THE CLOCK PROBLEM

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Abstract

The aim of the article is to study the angles that the clock hands form.

We will describe which angle the two clock hands form in a determined hour during the day and then we will analyze some particular situation.

Then, starting from the clock problem, we will study the planetary motion, we will find out the angular position and the time at which it would be more advantageous to launch a spaceship from one planet to reach another one with the shortest possible distance.

Finally we will propose the results of two simulations concerning the Solar System using Unity.

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Contents

Chapter 1

Introduction

Our research was born from the study of the angles that the clock hands form.

We asked ourselves which angle the two clock hands form in a determined hour during the day and then we analyzed some particular situation.

Finally we thought a clock as a model to study the planetary motion, starting form the clock problem we focused on the angular position and the time at which it would be more advantageous to launch a spaceship from one planet (i.e. the Earth) to reach another one (i.e. Mars) with the shortest possible distance. Another problem that we treated was the one regarding the trajectory of the midpoint between Mars and the Earth.

In the last Chapter we created two simulations concerning the Solar System using Unit.

Chapter 2

Angles between the clock hands

Let's calculate the clock hands' angular speed:

$$\omega_H = \frac{2\pi}{720\text{min}} = \frac{\pi}{360}\ \text{rad/min} \qquad \text{for the hours clock hand}$$
$$\omega_M = \frac{2\pi}{60\text{min}} = \frac{\pi}{30}\ \text{rad/min} \qquad \text{for the minutes clock hand}$$
$$\omega_S = \frac{2\pi}{1\text{min}} = 2\pi\ \text{rad/min} \qquad \text{for the seconds clock hand}$$

Let's now look for the functions which express the angle variation over time between two clock hands. To do this we searched the meeting times of each couple of clock hands. Starting from an overlaped situation we observed that the clock hands don't match during the first revolution of the fastest one. For instance in the case of minutes and hours hands we wrote the two equations of motion, when the fastest hand had already done a turn. We solved the system of the two equations:

$$\begin{cases} \Theta_H(t) = \dfrac{\pi}{6} + \omega_H t \\ \Theta_M(t) = \omega_M t \end{cases} \Longrightarrow \frac{\pi}{6} + \omega_H t = \omega_M t \Rightarrow t = \frac{60}{11}\,\text{min}$$

The time that pass between a meeting and the other is one hour and $\frac{60}{11}$ min, i.e.

$$60\,\text{min} + \frac{60}{11}\,\text{min} = \frac{720}{11}\,\text{min}$$

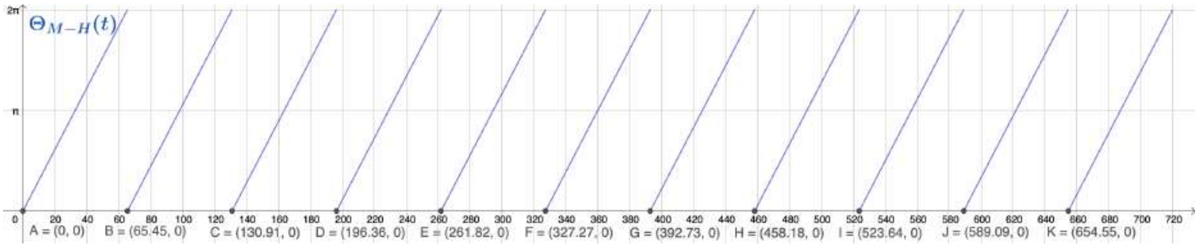
Given any instant we divide it by the meeting time and we take the floor of the result: this gives us the number of overlaps already happend.
In order to obtain an angle between 0 rad and 2π rad we subtract 2π for each overlap. The function that we were looking for is:

$$\Theta_{M-H}(t) = \frac{11}{360}\pi t - 2\pi \left\lfloor \frac{11}{720} t \right\rfloor$$

We can rewrite the equation above, with t measured in minutes:

$$\Theta_{M-H}(t) = 2\pi \left(\frac{11}{720} t - \left\lfloor \frac{11}{720} t \right\rfloor \right) = 2\pi \left\{ \frac{11}{720} t \right\} \tag{2.1}$$

This is a periodic function with period $T = \dfrac{720}{11}$ min



This is the graph of the function during half a day i.e. 720 min. The angle calculated by the fuction is the one betweeen the minute and hour hand therefore it can be a reflex angle.
In a similar way we get:

$$\Theta_{S-M}(t) = 2\pi \left\{ \frac{59}{60} t \right\} \tag{2.2}$$

$$\Theta_{S-H}(t) = 2\pi \left\{ \frac{719}{720} t \right\} \tag{2.3}$$

And finally the equations of the functions which indicate the angle between the hour/minute/second hands and the initial vertical position are

$$\Theta_{S}(t) = 2\pi \left\{ t \right\} \tag{2.4}$$

$$\Theta_{M}(t) = 2\pi \left\{ \frac{1}{60} t \right\} \tag{2.5}$$

$$\Theta_{H}(t) = 2\pi \left\{ \frac{1}{720} t \right\} \tag{2.6}$$

2.1 Overlapping times of two clock hands

In order to find when and where two clock hands meet (for example the minutes and hours hand) we have to solve the follow equation:

$$\Theta_{H}(t) = \Theta_{M}(t) \Longleftrightarrow \left\{ \frac{1}{720} t \right\} = \left\{ \frac{1}{60} t \right\} \tag{2.7}$$

If we call $t = 720x$ we get the easier equation

$$\{x\} = \{12x\}$$

We can seek the solutions in the interval $[0; 1[$ because of the periodicity of the fractional part function. We divide the interval $[0; 1[$ in 12 parts:

- if $0 \leq x < \dfrac{1}{12}$ the equation becomes $x = 12x$ and so $x = 0$

- if $\dfrac{1}{12} \leq x < \dfrac{2}{12}$ the equation becomes $x = 12x - 1$ and so $x = \dfrac{1}{11}$
- if $\dfrac{2}{12} \leq x < \dfrac{3}{12}$ the equation becomes $x = 12x - 2$ and so $x = \dfrac{2}{11}$
- ...
- if $\dfrac{11}{12} \leq x < \dfrac{12}{12}$ the equation becomes $x = 12x - 11$ and so $x = \dfrac{11}{11}$ (not in $[0;1[$).

Hence there are 11 solutions in $[0;1[$, i.e. the solution set is $S_x = \left\{\dfrac{k}{11}, \text{ with } k \in \mathbb{N} \mid k = 0, ..., 10\right\}$

Therefore the solution set of the eqaution 2.7 is $S_t = \left\{\dfrac{720}{11}k \mid 0 \leqslant k \leqslant 10\right\}$

This means that the 2 clock hands meet 11 times a day.

We call these times **overlapping times** t_k^{M-H}.

In the same way it's possible to find out the overlapping times of the hours and seconds hands and the minutes and seconds hands.

2.2 Overlapping times of the three clock hands

In order to find when and where the three hands meet we have to solve the following system:

$$\begin{cases} \Theta_H(t) = \Theta_M(t) \\ \Theta_H(t) = \Theta_S(t) \end{cases} \tag{2.8}$$

that is the system

$$\begin{cases} \left\{\dfrac{1}{720}t\right\} = \left\{\dfrac{1}{60}t\right\} \\ \left\{\dfrac{1}{720}t\right\} = \{t\} \end{cases}$$

If we call $t = 720x$ we get the easier system

$$\begin{cases} \{x\} = \{12x\} \\ \{x\} = \{720x\} \end{cases}$$

We have already solved the first equation in the previous section and we found in $[0;1[$ the solution set $S_1 = \left\{\dfrac{k}{11} \mid k = 0, ..., 10\right\}$. In the same way the solutions of the second equation in $[0;1[$ are 719 and the solution set is $S_2 = \left\{\dfrac{\lambda}{719} \mid \lambda = 0, ..., 718\right\}$.

The numbers 11 and 719 are coprime so $\dfrac{k}{11} = \dfrac{\lambda}{719}$ only for $k = \lambda = 0$ then the system has the only solution $x = 0$. Therefore the only solution of the system 2.8 is $t = 0$.

In conclusion we can say that the three hands clock meet just at 12 o'clock.

2.3 Right angle

In order to find out when two clock hands form a right angle, we set the equation that the describe the angle between two clock hands equal to $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ (the conjugate angle of the right angle). Let's analize the case with the hours and minutes hands:

$$\Theta_{M-H}(t) = \frac{\pi}{2} \quad \vee \quad \Theta_{M-H}(t) = \frac{3\pi}{2}$$

Let's now solve the two equations:

$$2\pi\left\{\frac{11}{720}t\right\} = \frac{\pi}{2} \quad \vee \quad 2\pi\left\{\frac{11}{720}t\right\} = \frac{3\pi}{2}$$

Which are:

$$\left\{\frac{11}{720}t\right\} = \frac{1}{4} \quad \vee \quad \left\{\frac{11}{720}t\right\} = \frac{3}{4}$$

Let's start with the first one, which means that $\frac{11}{720}t$ must be $\frac{1}{4}$ plus a natural number n:

$$\frac{11}{720}t = \frac{1}{4} + n \Rightarrow t = \frac{180}{11} + \frac{720}{11}n$$

As defined $0 \leqslant t < 720$ (t minutes in half a day) $\Rightarrow 0 \leqslant \frac{180}{11} + \frac{720}{11}n < 720$ that is:

$$\begin{cases} \frac{180}{11} + \frac{720}{11}n \geqslant 0 \Rightarrow n \geqslant -\frac{18}{72} \\ \\ \frac{180}{11} + \frac{720}{11}n < 720 \Rightarrow n < \frac{43}{4} \end{cases} \Longrightarrow 0 \leqslant n \leqslant 10$$

The solution set of the first equation is

$$S_{\frac{\pi}{2}} = \left\{\frac{180}{11} + \frac{720}{11}n, n \in \mathbb{N} \wedge n \leqslant 10\right\}$$

Following the same path for the second equation we obtain

$$S_{\frac{3\pi}{2}} = \left\{\frac{540}{11} + \frac{720}{11}n, n \in \mathbb{N} \wedge n \leqslant 10\right\}$$

So the solution set is the union of the two previous sets, i.e. the set

$$S = \left\{\frac{180}{11} + \frac{360}{11}n,\ n \in \mathbb{N}\, n \leqslant 21\right\}$$

Chapter 3

Geometrycal figures

3.1 Isosceles triangle

In order to discover when the triangle formed by two clock hands and the segment that connect the two end-points not in common is isosceles, we split the problem in two cases: the first one when the base of the triangle is the hours hand (h) and the two legs of the triangle are the minutes hand (m) and the segment, the second one when the base of the triangle is the minutes hand (m) and the hours hand (h) and the segment are the two legs of the triangle (we consider $h < m$).
In the first case the angles opposite to the legs are $\Theta_{M-H}(t) = \arccos\left(\frac{h}{2m}\right)$ or its conjugate while in the second case the angles are $\Theta_{M-H}(t) = \arccos\left(\frac{m}{2h}\right)$ or its conjugate. For instance, if $\frac{h}{m} = \frac{2}{3}$ than the angle becomes $\Theta_{M-H}(t) = \arccos\left(\frac{3}{4}\right) \vee \Theta_{M-H}(t) = \arccos\left(\frac{1}{3}\right)$.
As we find before the angle between the two clock hands as a function of time is:

$$\Theta_{M-H}(t) = 2\pi\left\{\frac{11}{720}t\right\}$$

So, we can write:

$$2\pi\left\{\frac{11}{720}t\right\} = \arccos\left(\frac{h}{2m}\right) \quad \vee \quad 2\pi\left\{\frac{11}{720}t\right\} = \arccos\left(\frac{m}{2h}\right)$$

$$t = \left(\frac{\arccos\left(\frac{h}{2m}\right)}{2\pi} + n\right)\frac{720}{11} \quad \vee \quad t = \left(\frac{\arccos\left(\frac{m}{2h}\right)}{2\pi} + n\right)\frac{720}{11}$$

Since t is defined for values among 0 and 720, we have:

$$0 < \left(\frac{\arccos\left(\frac{h}{2m}\right)}{2\pi} + n\right)\frac{720}{11} < 720 \vee 0 < \left(\frac{\arccos\left(\frac{m}{2h}\right)}{2\pi} + n\right)\frac{720}{11} < 720$$

$$n \leq 10$$

3.2 Rectangle

In order to find out when the hours and minutes hands act like two sides, and the seconds hand acts like the diagonal of this hypotetical rectangle we have to impose few conditions. First of all the angle between minute and hour's hand must be $\Theta_{M-H}(t) = \left(\frac{\pi}{2}\right) \vee \Theta_{M-H}(t) = \left(\frac{3\pi}{2}\right)$

We already find the solutions for this case, that are:

$$S = \left\{ t \in \mathbb{R} \wedge n \in \mathbb{N} \mid t = \frac{180}{11} + \frac{720}{11} n \vee t = \frac{540}{11} + \frac{720}{11} n \wedge n \leqslant 10 \right\}$$

As was said before the second hand is the diagonal and for that to happen we have to impose this condition:

$$s^2 = m^2 + h^2$$

We also have to impose:

$$s \sin(\Theta_{M-S}) = h \quad \vee \quad s \sin(\Theta_{H-S}) = m$$

$$\Theta_{M-S} = \arcsin\left(\frac{h}{s}\right) \quad \vee \quad \Theta_{M-S} = \arcsin\left(\frac{m}{s}\right)$$

So, knowing that:

$$\Theta_{M-S} = 2\pi \left\{ \frac{59}{60} t \right\}$$

We have that:

$$\arcsin\left(\frac{h}{s}\right) = 2\pi \left\{ \frac{59}{60} t \right\} \quad \vee \quad \arcsin\left(\frac{m}{s}\right) = 2\pi \left\{ \frac{719}{720} t \right\}$$

$$S = \left\{ t \in \mathbb{R} \wedge n \in \mathbb{N} \mid t = \frac{30 \arcsin\left(\frac{h}{s}\right)}{2\pi} + \frac{60}{59} n \quad \vee \quad t = \frac{30 \arcsin\left(\frac{m}{s}\right)}{2\pi} + \frac{60}{59} n \ \wedge \ n \leqslant 10 \right\}$$

Then, we can find a value of $\frac{h}{s}$ or a value of $\frac{m}{s}$ that make possible to find a value of t such that the rectangle could exist.

3.3 Parallelogram

We have three types of parallelograms that could be formed by the clock hands:

1 The sides are the hand of the minute and the seconds one, and one of the two diagonals is the hours hand.

2 The sides are the hand of the minute and the hours one, and one of the two diagonals is the seconds hand.

3 The sides are the hand of the seconds and the hours one, and one of the two diagonals is the minutes hand.

Before analizing every case let it be known that $s > m > h$.

Let's study the first case, where the diagonal is the hours hand:

$$\begin{cases} s^2 = m^2 + h^2 - 2mh\cos(\pi - \theta_{M-H}) \\ \theta_{M-H} = 2\pi\left\{\frac{11}{720}t\right\} \end{cases} \Rightarrow \begin{cases} \theta_{M-H} = \arccos\left(\frac{s^2 - m^2 - h^2}{2mh}\right) \\ \theta_{M-H} = 2\pi\left\{\frac{11}{720}t\right\} \end{cases}$$

so

$$2\pi\left\{\frac{11}{720}t\right\} = \arccos\left(\frac{s^2 - m^2 - h^2}{2mh}\right)$$

then

$$t = \frac{360\arccos\left(\frac{s^2 - m^2 - h^2}{2mh}\right)}{11\pi} + \frac{720}{11}n \quad \wedge \quad n \le 10$$

For example, if $\frac{s^2 - m^2 - h^2}{2mh} = -\frac{1}{2}$.
We have that:

$$2\pi\left\{\frac{11}{720}t\right\} = \frac{2}{3}\pi \quad \Rightarrow \quad \frac{11}{720}t = \frac{1}{3} + n \quad \Rightarrow \quad t = \frac{240}{11} + \frac{720}{11}n$$

In the second case we have:

$$\begin{cases} h^2 = s^2 + m^2 + 2ms\cos(\theta_{S-M}) \\ \Theta_{S-M}(t) = 2\pi\left\{\frac{59}{60}t\right\} \end{cases} \Rightarrow \begin{cases} \theta_{S-M} = \arccos\left(\frac{h^2 - s^2 - m^2}{2sm}\right) \\ \Theta_{S-M}(t) = 2\pi\left\{\frac{59}{60}t\right\} \end{cases}$$

so

$$2\pi\left\{\frac{59}{60}t\right\} = \arccos\left(\frac{h^2 - s^2 - m^2}{2sm}\right)$$

then

$$t = \frac{30 \arccos\left(\dfrac{h^2 - s^2 - m^2}{2sm}\right)}{59\pi} + \frac{60}{59}n \quad \wedge \quad n \leq 10$$

In the third case we have:

$$\begin{cases} m^2 = s^2 + h^2 + 2sh\cos\left(\theta_{S-H}\right) \\ \Theta_{S-H}(t) = 2\pi\left\{\dfrac{719}{720}t\right\} \end{cases} \qquad \Rightarrow \qquad \begin{cases} \theta_{S-H} = \arccos\left(\dfrac{m^2 + -s^2 - h^2}{2sh}\right) \\ \Theta_{S-H}(t) = 2\pi\left\{\dfrac{719}{720}t\right\} \end{cases}$$

so

$$2\pi\left\{\frac{719}{720}t\right\} = \arccos\left(\frac{m^2 + -s^2 - h^2}{2sh}\right)$$

then

$$t = \frac{360 \arccos\left(\dfrac{m^2 + -s^2 - h^2}{2sh}\right)}{719\pi} + \frac{720}{719}n \quad \wedge \quad n \leq 10$$

Chapter 4

Symmetries

Definition 1. *Given an angle α with $0 \leq \alpha < \pi$, r^α is the ray with origin in the clock center that forms an angle α with respect to the vertical line.* ***Symmetrycal time*** *t_k^α related to r^α is defined as the time in which the ray r^α or its opposite is the bisector of the angle between the hours and minutes clock hands.*

Definition 2. ***Symmetry axis*** *a^α is defined as the straight line to which the ray r^α belongs.*

4.1 Symmetry axis given a time

Given an hour, in order to find the symmetry axis between the two clock hands we have to calculate its inclination angle α that is the mean-value of the hands angles.

$$\alpha = \frac{\Theta_H + \Theta_M}{2}$$

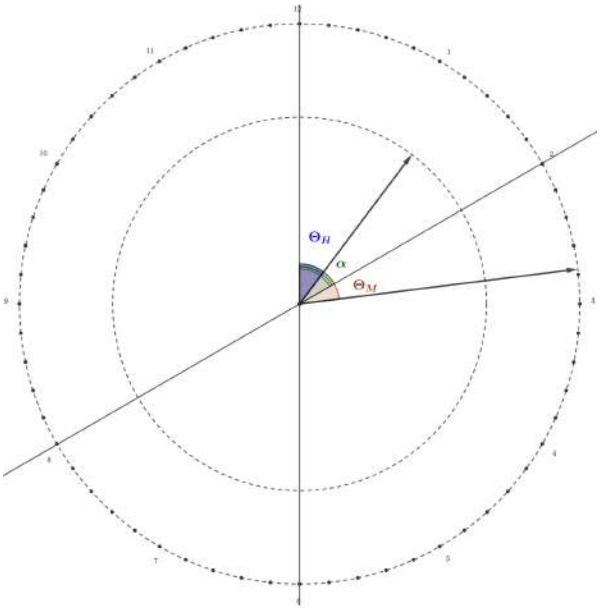
For each time there is only one symmetry axis.

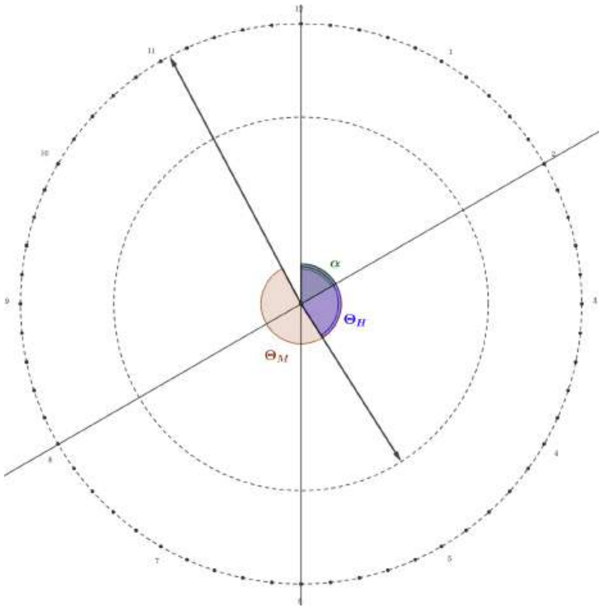
4.2 Symmetrycal times given an axis

Given an a^α, we want to find the symmetrycal times t_k^α.

Theorem 1. *Given a symmetry axis a^α, there are 13 different symmmetrycal time t_k^α, with k integer, $0 \leq k \leq 12$*

Proof. At the beginning we have to observe that there are 2 different situations: In the first case $\Theta_H + \Theta_M = 2\alpha$ and in the second case $\Theta_H + \Theta_M = 2\alpha + 2\pi$
As shown in the graphs below.





Therefore we have to solve the two equation:

$$2\pi\left\{\frac{1}{720}t\right\}+2\pi\left\{\frac{1}{60}t\right\}=2\alpha \quad \vee \quad 2\pi\left\{\frac{1}{720}t\right\}+2\pi\left\{\frac{1}{60}t\right\}=2\alpha+2\pi$$

$$\pi\left\{\frac{1}{720}t\right\}+\pi\left\{\frac{1}{60}t\right\}=\alpha$$

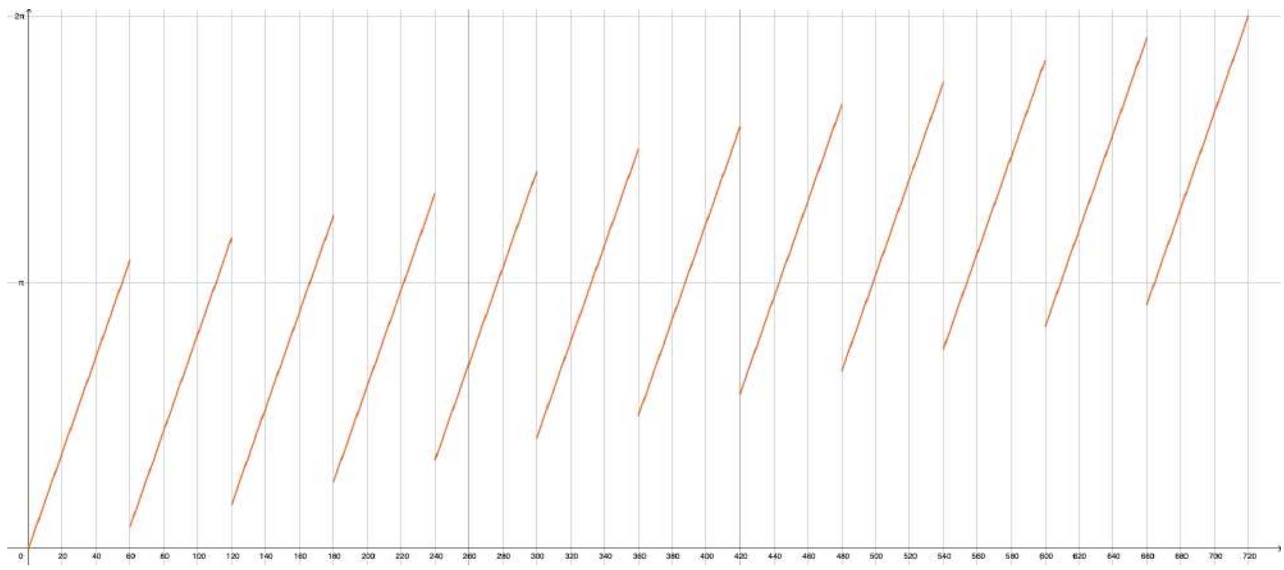
We divide the interval $[0;720[$ in 12 parts:

- if $0\leqslant t<60$, then: $\frac{13}{720}t=\frac{\alpha}{\pi}\Rightarrow t=\frac{720}{13}\left(\frac{\alpha}{\pi}\right)$ which is acceptable if $0\leqslant\alpha<\pi$
- if $60\leqslant t<120$, then: $\frac{13}{720}t=\frac{\alpha}{\pi}+1\Rightarrow t=\frac{720}{13}\left(\frac{\alpha}{\pi}+1\right)$ which is acceptable if $\frac{1}{12}\pi\leqslant\alpha<\pi$
- ...
- if $660\leqslant t<720$, then: $\frac{13}{720}t=\alpha+11\Rightarrow t=\frac{720}{13}\left(\frac{\alpha}{\pi}+11\right)$ which is acceptable if $\frac{11}{12}\leqslant\alpha<\pi$

  Therefore:

- if $0\leqslant\alpha<\frac{\pi}{12}$ there is 1 solution
- if $\frac{\pi}{12}\leqslant\alpha<\frac{2}{12}\pi$ there are 2 solutions
- ...
- if $\frac{11}{12}\pi\leqslant\alpha<\pi$ there are 12 solutions

Hence there are from 1 to 12 solutions depends on α.
Here is the graph of the function $y=\pi\left\{\frac{1}{720}x\right\}+\pi\left\{\frac{1}{60}x\right\}$ which has to be intersected by the graph of the function $y=\alpha$ with $0\leqslant\alpha<\pi$.



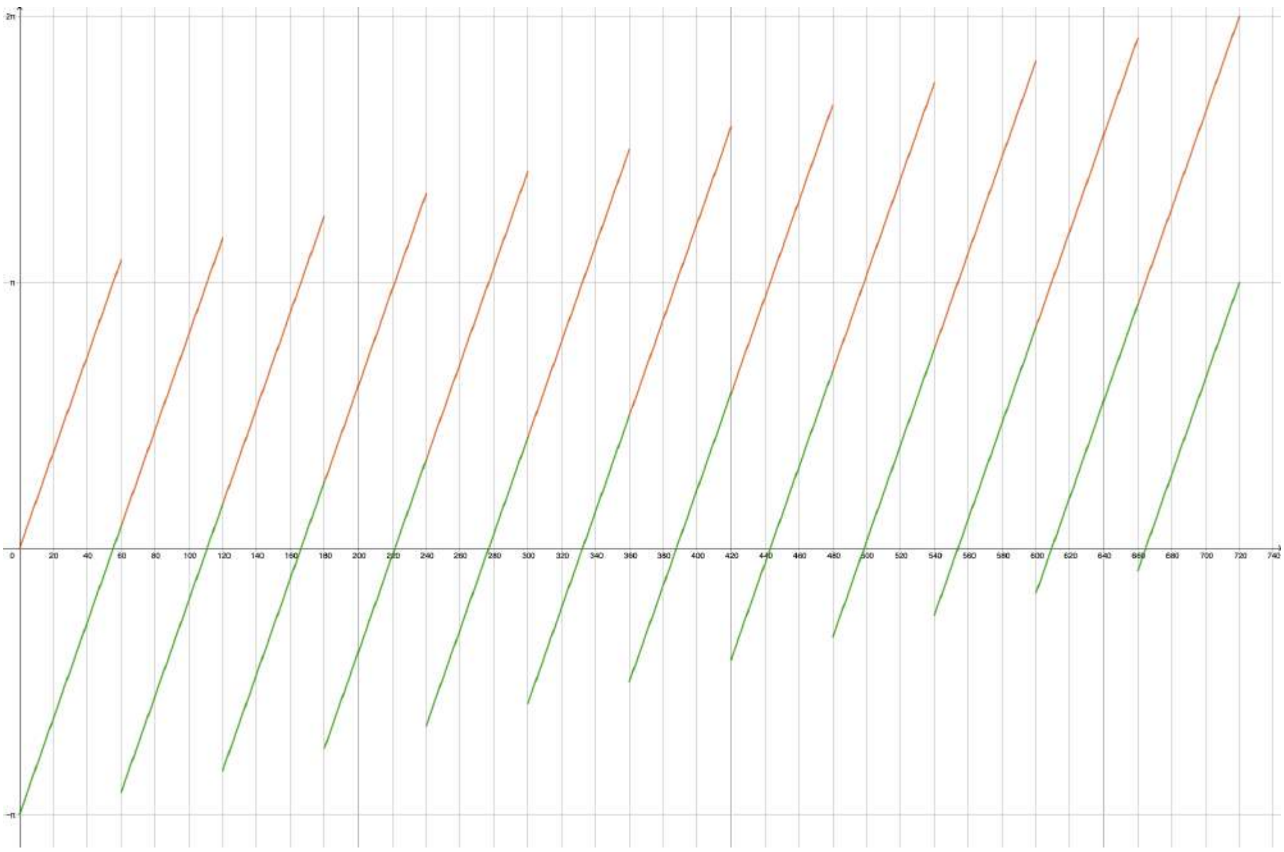
The second equation is $\pi\left\{\frac{1}{720}t\right\}+\pi\left\{\frac{1}{60}t\right\}=\alpha+\pi$ we divide the interval $[0,720[$ in 12 parts:

- if $0\leqslant\alpha<60$ then $t=\frac{720}{13}\left(\frac{\alpha}{\pi}+1\right)$ which is acceptable if $0\leqslant\alpha<\frac{1}{12}\pi$
- if $60\leqslant\alpha<120$ then $t=\frac{720}{13}\left(\frac{\alpha}{\pi}+2\right)$ which is acceptable if $0\leqslant\alpha<\frac{2}{12}\pi$
- ...
- if $660\leqslant\alpha<720$ then $t=\frac{720}{13}\left(\frac{\alpha}{\pi}+12\right)$ which is acceptable if $0\leqslant\alpha<\pi$

Therefore:

- if $0\leqslant\alpha<\frac{1}{12}\pi$ there are 12 solutions
- if $\frac{1}{12}\pi\leqslant\alpha<\frac{2}{12}\pi$ there are 11 solutions
- ...
- if $\frac{11}{12}\pi\leqslant\alpha<\pi$ there is 1 solution

Here are the graphs of the functions $y=\pi\left\{\frac{1}{720}x\right\}+\pi\left\{\frac{1}{60}x\right\}$ and $y=\pi\left\{\frac{1}{720}x\right\}+\pi\left\{\frac{1}{60}x\right\}-\pi$ that have to be intersected by the graph of the function $y=\alpha$ with $0\leqslant\alpha<\pi$.



Hence $\forall\alpha \mid 0 \leqslant \alpha < \pi$ there are 13 solutions of $\Theta_H + \Theta_M = 2\alpha \quad \vee \quad \Theta_H + \Theta_M = 2\alpha + 2\pi$ then there are 13 different symmetrycal times:

$$t_k^\alpha = \frac{720}{13}\left(\frac{\alpha}{\pi} + k\right) \text{ with } k \in \mathbb{N}\ k \leqslant 12$$

□

Theorem 2. *Let's define* $\eta_k^\alpha = \Theta_H(t_k^\alpha)$ *and* $\mu_k^\alpha = \Theta_M(t_k^\alpha)$*, given an axis* a^α*, given* h *and* m *the lenghts of the hours and minutes clock hands, the points* $A_k^\alpha(h\sin(\eta_k^\alpha); h\cos(\eta_k^\alpha))$ *are the vertices of a regular tridecagon inscribed in the circumenference of radius* h *with side* $2h\sin\left(\frac{\pi}{13}\right)$*. The points* $B_k^\alpha(m\sin\mu_k^\alpha); m\cos(\mu_k^\alpha))$ *are the vertices of a regular tridecagon inscribed in the circumenference of radius* m *with side* $2m\sin\left(\frac{\pi}{13}\right)$*.*

Proof. Given two symmetrycal times t_k^α and t_{k-1}^α let's calculate the difference between η_k^α and η_{k-1}^α

$$\eta_k^\alpha - \eta_{k-1}^\alpha = 2\pi\left\{\frac{1}{720}t_k^\alpha\right\} - 2\pi\left\{\frac{1}{720}t_{k-1}^\alpha\right\} =$$

$$2\pi\left\{\frac{1}{720}\cdot\frac{720}{13}\left(\frac{\alpha}{\pi}+k\right)\right\} - 2\pi\left\{\frac{1}{720}\cdot\frac{720}{13}\left(\frac{\alpha}{\pi}+k-1\right)\right\} =$$

$$2\pi\left\{\frac{\alpha}{13\pi}+\frac{k}{13}\right\} - 2\pi\left\{\frac{\alpha}{13\pi}+\frac{k}{13}-\frac{1}{13}\right\} =$$

$$2\pi\left(\frac{\alpha}{13\pi}+\frac{k}{13}-\left\lfloor\frac{\alpha}{13\pi}+\frac{k}{13}\right\rfloor - \frac{\alpha}{13\pi}-\frac{k}{13}+\frac{1}{13}+\left\lfloor\frac{\alpha}{13\pi}+\frac{k}{13}-\frac{1}{13}\right\rfloor\right) =$$

$$2\pi\left(\left\lfloor\frac{\alpha}{13\pi}+\frac{k}{13}-\frac{1}{13}\right\rfloor-\left\lfloor\frac{\alpha}{13\pi}+\frac{k}{13}\right\rfloor+\frac{1}{13}\right)$$

Because of $0\leqslant\alpha<\pi\ \wedge\ 0\leqslant k\leqslant 12$,

$$0\leqslant\frac{\alpha}{13\pi}+\frac{k}{13}<1\quad\wedge\quad-\frac{1}{13}\leqslant\frac{\alpha}{13\pi}+\frac{k}{13}-\frac{1}{13}<1$$

so

$$\left\lfloor\frac{\alpha}{13\pi}+\frac{k}{13}\right\rfloor=0\quad\wedge\quad\left\lfloor\frac{\alpha}{13\pi}+\frac{k}{13}-\frac{1}{13}\right\rfloor=0$$

Then the difference above becomes:

$$\eta_k^\alpha-\eta_{k-1}^\alpha=2\pi\left(\frac{1}{13}\right)=\frac{2}{13}\pi$$

This means that the 13 symmetrycal times divide the hours circumnference in 13 equal parts. Consequently A_k^α is after A_{k-1}^α in chronological order. For symmetry B_k^α is before B_{k-1}^α then let's now calculate the difference between μ_{k-1}^α and μ_k^α,

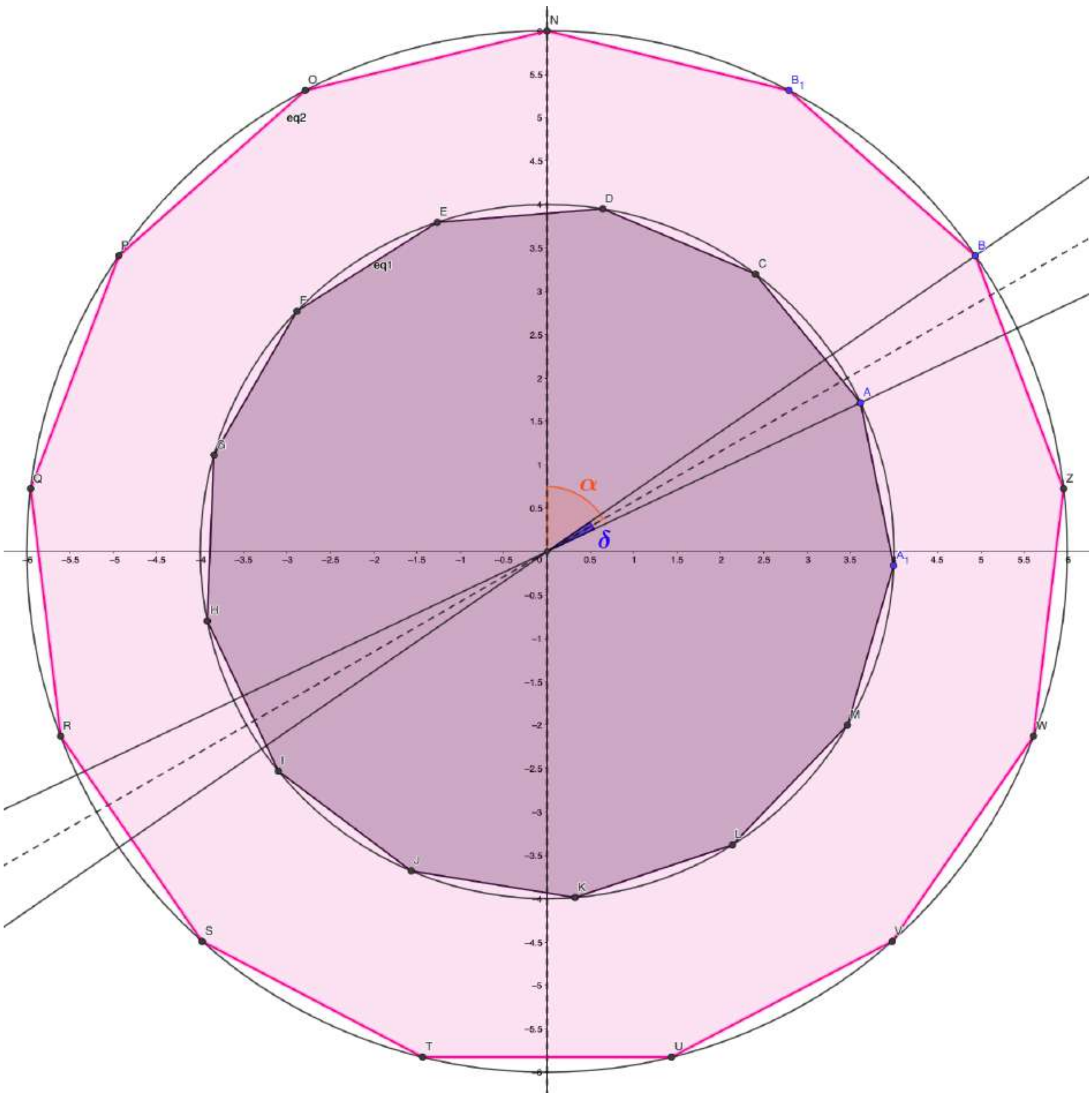
$$\mu_{k-1}^\alpha-\mu_k^\alpha=2\pi\left\{\frac{12}{13\pi}\alpha+\frac{12}{13}k-\frac{12}{13}\right\}-2\pi\left\{\frac{12}{13\pi}\alpha+\frac{12}{13}k\right\}=$$

$$2\pi\left(\left\lfloor\frac{12}{13\pi}\alpha+\frac{12}{13}k\right\rfloor-\left\lfloor\frac{12}{13\pi}\alpha+\frac{12}{13}k-\frac{12}{13}\right\rfloor-\frac{12}{13}\right)$$

Because the two arguments of the two floors differ of $-\frac{12}{13}$, the difference can be 0 or 1. So the difference above becomes:

$$2\pi\left(-\frac{12}{13}\right)=-\frac{24}{13}\pi\quad\vee\quad2\pi\left(\frac{1}{13}\right)=\frac{2}{13}\pi$$

Because of B_{k-1}^α is after B_k^α the difference $\mu_{k-1}^\alpha-\mu_k^\alpha$ can't be negative so $-\frac{24}{13}\pi$ is not accetable. This means that the 13 symmetrycal times divide the minutes circumnference in 13 equal parts.



In order to calculate the two side we use the chord theorem. So the chord, for the hour circumnference, is $2h \sin\left(\frac{\pi}{13}\right)$ and for the minutes circumnference is $2m \sin\left(\frac{\pi}{13}\right)$. □

Theorem 3. *The rotation angle δ between the two polygons is $\min\left\{\gamma, \frac{2}{13}\pi - \gamma\right\}$ with γ the angle between the first point A_k^α after r^α and the first point B_k before r^α.*

Proof. The first point A_k^α clockwise starting from the vertical line is A_0^α, so the angle $\eta_0^\alpha = \frac{2}{13}\alpha$.

Let $p = \left\lfloor \left(\frac{\alpha - \frac{2}{13}\alpha}{\frac{2}{13}\pi} \right) \right\rfloor = \left\lfloor \frac{11\alpha}{2\pi} \right\rfloor$, the point A_{p+1} is the first point after r_α and B_{p+1} is the first point before r^α, $\gamma = \eta_{p+1}^\alpha - \mu_{p+1}^\alpha$ then $\delta = \min\left\{\gamma, \frac{2}{13}\pi - \gamma\right\}$.

□

Corollary 1. *If one of the* 13 *different symmetrycal times is an overlapping time* t_k^{M-H} *then the rotation angle* $\delta = 0$.

Proof. In this case $\eta_{p+1}^{\alpha} = \mu_{p+1}^{\alpha}$ so $\gamma = 0$ and then $\delta = 0$. □

Notice that a time corresponds to a single axis meanwhile an axis corresponds to thirteen hours. This is the same relation between the 13^{th} power of a complex number and the thirteen roots of a complex number.

4.3 Hours and complex numbers

Theorem 4. *Given a symmetry axis* a^{α} *the points* A_k^{α} *in the hours circumnference are the 13 points in the circumneference of radius* h *in the Argand-Gauss plane which rapresent the thirteen complex roots of the complex number* z *with* $\arg(z) = \frac{13}{2}\pi - 2\alpha - 2q\pi$, $q = \left\lfloor \frac{13}{4} - \frac{\alpha}{\pi} \right\rfloor$, $|z| = h^{13}$ *and* B_k^{α} *are the 13 complex roots of the complex number* w *with* $\arg(w) = \arg(z) + 13\delta$, $|w| = m^{13}$.

Proof. In order to find the last A_k^{α} before the abscissa axis let's caluclate

$$q = \left\lfloor \frac{\frac{\pi}{2} - \frac{2}{13}\alpha}{\frac{2}{13}\pi} \right\rfloor = \left\lfloor \frac{13}{4} - \frac{\alpha}{\pi} \right\rfloor$$

then A_q^{α} is the last point before the abscissa axis.
If we consider the 13 roots of a number $z \in \mathbb{C}$ and we call z_0 the first root then $\arg(z_0) = \frac{\arg(z)}{13}$.
To find $\arg(z)$ of the complex number z whose roots form the same tridecagon formed by the 13 A_k^{α} let's equal the angles of z_0 and A_q^{α}:

$$t_q^{\alpha} = \frac{720}{13}\left(\frac{\alpha}{\pi} + q\right) \quad \Rightarrow \quad \Theta_H(t_q^{\alpha}) = \frac{2}{13}\alpha + \frac{2}{13}q\pi$$

$$\frac{2}{13}\alpha + \frac{2}{13}q\pi = \frac{\pi}{2} - \frac{\arg(z)}{13} \quad \Rightarrow \quad \arg(z) = \frac{13}{2}\pi - 2\alpha - 2q\pi$$

The first root w_0 of the complex number w has an angle $\frac{\arg(w)}{13}$ and it has to be the angle of z_0 plus the rotation angle δ in order to be a vertice of the minutes tridecagon.

$$\frac{\arg(w)}{13} = \frac{\arg(z)}{13} + \delta \Rightarrow \arg(w) = \arg(z) + 13\delta$$

The complex roots belong to a circumnference of radius $\sqrt[13]{z}$ (the real square) so $h = \sqrt[13]{z}$ then $|z| = h^{13}$ and in the same way $|w| = m^{13}$. □

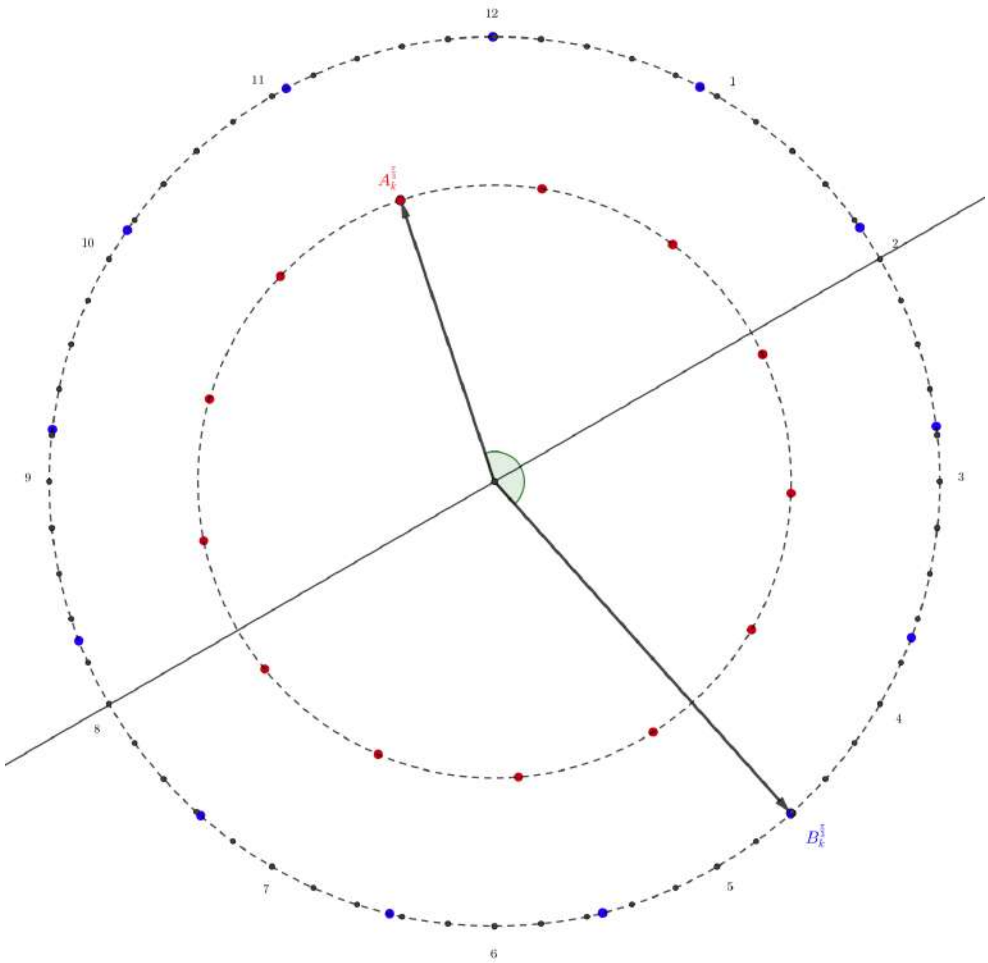
4.4 Examples

We are now going to analyze how it's possible to connect a given symmetry axis a^{α} with 2 complex numbers

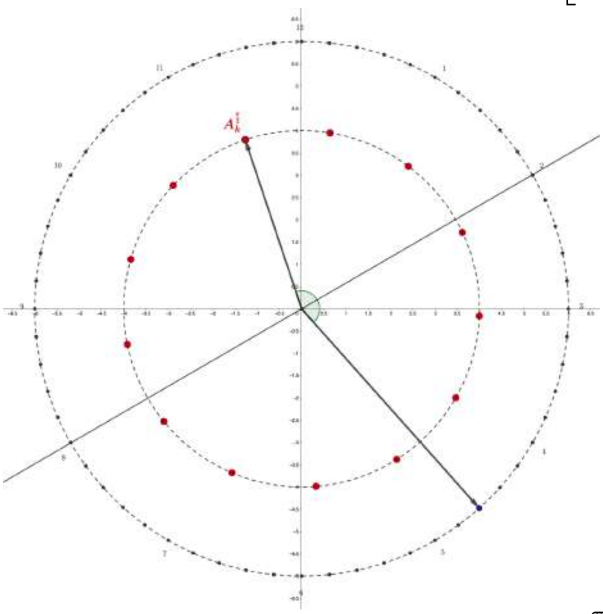
Example 1 For $\alpha = \dfrac{\pi}{3}$:

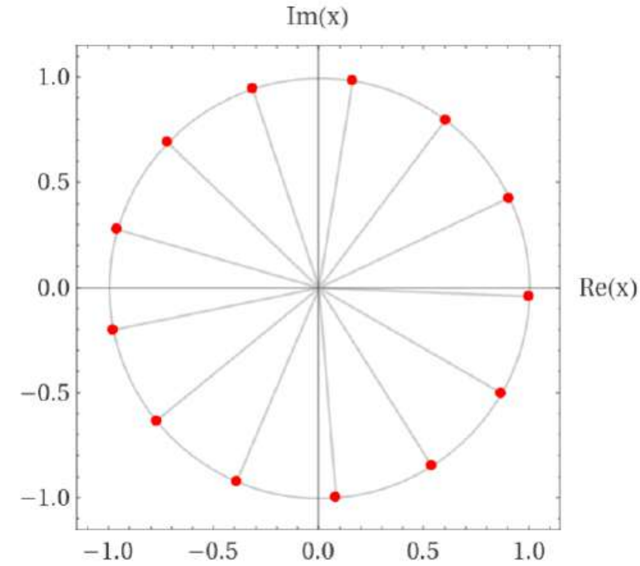
The 13 symmetrycal time are $t_k^{\frac{\pi}{3}} = \dfrac{720}{13}\left(\dfrac{1}{3} + k\right)$ with $0 \leqslant k \leqslant 12$

And we have drown them in the graph below:



Thanks to the theorem 4 we calculate: $q = \left\lfloor \frac{13}{4} - \frac{1}{3} \right\rfloor = 2$ and $\arg(z) = \frac{13}{2}\pi - \frac{2}{3}\pi - 4\pi = \frac{11}{6}\pi$



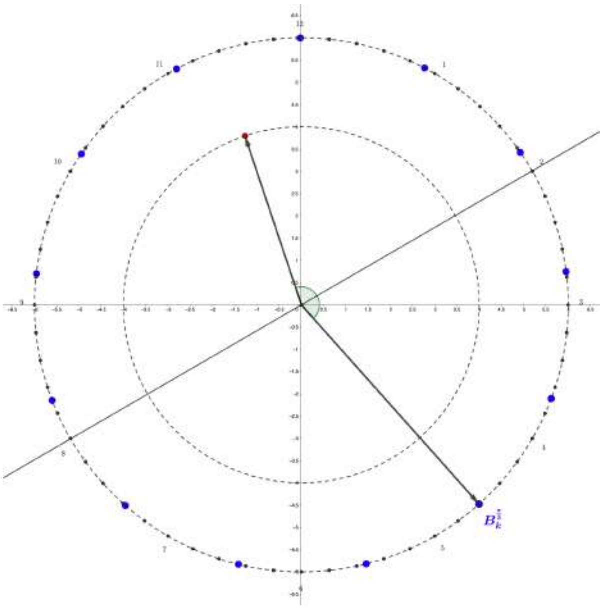


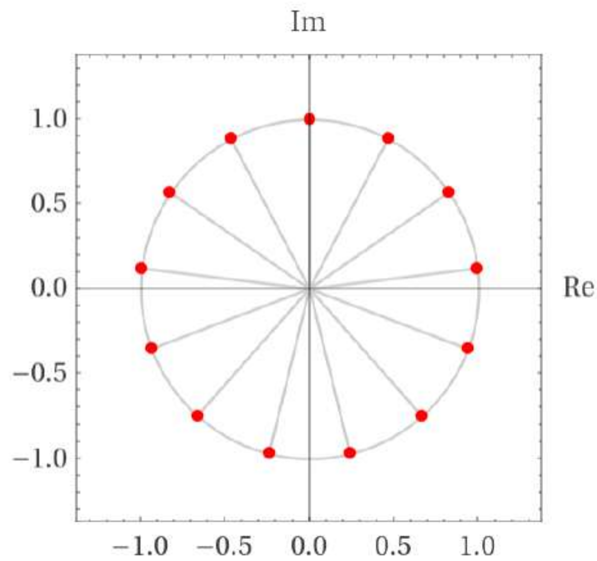
Thanks to theorem 3 let's calculate $p = \left\lfloor \frac{11 \cdot \frac{\pi}{3}}{2\pi} \right\rfloor = 1$

$$\gamma = \eta_2^{\frac{\pi}{3}} - \mu_2^{\frac{\pi}{3}} = 2\pi \left\{ \frac{1}{13} \left(\frac{1}{3} + 2 \right) \right\} - 2\pi \left\{ \frac{12}{13} \left(\frac{1}{3} + 2 \right) \right\} = \frac{14}{39}\pi - \frac{4}{13}\pi = \frac{2}{39}\pi$$

$$\delta = \min\left(\gamma; \frac{2}{13}\pi - \gamma \right) = \min\left(\frac{2}{39}\pi; \frac{4}{39}\pi \right) = \frac{2}{39}\pi$$

$$\arg(w) = \frac{11}{6}\pi + \frac{2}{3}\pi = \frac{5}{2}\pi$$

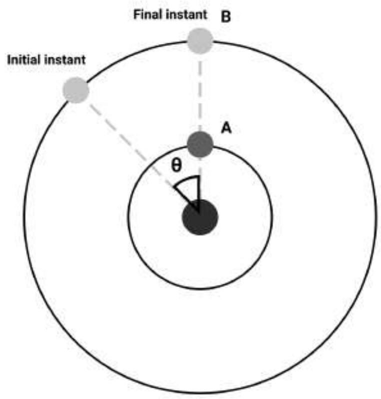




Chapter 5

The interplanetary voyage

Starting from the clock problem, we want to determine the angular position and the time at which it would be more advantageous to launch a spaceship from one planet (i.e. the Earth) to reach another (i.e. Mars) with the shortest possible distance. In the approximation of a circular orbit it is evident that the shortest path is radial. However, it must be considered that the target planet will move during the time of the journey; consequently the trip must be anticipated by the time necessary to cover this distance as illustrated below.



5.1 Spaceship in uniform motion

At first we dealt with this problem by assuming that the motion of the spaceship was uniform, thus neglecting the initial period of acceleration. The problem was treated in polar coordinates by writing the equations of motion in parametric form as a function of time, that is the distance $r(t)$ and the angle $\theta(t)$ with respect to the centre of the Sun. These equations were written separately for the spaceship that departed from Earth

$$\begin{cases} r(t) = R_{\oplus} + vt \\ \theta(t) = 0 \end{cases}$$

and for the planet Mars:

$$\begin{cases} r(t) = R_M \\ \theta(t) = \theta_0 + \omega_M t \end{cases}$$

In the equations above v is the constant speed of the spaceship, $R_\oplus$ the mean radius of the Earth's orbit, R_M the mean radius of the Martian orbit, θ_0 the angle of advance of Mars with respect to the Earth and ω_M the angular velocity of Mars. The travel time and the angle of advance can now be obtained by equating member to member the previous systems of equations, from which it follows

$$\begin{cases} R_\oplus + vt = R_M \\ 0 = \theta_0 + \omega_M t \end{cases}$$

So we obtain

$$\begin{cases} t = \dfrac{R_M - R_\oplus}{v} \\ \theta_0 = \dfrac{R_\oplus - R_M}{v}\,\omega_M \end{cases}$$

	Earth	**Mars**
Orbital circle $\mathcal{C}(km)$	924375700	1429000000
Orbital circle $\mathcal{C}(au)$	6,179	9,552
Orbital radius R(km)	147118962	227432414
Orbital radius R(au)	0,9834	1,52
Orbital period T(d)	365,26	686,96
Angular speed $\omega(rad/d)$	$1,72 \cdot 10^{-2}$	$9,14 \cdot 10^{-3}$

Table 5.1: Some orbital data for the Earth and Mars.

Considering the orbital for the Earth and Mars, summarized in table 5.1, and a speed of $1.1 \cdot 10^4 m/s$ (comparable with the escape velocity from the planet Earth and compatible with those currently reachable by existing space vectors), it yelds a travel time

$$t = \frac{R_M - R_\oplus}{v} \simeq 84\,d$$

and a lead angle

$$\theta_0 = \frac{R_\oplus - R_M}{v}\,\omega_M \simeq 0,765\,rad \simeq 44°$$

5.2 Escape velocity and acceleration

In the following it was considered the need to include an initial period of acceleration in the motion of the spaceship. Rather than dealing with the departure of the spaceship directly from the ground, we hypothesized that the same was previously placed in a stable orbit and subsequently launched at the appropriate time in the established direction.

In order to obtain a realistic estimate of the acceleration time needed, we determined that it was enough for the spaceship to reach the escape velocity from planet Earth. The latter is defined as the speed that a body must have to escape the gravitational field of a planet, that is, the speed it should have at the time of departure to reach an infinite distance from it (where its potential energy tends to zero) with a speed that is canceled asymptotically.

Considering the kinetic and potential energies of the spaceship at the time of departure, one obtains a total mechanical energy

$$E_{mi} = E_{ki} + U_{gi} = \frac{1}{2}mv_i^2 - \frac{GmM_\oplus}{R_\oplus + h_i}$$

where $M_\oplus$ and $R_\oplus$ point respectively out to the mass and the radius of the Earth, and G is the gravitational constant.

Requiring that both cancel when the spaceship is finally out of the influence of the Earth's gravitational field

$$E_{mf} = E_{kf} + U_{gf} = 0 + 0$$

by means of the conservation of mechanical energy we obtain that

$$\frac{1}{2}mv_i^2 - \frac{GM_\oplus m}{R_\oplus + h_i} = 0 \quad \Rightarrow \quad v_i = \sqrt{\frac{2GM_\oplus}{R_\oplus + h_i}}$$

Using the values in the following table

$G\left(\frac{N\,m^2}{kg^2}\right)$	$M_\oplus(kg)$	$R_\oplus(m)$
$6,67 \cdot 10^{-11}$	$5,9726 \cdot 10^{24}$	$6,372797 \cdot 10^6$

and an initial height $h_i = 400\,km = 4 \cdot 10^5\,m$, compatible with the average flight altitude of the International Space Station, one obtains

$$v_i = \sqrt{\frac{2GM_\oplus}{R_\oplus + h_i}} \simeq 10846\,m/s$$

which is in good agreement with the value previously used for the uniform motion.

In order for the orbit of the spaceship at the reference altitude to be stable, its initial speed must respect the equation of gravitational equilibrium, i.e. the corresponding centrifugal force must be equal and opposite to the gravitational attraction force:

$$\frac{mv_0^2}{R_\oplus + h_i} = \frac{GmM_\oplus}{(R_\oplus + h_i)^2} \quad \Rightarrow \quad v_0 = \sqrt{\frac{GM_\oplus}{R_\oplus + h_i}} = 7669,4\,m/s$$

Assuming that the departure of the spacecraft occurs when it, in its orbital motion around the Earth, is directed towards Mars, and neglecting the radius $R_\oplus + h_i$ of its orbit with respect to the distance between the two planets, the launch can be considered radial as assumed in section 5.

Considering a spaceship capable of maintaining a constant acceleration of about $3g$ (which is compatible with those currently reachable by existing space vectors), the time necessary to reach the required escape velocity is

$$t_{acc} = \frac{v_i - v_0}{3g} \simeq 108\,s$$

that is lesser than 2 minutes, and obviously negligible compared to the 84 days estimated for the journey. This justifies the fact that the motion of the spaceship can be considered uniform for the entire duration of the journey.

5.3 Equations of motion and angle between planets

Let us now try to obtain the hourly law which expresses the dependence on the time of the angle formed by the vector rays that identify the two planets with respect to the Sun.

Following the approach seen in chapter 2, starting from an instant t_0 in which the two planets are aligned with the Sun, we first want to find the angle traveled by the slowest of the two (Mars in our example) when the other one has already traveled a complete orbit. This *initial phase* is given by

$$\varphi = 2\pi \frac{\omega_M}{\omega_\oplus} = 2\pi \frac{T_\oplus}{T_M} = 3,34\, rad$$

having used the orbital periods of the planets shown in table 5.1. The time at which the next alignment of the planets will take place, measured from the moment when the fastest one (the Earth in our example) has already completed a revolution, will be obtained by equating the angles traveled

$$\varphi + \omega_M t_1 = \omega_\oplus t_1$$

from which it results:

$$t_1 = \frac{\varphi}{\omega_\oplus - \omega_M} = 2\pi \frac{T_\oplus/T_M}{\frac{2\pi}{T_\oplus} - \frac{2\pi}{T_M}} = \frac{2\pi}{2\pi} \frac{T_\oplus/T_M}{\frac{T_M - T_\oplus}{T_\oplus T_M}} = \frac{T_\oplus}{T_M} \frac{T_\oplus T_M}{T_M - T_\oplus} = \frac{T_\oplus^2}{T_M - T_\oplus} = 414,12\, d$$

The total time T_1 elapsed since the previous alignment will then be obtained by adding the period of the fastest planet to this

$$T_1 = t_1 + T_\oplus = \frac{T_\oplus^2 + T_M T_\oplus - T_\oplus^2}{T_M - T_\oplus} = \frac{T_M T_\oplus}{T_M - T_\oplus} = \left(\frac{1}{T_\oplus} - \frac{1}{T_M} \right)^{-1} = 779,38\, d \simeq 780\, d$$

The angle Θ between the planets will be covered with an angular velocity Ω such that

$$\Theta(t) = \Omega\, t$$

which must return a round angle after the time T_1. The resulting Ω is therefore given by:

$$\Omega = \frac{2\pi}{T_1} = 8,06 \cdot 10^{-3} \frac{rad}{d} \simeq \frac{8}{1000} \frac{rad}{d}$$

The angle θ we want to obtain in the end must be included in the $[0; 2\pi]$ range, that is the number N of round angles already covered must be subtracted from Θ. This last quantity can be expressed as a function of time as the whole part of the t/T_1 ratio:

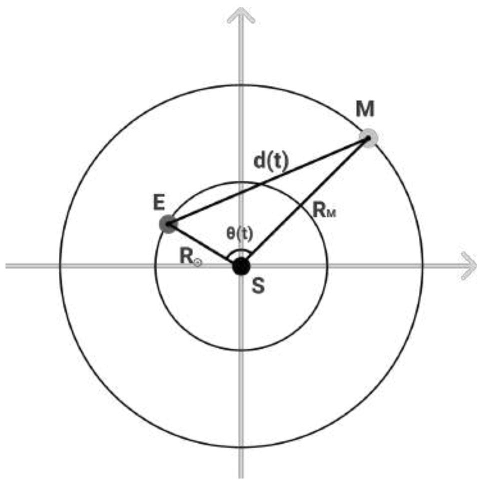
$$N = \left\lfloor \frac{t}{T_1} \right\rfloor$$

In conclusion the sought hourly law can be written as

$$\theta(t) = \Theta(t) - 2N\pi = \Omega\, t - 2\pi \left\lfloor \frac{t}{T_1} \right\rfloor \simeq \frac{8}{1000} \frac{rad}{d} \cdot t - 2\pi \left\lfloor \frac{t}{780\, d} \right\rfloor$$

5.4 Distance between planets

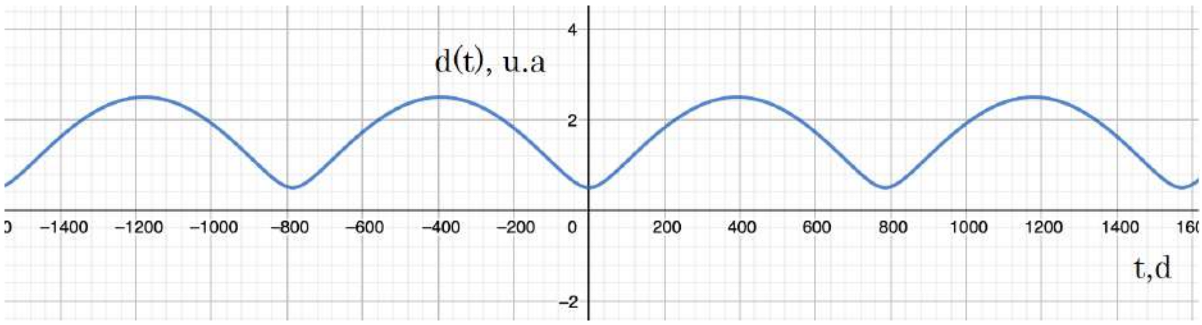
Another problem we faced was that of determining the distance between the two planets as a function of the angle $\theta(t)$. Let us consider the triangle SEM shown in the figure. It will have sides $R_\oplus$, R_M and d, where d is the Earth-Mars distance.



In the frame of reference in which the Earth is stationary with respect to the Sun the angle between the planets is just $\theta(t)$. By Carnot's theorem it follows that:

$$d = \sqrt{R_\oplus^2 + R_M^2 - 2R_\oplus R_M \cos\theta(t)}$$

The distance between the two planets thus obtained is represented below in astronomical units as a function of time measured in days.



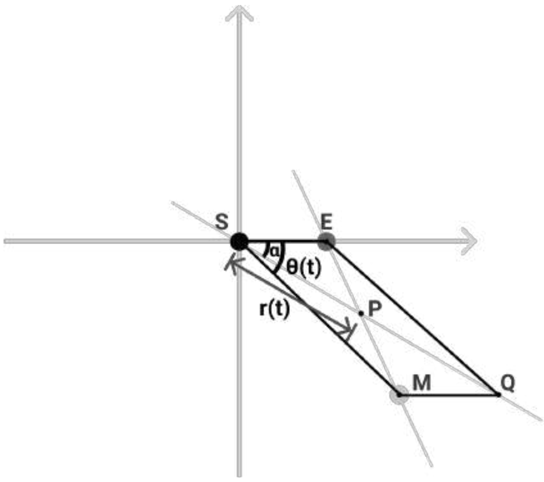
As it is evident, the Earth-Mars distance goes between a minimum of $\simeq 0,5\,a.u.$ when the two planets are aligned on the same side of the Sun, and a maximum of $\simeq 2,5\,a.u.$ when instead they are on opposite sides.

5.5 Trajectory of the midpoint

A more complex problem related to the previous one may be that of determining the trajectory of the Earth-Mars midpoint. The instantaneous position of the midpoint could be obtained in Cartesian coordinates as the average of those of the two planets.

$$\begin{cases} x(t) = \dfrac{x_\oplus + x_M}{2} = \dfrac{R_\oplus \cos\theta_\oplus + R_M \cos\theta_M}{2} = \dfrac{R_\oplus \cos(\omega_\oplus t) + R_M \cos(\omega_M t)}{2} \\ y(t) = \dfrac{y_\oplus + y_M}{2} = \dfrac{R_\oplus \sin\theta_\oplus + R_M \sin\theta_M}{2} = \dfrac{R_\oplus \sin(\omega_\oplus t) + R_M \sin(\omega_M t)}{2} \end{cases}$$

On the other hand, this approach would not have an obvious link with the original problem. Therefore we preferred to determine the position of the midpoint in polar coordinates $(r(t); \alpha(t))$ in the reference system in which the Earth is stationary with respect to the Sun.



In this frame of reference the Sun is placed on the origin and the Earth in the position $(1\,a.u.;\,0)$ while Mars orbits on a circle of radius $R_M \simeq 1,5\,a.u.$ The angle $\theta(t)$ must also be considered negative since the angular velocity of Mars is less than that of the Earth. The midpoint will be given by the intersection between the diagonals of the parallelogram of sides SE and SM. The distance $r(t)$ of the midpoint from the Sun is given by half of the diagonal SQ and can be obtained again using Carnot's theorem on the triangle SEQ.

$$r(t) = \frac{\overline{SQ}}{2} = \frac{\sqrt{R_\oplus^2 + R_M^2 - 2R_\oplus R_M \cos(\pi - \theta(t))}}{2} = \frac{\sqrt{R_\oplus^2 + R_M^2 + 2R_\oplus R_M \cos\theta(t)}}{2}$$

To find the position of the midpoint in polar coordinates, it is also necessary to determine the angle $\alpha(t)$ formed by the segment between the Sun and the midpoint and the Sun-Earth vector radius. Once again it is possible to use Carnot's theorem on the triangle QSE

$$R_M^2 = R_\oplus^2 + \overline{SQ}^2 - 2R_\oplus \overline{SQ} \cos\alpha(t)$$

which gives

$$\cos\alpha(t) = \frac{R_\oplus^2 - R_M^2 + \overline{SQ}^2}{2R_\oplus \overline{SQ}} = \frac{R_\oplus^2 - R_M^2 + 4r^2(t)}{4R_\oplus r(t)} = \frac{R_\oplus + R_M \cos\theta(t)}{\sqrt{R_\oplus^2 + R_M^2 + 2R_\oplus R_M \cos\theta(t)}}$$

The Cartesian coordinates $x(t)$ and $y(t)$ of the midpoint will be given by

$$\begin{cases} x(t) = r(t)\cos\alpha(t) \\ y(t) = r(t)\sin\alpha(t) \end{cases}$$

Consequently, it will be necessary to express $\sin\alpha(t)$ as a function of the cosine of the angle. Using the fundamental identity of goniometry, we obtain

$$\sin\alpha(t) = \pm\sqrt{1 - \cos^2\alpha(t)} = \pm\sqrt{1 - \frac{R_\oplus^2 + R_M^2 \cos^2\theta(t) + 2R_\oplus R_M \cos\theta(t)}{R_\oplus^2 + R_M^2 + 2R_\oplus R_M \cos\theta(t)}} =$$

$$= \pm\sqrt{\frac{R_M^2(1 - \cos^2\theta(t))}{R_\oplus^2 + R_M^2 + 2R_\oplus R_M \cos\theta(t)}} = \pm\frac{R_M \sin\theta(t)}{\sqrt{R_\oplus^2 + R_M^2 + 2R_\oplus R_M \cos\theta(t)}}$$

It makes no sense to consider negative roots as the sine of $\alpha(t)$ is definitely positive as long as the angle between SE and SP is between 0 and π. But this is always the case since the midpoint P is inside the convex angle ESM. In this way the Cartesian coordinates result:

$$\begin{cases} x(t) = r(t)\cos\alpha(t) = \dfrac{R_\oplus + R_M\cos\theta(t)}{2} \\ y(t) = r(t)\sin\alpha(t) = \dfrac{R_M\sin\theta(t)}{2} \end{cases} \tag{5.1}$$

The corresponding Cartesian equation can be easily obtained as it follows:

$$\begin{cases} x(t) = r(t)\cos\alpha(t) = \dfrac{R_\oplus + R_M\cos\theta(t)}{2} \\ y(t) = r(t)\sin\alpha(t) = \dfrac{R_M\sin\theta(t)}{2} \end{cases} \Rightarrow \begin{cases} 2x(t) - R_\oplus = R_M\cos\theta(t) \\ 2y(t) = R_M\sin\theta(t) \end{cases}$$

$$\begin{cases} 4x^2 - 4R_\oplus x + R_\oplus^2 = R_M^2\cos^2\theta(t) \\ 4y^2 = R_M^2\sin^2\theta(t) \end{cases} \Rightarrow \quad 4x^2 + 4y^2 - 4R_\oplus x = R_M^2 - R_\oplus^2$$

This is obviously the equation of a circle, which can be written as

$$x^2 - R_\oplus x + \frac{1}{4}R_\oplus^2 + y^2 = \frac{R_M^2 - R_\oplus^2}{4} + \frac{1}{4}R_\oplus^2 \quad \Rightarrow \quad \left(x - \frac{1}{2}R_\oplus\right)^2 + y^2 = \frac{R_M^2}{4}$$

In this form it is clear that the center of the circumference has coordinates $C_P\left(\frac{1}{2}R_\oplus\,, 0\right)$ and its radius is equal to $R_P = \frac{1}{2}R_M$.

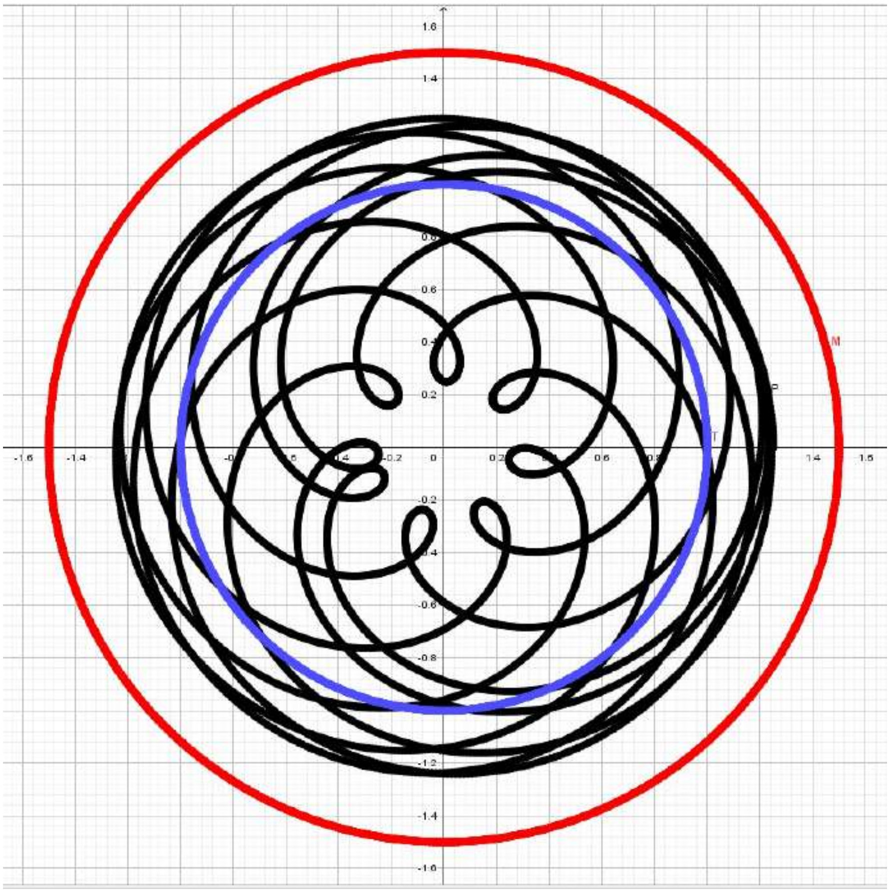
If we now also want to consider the rotation of the Earth, the new coordinates of the midpoint will be obtained by adding the Earth's rotation angle to α.

$$\begin{cases} x(t) = r(t)\cos(\alpha(t) + \omega_T t) \\ y(t) = r(t)\sin(\alpha(t) + \omega_T t) \end{cases}$$

By expanding sine and cosine with the angle addition formulas and using the relations (5.1) it follows that

$$\begin{cases} x(t) = \dfrac{1}{2}\left(R_\oplus + R_M\cos\theta(t)\right)\cos(\omega_T t) - \dfrac{1}{2}R_M\sin\theta(t)\sin(\omega_T t) \\ y(t) = \dfrac{1}{2}\left(R_\oplus + R_M\cos\theta(t)\right)\sin(\omega_T t) + \dfrac{1}{2}R_M\sin\theta(t)\cos(\omega_T t) \end{cases}$$

These parametric equations produce the curve represented (in black) in the following figure, together with the Earth's orbit (in blue) and the Martian one (in red).



Furthermore, by expanding the term in parentheses, one obtains

$$\begin{cases} x(t) = \frac{1}{2}R_\oplus \cos(\omega_T t) + \frac{1}{2}R_M \cos\theta(t)\cos(\omega_T t) - \frac{1}{2}R_M \sin\theta(t)\sin(\omega_T t) \\ y(t) = \frac{1}{2}R_\oplus \sin(\omega_T t) + \frac{1}{2}R_M \cos\theta(t)\sin(\omega_T t) + \frac{1}{2}R_M \sin\theta(t)\cos(\omega_T t) \end{cases}$$

which can also be written as

$$\begin{cases} x(t) = \frac{1}{2}R_M \cos(\theta(t) + \omega_T t) + \frac{1}{2}R_\oplus \cos(\omega_T t) \\ y(t) = \frac{1}{2}R_M \sin(\theta(t) + \omega_T t) + \frac{1}{2}R_\oplus \sin(\omega_T t) \end{cases}$$

Finally, by noticing that $\theta(t) + \omega_T t$ is the rotation angle of Mars $\omega_M t$ it results

$$\begin{cases} x(t) = \frac{1}{2}R_M \cos(\omega_M t) + \frac{1}{2}R_\oplus \cos(\omega_T t) = \frac{R_M \cos(\omega_M t) + R_\oplus \cos(\omega_T t)}{2} \\ y(t) = \frac{1}{2}R_M \sin(\omega_M t) + \frac{1}{2}R_\oplus \sin(\omega_T t) = \frac{R_M \sin(\omega_M t) + R_\oplus \sin(\omega_T t)}{2} \end{cases}$$

which is exactly the arithmetic mean between the positions of the Earth and Mars (5.5).

Chapter 6

The simulations

In this section we have created two simulations concerning the Solar System using Unity, a widespread graphics software for creating animations, and also employing programming parts in the $C\sharp$ language (C sharp). In the first simulation we have drawn the graph of the midpoint of a segment which connects two planets orbiting around the Sun. In the second one we have simulated, in a approximate way, an interplanetary voyage from Earth to another planet of the Solar System.

6.1 First simulation: graph of the midpoint

The first aim is to plot the midpoint of an imaginary segment between two planets in the Solar System. During the creation of this simulation we analyzed fours aspects: - Creation of a 3D scaled model of the Solar System - Programming of the motion of the planets - Creation of the segment and plotting of its midpoint - Programming of the various menus for the selection and choice of the planets.

In developing these four sections we created many pieces of code, especially for the motion of the planets by configuring parameters such as the rotations and the revolution of the planets. For the midpoint, instead, we used packages already present on Unity.

6.2 Second simulation: the interplanetary voyage

For the second simulation we started from the results obtained in the previous chapters concerning the interplanetary voyage. We decide to leave out the calculation of fuel consumption of the spaceship or the long distances in order to simplify the analysis. We set ourselves these goals: - Graphics and 3D models - Algorithms and programming - Real aspects

Regarding the graphics and algorithms we reused parts of the previous simulation, such as the planetary motion algorithms, modifying some image rendering factors in order to make the simulation look more photorealistic. As regards the part of physics we focused on the "gravitational slingshot". It can be used to accelerate a spacecraft, that is, increase or decrease its speed or redirect its path. To create this section we therefore used the "IA" package of Unity which allows the automatic movement of objects which we optimized by inserting the orbits so that the spaceship could use them to recreate the gravitational slingshot and reach the destination. For the development of these orbits we used parameters similar to the real one (for example the orbit of Jupiter is far larger than that of the Earth)

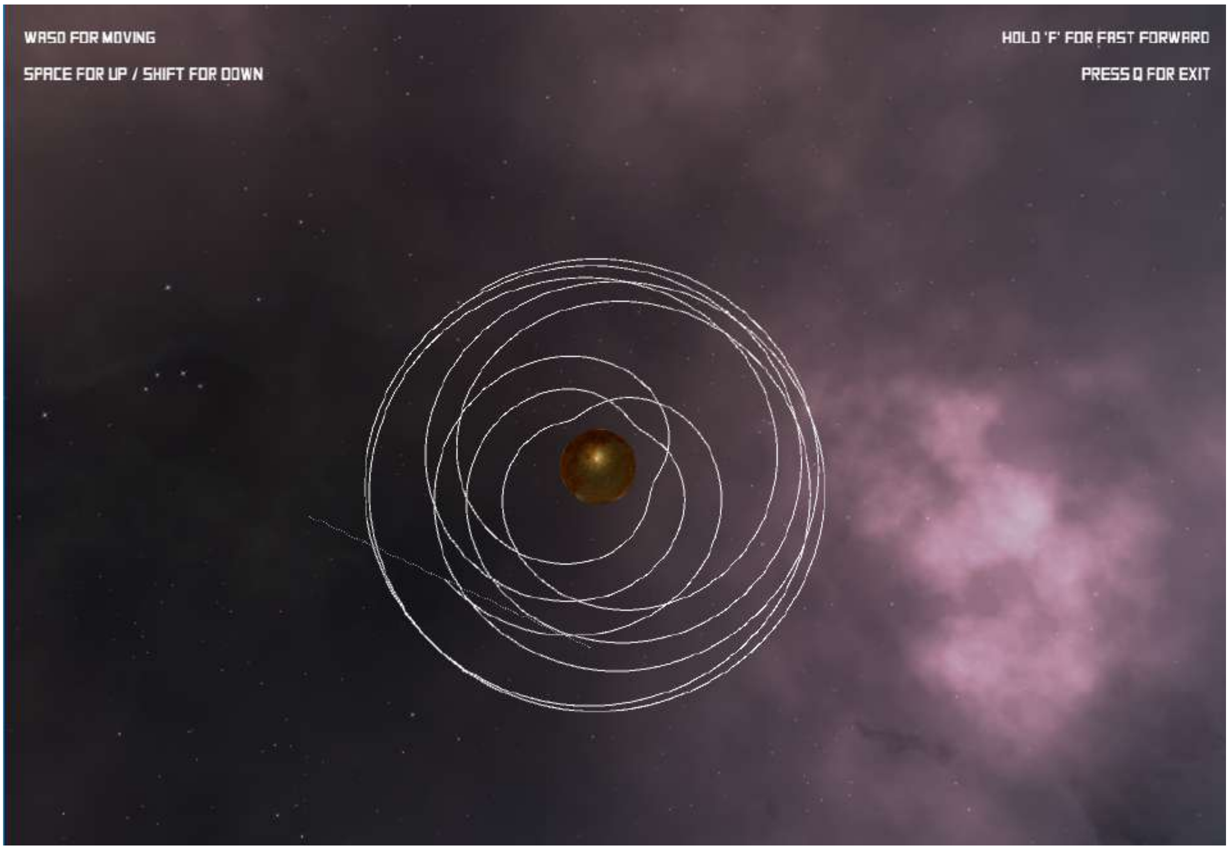


Figure 6.1: Graph under development of the midpoint of a segment connecting Venus and Mercury

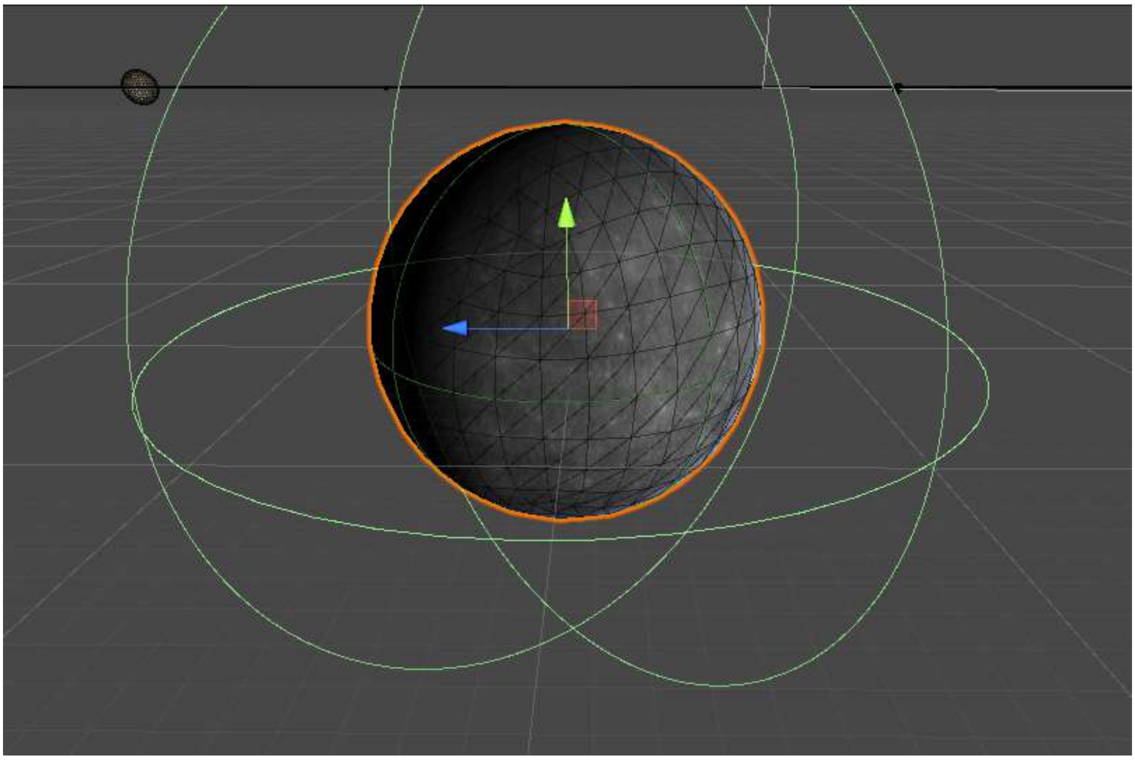


Figure 6.2: Representation of the orbit of Mercury

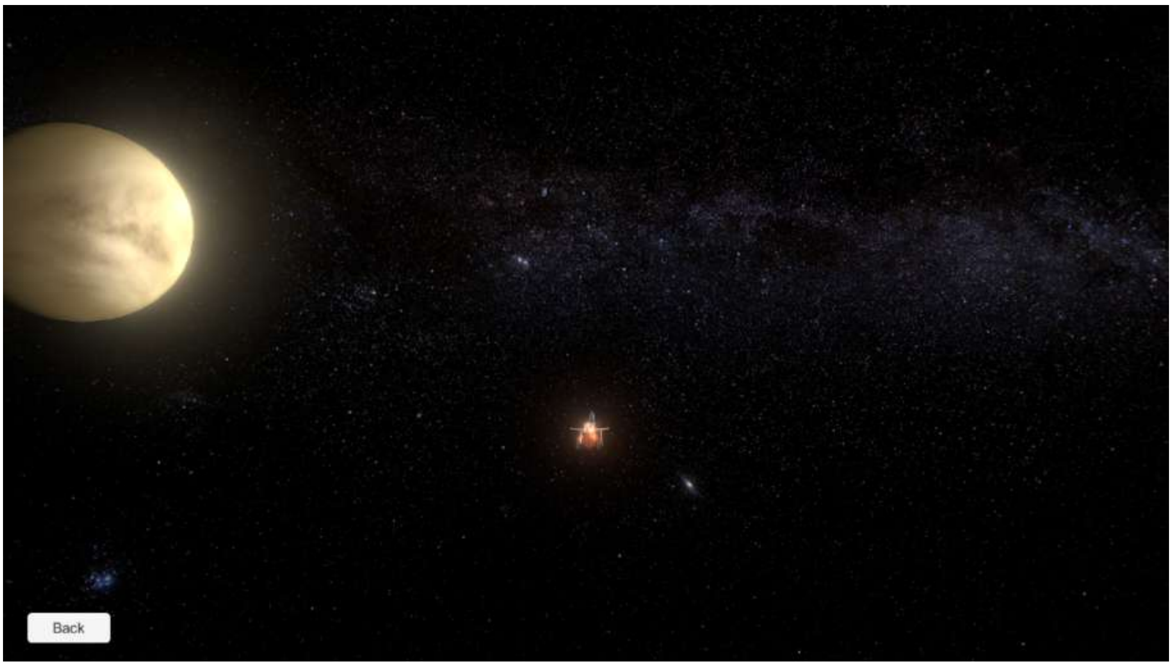


Figure 6.3: Second simulation image

We realized that this system of representation of physical phenomena is effective for the understanding of theoretical concepts. However, we found many difficulties, in particular with regard to the motion of the spaceship in the second simulation. We could have improved if we had had more time available, adding more parameters allowing us to create a model as close as possible to the real one.