This article is written by students. It may have omissions and imperfections, reported as far as possible by our reviewers in the editing notes.

Volume and density of a tree

2020-2021

Name and level of students : Apahidean Luca Teodor, Cotrău Marcus, Pop Claudiu Gabriel, students in the XI-th grade ; Radu Tudor, Hațegan Amalia, Gavrilă Cătălin, students in the X-th grade.

School : Colegiul Național "Emil Racoviță" Cluj-Napoca

Teacher : Văcărețu Ariana Stanca

Researcher : Lorand Parajdi, Babes-Bolyai University

1 Introducing the research topic

Our task was to find a method for calculating the volume of a tree and its density.

2 Volume of the tree

Entry dates : L (Lenght) R (Diameter of big base) r (Diameter of small base). [1]
If we assume that the volume of the trunk is approximately like a cone trunk, then we can use the following formula :

$$V = \pi \cdot \frac{L}{3}\left(\frac{R^2}{4} + \frac{R \cdot r}{4} + \frac{r^2}{4}\right)$$

2.1 Calculating R and r

In order to calculate R and r we must use a compass.

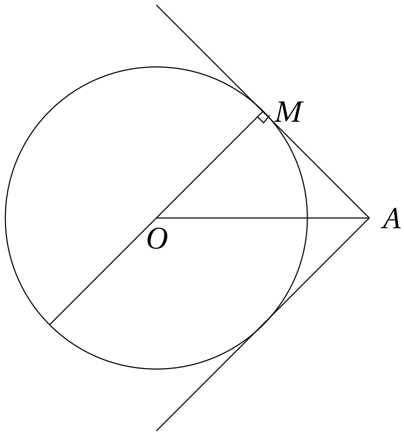


Figure 1

Since $\triangle OMA$ is a right triangle, then $tan(\angle MAO) = \frac{R}{2MA}$. From here we obtain : $R = 2MA \cdot tan(\angle MAO)$. [2] For r we use the same reasoning.

2.2 Volume of the entire tree

For this task we used fractals to calculate the entire tree volume. Let be the following fractal :

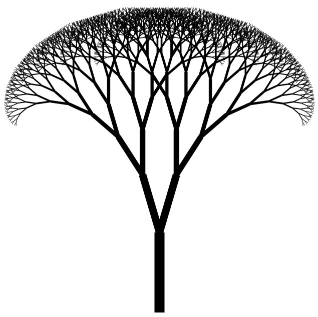


Figure 2 - Fractal

We construct the fractal by forming from each branch two "smaller" branches forming an "Y". We say that a transformation is the process that forms from each branch two new branches. We say that a branch is n-th order if it was formed after exactly n transformations. Let q be the ratio of two consecutive branches [3]. Two branches are consecutive if one is "formed" from another one. Now let be $(L)_n$ a series representing the length and $(R)_n$, $(r)_n$ the base diameter, respectively the diameter for the small base of an n order branch. Now we can define the following recurrences :

$L_n = \dfrac{L_{n-1}}{q}\ \forall n \geq 1 ;\ L_0 = L$

$R_n = \dfrac{R_{n-1}}{q}\ \forall n \geq 1 ;\ R_0 = R$

$r_n = \dfrac{r_{n-1}}{q}$ $\forall n \geq 1$; $r_0 = r$
Now it's easy to see that :
$L_n = \dfrac{L}{q^n}$, $R_n = \dfrac{R}{q^n}$ and $r_n = \dfrac{r}{q^n}$ $\forall n \geq 0$

Now let's count the number of branches that appear at each new transformation. By letting $(a)_n$ be a sequence of natural numbers, we can set $a_0 = 1$ because we have initially one branch(the tree trunk), and we can see that $a_n = 2a_{n-1}$ because the number of branches that are formed doubles each time. So we can say that $a_n = 2^n$ $\forall n \geq 0$
For calculating the volume of the entire tree, we have to calculate the volume for the total number of n-th order branches. If we set the following sequence $(V)_n$; $V_0 = V$ where V is the volume only of the tree trunk.
We can see that : $V_n = V_{n-1} + \pi a_n \cdot \dfrac{L_n}{3}\left(\dfrac{{R_n}^2}{4} + \dfrac{R_n \cdot r_n}{4} + \dfrac{{r_n}^2}{4}\right)$ $\forall n \geq 1$ (1)
Now if we let i take values from 1 up to n, and for each i we put it in (1), and after we sum everything up we get :

$$V_n = V_0 + \sum_{i=1}^{n} \left[\pi a_i \cdot \frac{L_i}{3}\left(\frac{{R_i}^2}{4} + \frac{R_i \cdot r_i}{4} + \frac{{r_i}^2}{4}\right)\right] \ \forall n \geq 1 \ \Leftrightarrow$$

$$V_n = V + \frac{\pi}{12} \sum_{i=1}^{n} \left[a_i \cdot L_i \left({R_i}^2 + R_i \cdot r_i + {r_i}^2\right)\right] \ \forall n \geq 1$$

If we substitute for each sequence the formulas that we deduces earlier we have :

$$V_n = V + \frac{\pi}{12} \sum_{i=1}^{n} \left[2^i \cdot \frac{L}{q^i}\left(\left(\frac{R}{q^i}\right)^2 + \left(\frac{R}{q^i}\right)\cdot\left(\frac{r}{q^i}\right) + \left(\frac{r}{q^i}\right)^2\right)\right] \ \forall n \geq 1$$

If we factor out a $\dfrac{1}{q^{3i}}$ we get :

$$V_n = V + \frac{\pi}{12} \sum_{i=1}^{n} \frac{2^i}{q^{3i}} \cdot L(R^2 + R \cdot r + r^2) \ \forall n \geq 1$$

Since $V = \dfrac{\pi}{12} \cdot L(R^2 + R \cdot r + r^2)$ we can see that $V_n = V \displaystyle\sum_{i=0}^{n} \frac{2^i}{q^{3i}}$ $\forall n \geq 1$ Let be U the volume for the entire tree. We can say that in order to get a closer approximation for U we should let n go to infinity, and (V_n) must be convergent. So : $U = \lim\limits_{n\to\infty} V_n$

Proposition 1

Let r be a real number such that $|\, r \,| < 1$, then $\lim\limits_{n\to\infty} \displaystyle\sum_{i=0}^{n} r^i = \frac{1}{1-r}$

Now if we apply this for $r = \dfrac{2}{q^3}$, $|\, r \,| < 1$ we have that $U = \lim\limits_{n\to\infty} V_n =$

$$= \lim_{n\to\infty} V \sum_{i=0}^{n} \left(\frac{2}{q^3}\right)^i = V \cdot \frac{1}{1 - \frac{2}{q^3}} = V \frac{q^3}{q^3 - 2}$$

$$U = V\frac{q^3}{q^3-2}$$

So we can see that the volume of the entire tree depends on the ratio q. This is the graph for the function $f : \left(\sqrt[3]{2},\infty\right) \to \mathbb{R};\ f(x) = \dfrac{x^3}{x^3-2}$

3 Density of the tree

First we can assume that every part of the tree has the same density as the whole tree. Using this fact, we can cut off a part from the tree and we can calculate its volume using the method presented above, and since it's a small part, we can calculate its weight, and knowing these values, we can calculate the density using the following formula : $\rho = \frac{m}{V}$, where ρ is the density, m stands for mass and V represents the volume.

Let be r representing the percentage of water from the volume of the wet tree. Let be ρ the density of the wet tree, calculated above. We can establish the following relationship : $\rho = (1-r)\cdot\rho_{tree} + r\cdot\rho_{water}$; $\quad$ (ρ_{tree} is the density of the dry tree.)

Proof for relation above :
Let be V_w be the volume of the water contained in the tree, and V_t the volume of the dry tree. We have that : $V_w = rV$ and $V_t = (1-r)V$, where V is the volume of the wet tree. Now if we set m_w be the mass of the water from the tree and let m_t be the mass of the dry tree and we obtain : $m_t = \rho_{tree}\cdot V_t$ and $m_w = \rho_{water}\cdot V_w$. Since $\rho = \frac{m_w+m_t}{V_w+V_t}$, we obtain : $\rho = \frac{V(r\rho_{water}+(1-r)\rho_{tree})}{V}$. After simplification we obtain the relationship mentioned above.
We know that $\rho_{water} = 1000\frac{kg}{m^3}$.

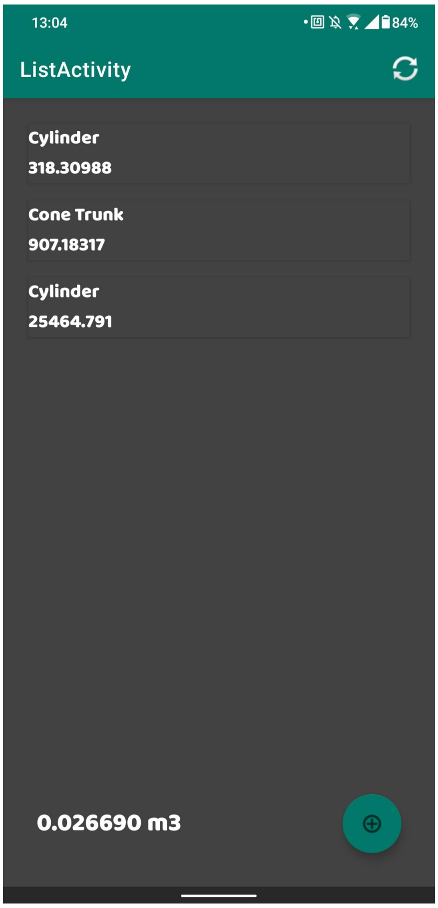
And we obtain : $\rho_{tree} = \dfrac{\rho-1000r}{1-r}\ \dfrac{kg}{m^3}$

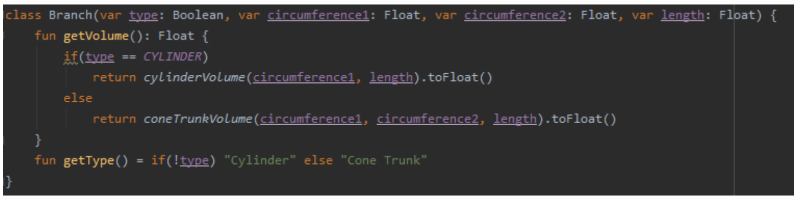
We must mention that since r represents a percent, then $r \in (0;1)$.

4 Android Application

Complementary to the research, we've also built an Android application that facilitates the calculation of the tree volume. The app takes as parameters the types of branches (cylindrical or cone-trunk shaped), their lengths and their circumferences and adds the volumes of each branch resulting in the total volume of the tree, which is output in cubic meters.

The application was built using the Android Software Development Kit with the Kotlin programming language.





```
class Branch(var type: Boolean, var circumference1: Float, var circumference2: Float, var length: Float) {
    fun getVolume(): Float {
        if(type == CYLINDER)
            return cylinderVolume(circumference1, length).toFloat()
        else
            return coneTrunkVolume(circumference1, circumference2, length).toFloat()
    }
    fun getType() = if(!type) "Cylinder" else "Cone Trunk"
}
```

5 Conclusion

The production of lumber is one of the most important sectors that implies the use of wood. The biggest producers of lumber are the USA (106 mil.m^3) , Japan(27,5 mil.m^3), Sweden (12,7 mil.m^3), Germany, Finland, France , Poland, Austria. Recent studies have found that, each year, the roundwood industry consumes 2.4B m^3. Therefore, the understanding of the tree volume and its density is an important topic.

EDITION NOTES

[1] Calling R and r the diameters of the big and small bases might be misleading. The notation D and d would have been preferable.

[2] This equality is never used in the paper and holds for every circle and any external point A. It is presumably meant to provide a practical tool for calculating the diameter of the tree when there is no way of determining it by a direct measurement. If this is the case, some comments or explanations would be needed.

[3] The possible values of the ratio q should have been discussed in some way. To make sense of the problem, the quantity $V_n - V_{n-1}$ must be strictly decreasing, otherwise, the limit of V_n would be trivially $+\infty$.
Every branch B in the tree generates two equal branches B_1 and B_2. Then. the quantity $V_n - V_{n-1}$ is strictly decreasing if and only if the volume of B is greater than the sum of the volumes of B_1 and B_2 :

$$V(B) > V(B_1) + V(B_2) = 2V(B_1) \qquad (*)$$

Since the ratio between the linear measures of B and B_1 is q, we have $V(B_1) = \frac{1}{q^3}V(B)$, and hence $(*)$ is equivalent to $\frac{2}{q^3} < 1$. Indeed, the same inequality appears at the end of p. 3, as a necessary and sufficient condition for the convergence of the sequence V_n.