A Sweet Problem

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[Presentation of the research topic]

A cake as a cuboid has to be served by 100 persons at a birthday party. The person who celebrates his birthday has to cut the cake such that everybody who is there can eat a piece in order to taste the sweet cake. Pieces can be every shape or size; they do not have to be equal, but they cannot be rearranged after any cut. The cake can be cut every direction.

a) Which one is the minimum number of cuts the celebrated person should do in order to satisfy all guests (everybody tastes the cake)?
b) Which one is the minimum number of cuts in order to get equal pieces?
c) The cake is covered by chocolate (except the base). The aria covered by chocolate is equal to 0.4 m^2. Find the dimensions of the cake that has maximum volume. The thickness of the glaze is neglected.
d) A company produces packs for cakes. The pack has to be a cuboid made of cardboard with a volume of 0.03 m^3, with a double base as a square. The price of cardboard is 0.5 USD for 1 m^2. Build the cheapest box.

[The text of the article]

Part a)

Which one is the minimum number of cuts the celebrated person should do in order to satisfy all guests (everybody tastes the cake)?

Solution

Number of cuts	0	1	2	3	4…
Number of pieces	1	2	4	8	15…

Below, we write the differences, the difference of differences and so on, for the consecutive values of a_n.

1 2 4 8 15…

1 2 4 7

1 2 3

1 1

We observe that after three iterations we arrive at a constant sequence. Therefore, the general term of the sequence will be of the form:

$$a_n = an^3 + bn^2 + cn + d$$

By writing the first four terms of the sequence, we have:

$$a_0 = a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d = 1 \Leftrightarrow d = 1$$

$$a_1 = a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d = 2 \Leftrightarrow a + b + c = 1$$

$$a_2 = a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d = 4 \Leftrightarrow 8a + 4b + 2c = 3$$

$$a_3 = a \cdot 3^3 + b \cdot 3^2 + c \cdot 3 + d = 8 \Leftrightarrow 27a + 9b + 3c = 7$$

Therefore, we solve the system for *a, b, c, d* and we obtain :

$$a = \frac{1}{6},\ b = 0,\ c = \frac{5}{6}.$$

As a consequence, $a_n = \frac{1}{6}n^3 + \frac{5}{6}n + 1.$

Calculating, we get :

$a_5 = 26,\ a_6 = 42,\ a_7 = 64,\ a_8 = 93,\ a_9 = 130.$

Thus, we need 9 cuts in order to have at least 100 pieces of cake.

Part b)

Which one is the minimum number of cuts in order to get equal pieces?

We found a rule for the number of cuts in order to get 100 equal pieces for the guests at that party.

For instance, we can cut the length of the cake into 5 equal horizontal columns (4 cuts), the width into 4 equal rows (3 cuts) and the height into 5 equal layers (4 cuts). Any other similar way of cutting will lead us to 100 equal pieces of cake.

Therefore, we need to cut the cake 11 times in the way mentioned before in order to get 100 equal pieces of cake.

Part c)

The cake is covered by chocolate (except the base). The aria covered by chocolate is equal to 0.4 m^2. Find the dimensions of the cake that has maximum volume. The thickness of the glaze is neglected.

We denote by a the length of the cake, b the width of the cuboid cake, and c the height of it. The volume of this cuboid is

$$V = a\cdot b\cdot c$$

and the area covered by chocolate is equal to $A = a\cdot b + 2a\cdot c + 2b\cdot c$. In order to get the value of dimensions of the cake for maximum volume when the area is given (**0.4 m^2**), we use the inequality of mean (the arithmetic mean is greater than or equal to geometric mean).

More precisely,

$$A = \frac{a\cdot b}{2} + \frac{a\cdot b}{2} + a\cdot c + a\cdot c + b\cdot c + b\cdot c = 2\left(\frac{a\cdot b}{2} + a\cdot c + b\cdot c\right).$$

$$A \geq 2\cdot 3\cdot \sqrt[3]{\frac{a\cdot b}{2}\cdot a\cdot c\cdot b\cdot c} = \frac{6}{\sqrt[3]{2}}(a\cdot b\cdot c)^{2/3} = 3\sqrt[3]{4}\cdot V^{2/3}.$$

$$0,4 \geq 3\sqrt[3]{4}\cdot V^{2/3}.$$

Therefore, the maximum value of volume is equal to

$$V = \frac{4}{30\sqrt{30}} \text{ m}^3$$

and it is reached when $a = b = 2c$.

Thus, $a = b = \frac{\sqrt{30}}{15}$ m and $c = \frac{\sqrt{30}}{30}$ m, which means, approximating, $a = b = 36.5$ cm and $c = 18,25$ cm.

Part d)

A company produces packs for cakes. The pack has to be a cuboid made of cardboard with a volume of 0.03 m^3, with a double base as a square. The price of cardboard is 0.5 USD for 1 m^2. Build the cheapest box.

We will split the problem into two cases :

I. Base is a square.

II. Base is a square with a double base..

I. Base is a square with a simple base.

We assume, in this case, that $a = b$, and this means that the area is equal to $A = 2a^2 + 4a \cdot c$ and the volume is equal to $V = a^2 \cdot c = 0.03\,\text{m}^3$.

Using same technique as in c), we get

$$A = 2a \cdot c + 2a \cdot c + 2a^2 \geq 3 \cdot \sqrt[3]{2a \cdot c \cdot 2a \cdot c \cdot 2a^2} = 6 \cdot V^{2/3}.$$

The minimum area is reached when $a = c$, which means that $a^3 = 0.03 \Leftrightarrow a = \sqrt[3]{0.03}$ m. Thus, area is equal to $A = 6 \cdot V^{2/3} = 6 \cdot 0.03^{2/3}$ m^2, which means approximately $A \approx 0.579$ m^2. Taking into consideration the price for 1 m^2 of cupboard, we obtain the price of the cheapest box, more precisely about $0.579 \cdot 0.5 \approx 0.29$ USD, which means 29 cents.

II. Base is a square with a double base.

The area is equal to $A = 3a^2 + 4a \cdot c$ and the volume is equal to $V = a^2 \cdot c = 0.03\,\text{m}^3$.
Using same technique as in c), we get

$$A = 2a \cdot c + 2a \cdot c + 3a^2 \geq 3 \cdot \sqrt[3]{2a \cdot c \cdot 2a \cdot c \cdot 3a^2} = 3 \cdot \sqrt[3]{12} \cdot V^{2/3}.$$

The minimum area is reached when $c = \frac{3a}{2}$, which means that $a^3 = 0.02 \Leftrightarrow a = \sqrt[3]{0.02}$ m. Thus, area is equal to $A = 9 \cdot a^2 = 9 \cdot 0.02^{2/3}\,\text{m}^2$. Taking into consideration the price for 1 m^2 of cupboard, we obtain the price of the cheapest box in this case, more precisely about $9 \cdot 0.02^{2/3} \cdot 0.5 = 4.5 \cdot 0.02^{2/3}$ USD.