is not compatible with A_2:
If $b=(a+c)/2$ and $a=c$, $\Rightarrow b=(a+a)/2=a \Rightarrow y=b-a=0$. But y cannot be equal to zero.
We can do the same thing with B_1 and B_2.
A_1 is not compatible with B_1:
If $b=(a+c)/2$ and $c=(b+d)/2$, $a \neq b \neq c \neq d$ but it's not possible to have a line like that before the lines of y and zeros (we know that because here, we only use the first two cases but all the cases can be used for all the numbers) (8).
There are three cases left:

- $(A_1$ & $B_2)$ $b=d$ and $d=(a+c)/2$ **(Thesis Case 2.1)** (The difference between b and a or d and a is $|a-b|=y$)
- $(A_2$ & $B_1)$ $a=c$ and $c=(b+d)/2$ **(Thesis Case 2.2)** (The difference between a and b or c and b is $|b-c|=y$)
- $(A_2$ & $B_2)$ $a=c$ and $b=d$ **(Thesis Case 1)** (The difference between a and b or c and d is $|a-b|=y$)

4 Conclusion

Here's what we proved:

- If n is odd:
  The 0-row will never be reached, unless the numbers are equal.

- If n is even but not a power of two:
  In some cases it is impossible to reach the 0-row.

- If n is a power of two:
  For natural numbers, integers and rational numbers the 0-row is reached, any number being randomly chosen.
  For irrational numbers there are cases in which the 0-row is never reached

- The Third Line Thesis (3.2.2)

5 References

- *Game of differences: https://www.mathenjeans.fr/content/game-differences-colegiul-national-din-iasi-iasi-roumanie*

- *Elementary cellular automaton: http://mathworld.wolfram.com/ElementaryCellularAutomaton.html*

- *Rule 102: http://mathworld.wolfram.com/Rule102.html*

- *Sierpiński's Sieve: http://mathworld.wolfram.com/SierpinskiSieve.html*

- *Pascal's Triangle: http://mathworld.wolfram.com/PascalsTriangle.html*

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6 Notes d'édition

(1) (p, q) denotes the greatest common divisor of two integers p and q, so $(p, q) = 1$ means that they are relatively prime.
(2) Indeed, $f(1) = 1 - (n-1) = 2 - n < 0$ and $f(2) = 1 > 0$ since $2^{n-2} + 2^{n-3} + \cdots + 1 = 2^{n-1} - 1$.
(3) In the case of 4 numbers, the fact that the real root of the polynomial is irrational follows the result that the null row is obtained in the rational case, but it can also be proved in the general case of n numbers.
(4) If all the numbers of the first row are odd, we do not change the next row by subtracting 1 to each of them; so we can suppose that they are even and then the previous argument tells us that either all of them are 0 modulo 4 or all of them are 2 modulo 4 and in the initial row all of the numbers still have the same remainder modulo 4. Continuing with successive powers of 2, we see that all the numbers of the row must be equal.
(5) Following the same proof as in section 2.5.1, if we have a row $(a_1, \ldots, a_n)$ of 1's and 0's with an even number of ones, we can construct a row $(b_1, \ldots, b_n)$ having a change at each index i such that $a_i = 1$. Note also that starting with the null row, the backtracking cannot come back to the null row and thus not to any row that was already seen, so it can stop only when we get an odd number of ones.
(6) These configurations must be completed with complementary rows and circular permutations: in fact we obtain all sequences $(a_1, \ldots, a_n)$ of period 4 ($a_{i+4} = a_i$ for $1 \leq i \leq 8$). In the general case $n = 2^k p$ where p is odd, if a sequence $(a_1, \ldots, a_n)$ is 2^k-periodic and has m ones between the indices 1 and 2^k, the total number of ones is mp and it is even iff m is. Then, constructing a row $(b_1, \ldots, b_n)$ above $(a_1, \ldots, a_n)$, we have an even number of changes between b_1 and b_{2^k+1}, whence $b_{2^k+1} = b_1$, and continuing it will follow that $(b_1, \ldots, b_n)$ is still 2^k-periodic. Since the null row is 2^k-periodic we get by induction that every row found in the bactracking is 2^k-periodic. Therefore any row where the remainders modulo 2 are not 2^k-periodic cannot reach the null row.
(7) Actually it is impossible to test by computer all the rows of zeros and ones for *all* positive integers n.
(8) More precisely, C would imply either $a = c$ contradicting $A1$ or $d = (a + c)/2 = b$ contradicting $B1$.