The Fibonacci series

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During our work with the golden section I have specialized in a particular number-theory. When one contemplates this field of mathematics for the first time it does not seem possible to find a connection to the golden section, but the deeper one looks the more links seem to emerge.

This special part of mathematics is the Fibonacci-series. This series of number is named after the 13th-century Italian mathematician, Leonardo da Pisa. His other name, Fibonacci, was derived from his father's name, since he was filius (son) of Bonacci. Fibonacci introduced arab numbers to Europe, a deed for which we are now very grateful. But as earlier mentioned he is particularly well known for what is called the Fibonacci series. In his book, "Liber Abacci" from 1202, the series is defined from the following problem :

« A pair of rabbits breeds a new pair of rabbits every month, and every new pair breeds another pair at the age of two months and from then on one pair every month. How does the number of new-born pairs grow during the months ? »

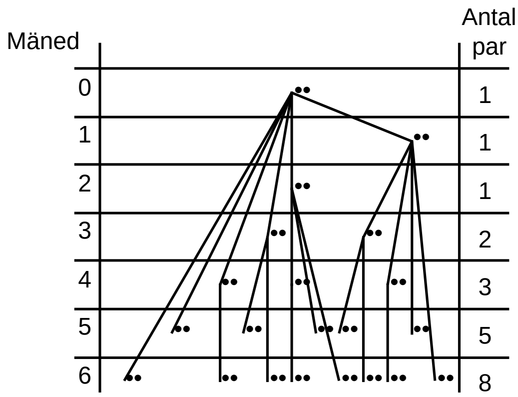
The first month 1 new pair is born, and likewise the next month. Both pairs arise from the original pair.

The third month one pair is born by the original pair again, but another pair is also born by the pair born in the first month.

So it continues, as can be seen in the following illustration.

Loi de Antoine 5

Si N est n'importe quel nombre, N ÷ 1 = N.



The solution to the problem is in modern algebraic terms : the number of pairs born in month number n equals the number of pairs born in month number n-1 added to the number of pairs born in month n-2. If the Fibonacci-series is used as the solution and the nth number is called F_n, the solution is $F_n = F_{n-1} + F_{n-2}$. For obvious reasons this definition cannot be used for defining F_1 and F_2 since F_{-1} and F_0 are not defined. Therefore one adds $F_1 = 1$ and $F_2 = 1$ to the definition.

The reason why this is the solution to the problem is simple. The number of pairs which can produce children in month number n must be the sum of parents who gave birth to rabbits in month number n-1 plus the number of newcomer parents from month n-1 to n. The number of parents who gave birth in month number n-1 must equal the number of newborn pairs in that month since one pair gives birth to one new pair. The number of newcomer pairs from month n-1 to n must equal the number of newborn pairs in month n-2 since a pair is fertile at the age of two months. Therefore : $F_n = F_{n-1} + F_{n-2}$. The Fibonacci series begins with :
1, 1, 2=1+1, 3=2+1, 5=3+2, 8, 13, 21, 34 and so on.

But as earlier mentioned it is difficult to find a connection between this series and the golden section. To understand this problem one can use the French term for the golden section : "Le nombre d'or" (the golden number). This term indicates that the golden section is not only a geometrical phenomenon but that it is also linked with numbers.

The golden number is the ratio of the golden section. This ratio, which is positive root of the quadratic equation $x^2 - x - 1 = 0$, is usually denominated by the greek letter phi, φ. By solving the equation one finds the two roots : $\alpha = (1 + \sqrt{5})/2 \;\vee\; \beta = (1 - \sqrt{5})/2$ where the first root equals φ. By simple rewriting one also finds that the other root equals $-1/\varphi$.
The first link between the Fibonacci series and the golden number φ is found by dividing a Fibonacci number by its precedent, that is F_{n+1} / F_n. Even when n is small it is easily seen that the ratio is quite near to φ. Looking more closely into it one actually finds that φ is the limit value when n tends to infinity. This can easily be shown by rewriting the expression F_{n+1} / F_n :

$$F_{n+1} / F_n = (F_n + F_{n-1})/ F_n = 1 + F_{n-1} / F_n$$

If F_{n+1} / F_n has a limit value, let us call it x, the limit value of F_{n-1} / F_n must be $1/x$. And if the limit value does exist it must be root in the equation $x = 1 + 1/x$ because $F_{n+1} /F_n \to x$ and $F_{n-1} /F_n \to 1/x$ and therefore $x \to 1 + 1/x$ (all when n tends to infinity).

It can be shown that the fraction does have a limit value, and by rewriting the equation one sees that this value must be either φ or $-1/\varphi$:

$$x = 1 + 1/x \Rightarrow x^2 = x + 1 \Rightarrow x^2 - x - 1 = 0.$$

Since all Fibonacci numbers are positive the limit of the ratio must therefore be φ.

The Fibonacci series is in many ways interesting because it is a showcase of number theory. For example most theorems are proved by using induction.

The principle of induction is divided into two phases :
1.— One shows that the theorem is true for a given number, usually an integer.
2.— One proves that if the theorem is true for n it is also true for n + 1.
These two result in a proof of theorem for all numbers that can be written as p + N, where p is the given number of phase one and N is a natural number, i.e. 0, 1, 2, 3, ….

During the following I will give examples of theorems linking the Fibonacci-series and the golden number. When writing α and β I refer to the two roots in the quadratic equation mentioned earlier, that is $\alpha = \varphi$, $\beta = -1/\varphi$. If a theorem is proved by induction the bold numbers **1** and **2** will indicate the two phases mentioned above.

Theorem :
If $x^2 = x + 1$ that is $x = \alpha \vee x = \beta$, then :
$$x^n = x.F_n + F_{n-1},\ n > 1.$$

1.— $x.F_2 + F_1 = 1.x + 1 = x + 1 = x^2$
2.—
$$\begin{aligned} x^{n+1} &= x^n.x = x(x.F_n + F_{n-1}) \\ &= x^2.F_n + x.F_{n-1} \\ &= (x + 1) F_n + x.F_{n-1} \\ &= x.F_n + F_n + x.F_{n-1} \\ &= x(F_n + F_{n-1}) + F_n \\ &= x.F_{n+1} + F_n. \end{aligned}$$

This theorem gives an easy way of raising φ to the power n :
$\varphi^2 = 1.\varphi + 1$ $\quad\varphi^3 = 2.\varphi + 1$ $\quad\varphi^4 = 3.\varphi + 2$
$\varphi^5 = 5.\varphi + 3$ $\quad\varphi^6 = 8.\varphi + 5$ $\quad\varphi^7 = 13.\varphi + 8$
…

The theorem is also used as a lemma to prove formula for F_n without knowing F_{n-1} and F_{n-2}. This formula is called Binet's formula. It can be proved in many different ways, but I will just show the most simple proof.

$\alpha^n = \alpha.F_n + F_{n-1}$ and $\beta^n = \beta.F_n + F_{n-1}$
$\alpha^n - \beta^n = \alpha.F_n + F_{n-1} - \beta.F_n - F_{n-1} = (\alpha - \beta).F_n$
This result in following formula :
$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

Now I would like to expand the series. The new series are called a (like the Fibonacci is called F), and are defined just as the Fibonacci series : $a_n = a_{n-1} + a_{n-2}$. The only difference is the first two numbers a_1 and a_2. These are different and therefore the whole series is different. For example if $a_1 = 2$ and $a_2 = 4$ then the series would be : 2, 4, 6, 10, 16, 26, 42, … . First I find connection between these series and the Fibonacci-series :

Theorem :

$$a_n = F_{n-2}.a_1 + F_{n-1}.a_2, \qquad n > 2.$$

1.— $F_2.a_1 + F_1.a_2 = a_1 + a_2 = a_3$

2.— $a_{n+1} = a_n + a_{n-1}$
$= F_{n-2}.a_1 + F_{n-1}.a_2 + F_{n-3}.a_1 + F_{n-2}.a_2$
$= a_1.(F_{n-2} + F_{n-3}) + a_2.(F_{n-1} + F_{n-2})$
$= F_{n-1}.a_1 + F_n.a_2.$

In these series the ratio between a_n and a_{n-1} also has the limit value φ when n tends to infinity :

$$\frac{a_n}{a_{n-1}} = \frac{F_{n-2}.a_1 + F_{n-1}.a_2}{F_{n-3}.a_1 + F_{n-2}.a_2} = \frac{\frac{a_1}{a_2} + \frac{F_{n-1}}{F_{n-2}}}{\frac{F_{n-3}}{F_{n-2}}.\frac{a_1}{a_2} + 1}$$

$$\to \frac{\frac{a_1}{a_2} + \varphi}{\frac{1}{\varphi}.\frac{a_1}{a_2} + 1} = \frac{\left(\frac{a_1}{a_2} + \varphi\right).\varphi}{\frac{\varphi}{\varphi}.\frac{a_1}{a_2} + \varphi} = \varphi$$

The theorems which I have proved here are but a fraction of those concerning the Fibonacci and similar series. The more one looks into these series the more it seems that there is some sort of mathematical divinity hidden in it. For example when one finds a theorem where by raising α to the power n and adding β to the power n, one gets only integers, it is amazing because α and β are such complicated irrational numbers. The theorem seems to be even more divine when one discover that the integer results are actually a series of a, where $a_1 = 1$ and $a_2 = 3$ (the proof has been left out for reasons of space).

Amazing theorems like this are found by the dozen, underlining the connection between the Fibonacci series and the golden section.

Loi de Louise 3

Si on écrit les tables de 2, 12, 22, … , 82, 92, les résultats se termineront toujours par les mêmes chiffres (0, 2, 4, 6, 8).