Winning Bets

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The research topic

At a casino, people are betting on the results of 7 matches. Each match can have 2 outcomes (0-the guest team wins, 1-the away team wins). The first prize is won for guessing all results, the second one for guessing all except one. A playslip consists of the choices of 7 results.

- A: Which is the probability of winning the first prize (respectively the second one) for a randomly completed playslip?
- B: What is the minimum number of playslips which have to be completed in order to surely win the first prize? What is the minimum number of playslips which have to be completed in order to win at least a first prize or a second prize? Give such an example of completing the playslips.
- C: Same questions as above but in the hypothesis the matches can have 3 results (0,x,1) and that there are 13 matches(Sweepstakes game).
- D: Generalization.

The conjectures and results obtained

A:

The numbers from the playslips have the following form: 1111111, 1101101; 7-digit numbers, the digits can be either 0 or 1.

Therefore, we have in total 2^7 possible playslips. So, the probability of winning the first prize with a randomly completed playslip is 1/128(0.78105%) (favorable cases/total number of cases).

The probability of winning the second prize is 7/128(5.46875%).

We have 7 possibilities of winning the second prize, guessing all the results but one. Being 7 matches there are 7 cases of mistaking the result of one match. For example, if the number 1111111 would be the winning one, we can win the second prize with the following 7 playslips:

- 0111111
- 1011111
- 1101111
- 1110111
- 1111011
- 1111101
- 1111110

B:

For being sure that we will win the first prize the minimum number of playslips which we must complete is exactly the number of possible combinations: 128.

Instead, in order to determine the minimum number of playslips which we have to complete for winning at least a prize(first or second) we have to analyze more carefully the problem.

On the same example: 1111111 we can easily observe that it gives us a possibility of winning the first prize (1111111), and 7 possibilities of winning the second one, the list of the needed numbers was presented in the previous stage of this problem.

So, we deduced the fact that if we complete a playslip it will take the place, in reality, of 8 completed playslips, offering a chance of winning the first prize and 7 of winning the second prize.

Therefore, for winning at least a prize we need 128/8=16 completed playslips. Their filling is, although, difficult, as we have to take into account the following aspect: The chosen numbers can't cover the same number (option) more than once.

For example, if the winning number is 1111111, writing 1111110 and 0111111 will not be in our advantage. If we switch the last bit of 1111110 and the first one of 0111111 into the opposite bit, they both cover the same option, 1111111. Therefore, we will get to the following conclusion: the chosen numbers has to differ one from the other in at least 3 places. Numbers of the type 111abcd, 000efgh will surely not cover the same second prize option.

The selection of the numbers:

Another part of the problem is finding the playslips that will bring us the win. Thus, when we take our example, we have 2 alternative digits and seven positions. So, 128 tickets in total, of which, as already mentioned we choose 16 in order to obtain at least the 2^{nd} prize.

The first step is choosing one of the numbers from the playslip. Let's pick, for example,0. We divide, then, the numbers in 8 groups, each containing all the numbers that figure 0 as many times as is the number of the group (eg Group 3 will contain all the playslips with 3 variants 0, Group 4 will contain all the playslips with 4 variants 0, etc).

The next step is to see how many and which positions a playslip from every group occupies. Let's choose for example the number from group number 3: 0110011. If we pick a bit which has the value 1 and change it into 0, the obtained playslip will have 4 zeros, thus positioning itself in group number 4. So, if our playslip has 4 bits with value 1, changing these values into 0, this playslip will cover 4 other ones from group number 4. With the same reasoning we can change the three 0-valued bits into the opposite one, thus covering 3 new playslips from group number 2. Applying this reasoning for each of the 8 groups we get the following table:

The group number	0	I	II	III	IV	V	VI	VII
0	1	7	0	0	0	0	0	0
I	1	1	6	0	0	0	0	0
II	0	2	1	5	0	0	0	0
III	0	0	3	1	4	0	0	0
IV	0	0	0	4	1	3	0	0
V	0	0	0	0	5	1	2	0
VI	0	0	0	0	0	6	1	1
VII	0	0	0	0	0	0	7	1

The table shows us how many playslips from a certain group does a randomly chosen one from each group cover (the groups that we choose from are on the vertical line, and the groups with the covered playslips are on the horizontal lines). For example, we saw that a playslip from group number 3 can cover 3 other ones from group 2 and 4 other ones from group 3, from all the other groups it can't cover anything just by changing a bit, so the rest of the numbers are 0. For a playslip from group 2, we can see that it covers itself, 2 other ones from group 1 (by changing the 0-bits into ones) and 5 other ones from group 3 (by changing the 1-bits into 0).

In each group we have the following number of playslips:

0	1 $\binom{7}{0}$
I	7 $\binom{7}{1}$
II	21 $\binom{7}{2}$
III	35 $\binom{7}{3}$
IV	35 $\binom{7}{4}$
V	21 $\binom{7}{5}$
VI	7 $\binom{7}{6}$
VII	1 $\binom{7}{7}$

In order for every playslip to be covered by the other chosen playslips, we have to be careful at the distribution of the 16 playslips in the 8 groups. Because group number 7 can be covered by playslips from groups 6 and 7, the first 2 playslips chosen will be from the groups 7 and 0 (because of the symmetry): 0000000, 111111. Choosing these ones we will have covered all the playslips from the 1 and 6 groups.

We observe that the groups 4 and 5 have 35 and 21 playslips, which both are multiples of 7. Thus, in order to make sure that all the playslips from groups 5 and 2 will be covered will choose 7 from the fourth group and 7 from the third one. In total we will have 16 playslips from all the groups:

Group number	0	I	II	III	IV	V	VI	VII
0	1	7	0	0	0	0	0	0
III	0	0	21	7	28	0	0	0
IV	0	0	0	28	7	21	0	0
VII	0	0	0	0	0	0	7	1
Total	1	7	21	35	35	21	7	1

The groups that we can select the playslips from are as we have shown 0,3,4 and 7 (listed on the vertical line). We know that we need 1 playslip from group number 1, 7 from groups 3 and 4 and another one from group 7, the playslips not covering the same thing twice. Combining the previous table that distributed the number of covered playslips from each group with the fact that we need 1,7,7 and 1 from these groups, we get for each one of them the following table. On the final row we summed the total number of covered playslips from each group, observing that the row is identical with the total number of playslips from each group, thus proving that this distribution can guarantee us the second prize.

We saw from which groups we have to choose the numbers. Now we have to see which are the actual numbers. Let's analyze the third group. Each playslip covers four other playslips from the fourth group. The condition is that the chosen playslips can't cover the same thing. So, they have to be different in at least 3 positions. Applying this condition we have the numbers from the third group:

- 0011101; 0101110;
- 0110011; 1001011;
- 1010110; 1100101;
- 1111000.

We will choose the numbers from the fourth group by symmetry.

In conclusion, the numbers chosen are:

- 0011101; 0101110;
- 0110011; 1001011;
- 1010110; 1100101;
- 1111000; 1100010;
- 1010001; 1001100;
- 0110100; 0101001;
- 0011010; 0000111;
- 1111111; 0000000.

C:

The probability of winning the first prize it's the number of favorable cases divided by the number of all cases. So it's $1/3^{13}$ (0.0000607%). For the second prize, the chance is 26 (number of possible cases)/3^{13} (0.0016307%). Now, for completing one playslip randomly we have one possibility of winning the first prize and 26 possibility of winning the second prize. For example, if the winning number is 1111111111111 we can win the first prize by completing this number and second prize by completing any of the following numbers:

- 0111111111111 x111111111111
- 1011111111111 1x11111111111
- 1101111111111 11x1111111111
- 1110111111111 111x111111111
- 1111011111111 1111x11111111
- 1111101111111 11111x1111111
- 1111110111111 111111x111111
- 1111111011111 1111111x11111
- 1111111101111 11111111x1111
- 1111111110111 111111111x111
- 1111111111011 1111111111x11
- 1111111111101 11111111111x1
- 1111111111110 111111111111x

Thus, the minimum number for winning surely one prize is the number of all possible cases/the number of cases in which you win the first prize(1) + the number of cases which you win the second prize(26). So, it's $3^{13}/27=3^{10}$.

D:

We will define the number of the matches in a playslip with "n" and with x the number of possible outcomes of a match.

Therefore, the total number of possible playslips will be x^n. The chance of winning the first prize is $1/x^n$.(and for winning surely the first prize we have to complete obviously x^n.

Now, a completed playslip offers us a chance of winning the first prize, and (x-1)*n possibilities of winning the second prize. So, for a chance of winning the first or the second prize we have to complete:

$$\frac{x^n}{(x-1)*n+1}$$

Conclusions:

- Each playslip can cover other playslips that can bring lower prizes;
- If we would have "x" outcomes for a match and "n" matches we would need $\frac{x^n}{(x-1)*n+1}$ completed playslips to be sure we would win any of the first two prizes;
- If we want to find the lucky playslips we need to group all the possible ones and choose the calculated number of playslips so that they will not cover the same playslip.